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A Modern Theory of Integration

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in Mathematics

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ABSTRACT. This book gives an introduction to integration theory via the "generalized Riemann integral" due to Henstock and Kurzweil. The class of integrable functions coincides with those of Denjoy and Perron and includes all conditionally convergent improper integrals as well as the Lebesgue integrable functions. Using this general integral the author gives a full treatment of the Lebesgue integral on the line.

The book is designed for students of mathematics and of the natural sciences and economics. An understanding of elementary real analysis is assumed, but no familiarity with topology or measure theory is needed. The author provides many examples and a large collection of exercises—many with solutions.

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To Carolyn
with love and thanks

Preface

It is hardly possible to overemphasize the importance of the theory of integration to mathematical analysis; indeed, it is one of the twin pillars on which analysis is built. Granting that, it is surprising that new developments continue to arise in this theory, which was originated by the great Newton and Leibniz over three centuries ago, made rigorous by Riemann in the middle of the nineteenth century, and extended by Lebesgue at the beginning of the twentieth century.

The purpose of this monograph is to present an exposition of a relatively new theory of the integral (variously called the "generalized Riemann integral", the "gauge integral", the "Henstock-Kurzweil integral", etc.) that corrects the defects in the classical Riemann theory and both simplifies and extends the Lebesgue theory of integration. Not wishing to tell only the easy part of the story, we give here a complete exposition of a theory of integration, initiated around 1960 by Jaroslav Kurzweil and Ralph Henstock. Although much of this theory is at the level of an undergraduate course in real analysis, we are aware that some of the more subtle aspects go slightly beyond that level. Hence this monograph is probably most suitable as a text in a first-year graduate course, although much of it can be readily mastered by less advanced students, or a teacher may simply skip over certain proofs.

The principal defects in the Riemann integral are several. The most serious one is that the class of Riemann integrable functions is too small. Even in calculus courses, one needs to extend the integral by defining "improper integrals", either because the integrand has a singularity, or because the interval of integration is infinite. In addition, by taking pointwise limits of Riemann integrable functions, one quickly encounters functions that are no

longer Riemann integrable. Even when one requires uniform convergence, there are problems on infinite intervals.

Other difficulties center around the Fundamental Theorem(s) of Calculus. The Newton-Leibniz formula that we learn in calculus is that

$$\int_a^x f(t)\,dt = F(x) - F(a) \qquad \text{for all} \quad x \in [a,b],$$

when f and F are related by the formula $F'(x) = f(x)$ for all $x \in [a,b]$; that is, when F is a *primitive* (or *antiderivative*) of f on $[a,b]$. Unfortunately, this "theorem" is not always valid; or at least, it requires further hypotheses to be satisfied. The first disappointment a student encounters is that not every Riemann integrable function has a primitive — not only that he or she can't find one, but that such a primitive may not exist. The second potential disappointment (often not learned), is that even when a function has a primitive on $[a,b]$, the function may not be Riemann integrable. Thus, not only is the derivative of the integral not always the function in the integrand (which is perhaps not such a surprise if integration is to be a "smoothing process"), but the integral of the derivative does not always exist.

Towards the end of the nineteenth century, many mathematicians attempted to remedy some of these defects. The most successful was Henri Lebesgue, whose theory enabled one to remove the restriction that the integrand be bounded and the interval be compact. In addition his theory enlarged the class of integrable functions, and gave more satisfactory conditions under which one could take limits, or differentiate under the integral sign.

Unfortunately, Lebesgue's theory did little to simplify the Fundamental Theorem. Spurred by the desire to get an integral in which every derivative was integrable, in the early part of the twentieth century Arnaud Denjoy and Oskar Perron developed integrals that solved this problem — in two very different ways. Surprisingly, their integrals turned out to be equivalent! Moreover, the Denjoy-Perron integrable functions also include conditionally convergent integrals, such as the important Dirichlet integral

$$\int_0^\infty \frac{\sin x}{x}\,dx,$$

that are not included in the Lebesgue theory.

However, there is a price that had to be paid even for the Lebesgue integral: one must first construct a rather considerable theory of measure of sets in $\mathbb{R}$. Consequently, it has long been thought that an adequate theory of

integration is necessarily based on notions that are beyond the undergraduate level of real analysis. (The demands imposed by the Denjoy or Perron theories are considerably greater!) However, Kurzweil and Henstock's integral, which is equivalent to the Denjoy-Perron integral, has a definition that is a slight modification of the definition of the classical Riemann integral. This new integral, which is still not well known, often comes as a surprise to mathematicians whose work is based on the Lebesgue theory.

One of the virtues of the presentation here is that *no measure theory* and virtually *no topology* is required. While some familiarity with the Riemann theory is anticipated as a background, we do not require a mastery of that theory. The only prerequisites are that the reader have good understanding of ε-δ arguments common in a first serious course in real analysis — at the level of the book by the author and D. R. Sherbert [B-S], for example. It will be seen that, by modifying very slightly the definition of the Riemann integral, one obtains an integral that (1) integrates all functions that have primitives, (2) integrates all Riemann integrable functions, (3) integrates all Lebesgue integrable functions, (4) integrates (without further limiting processes) all functions that can be obtained as "improper integrals", and (5) integrates all Denjoy-Perron integrable functions. In addition, this integral has theorems that generalize the Monotone Convergence Theorem and the Dominated Convergence Theorems associated with the Lebesgue integral; thus, it possesses satisfactory convergence theorems.

Although the author has long been familiar with the Riemann and Lebesgue integrals, he has become acquainted only recently with the theory presented here by reading the (relatively) few expositions of it. Most notable of these are: the monograph of McLeod [McL], the relevant chapters in the book of DePree and Swartz [DP-S], the booklets of Henstock [H-5] and P.-Y. Lee [Le-1] and the treatise of Mawhin [M]. In addition, some research articles have been found to be useful to the author. Since work on this monograph was started, the books of Gordon [G], Pfeffer [P], Schechter [Sch] and Lee and Výborný [L-V] have been published; we strongly recommend these books. The author makes few claims for originality, and will be satisfied if this monograph is successful in helping to make this theory better known to the mathematical world.

* * *

In answer to questions about the title of the book, we chose the word "modern" to suggest that we think the theory given here is appropriate for present-day students who will need to combine important concepts from the past with their new ideas. It is not likely that these students will be able to make significant progress in analysis by successive abstraction or further

axiomatization. It is our opinion that a student who thinks of the integral only as a linear functional on a class of functions, but who doesn't know what AC and BV mean has been deprived of fundamental tools from the past. We also think that those whose integration theory does not include the Dirichlet integral are doomed to miss some of the most interesting parts of analysis.

* * *

A few words about the structure of this book are in order. We have chosen to develop rather fully the theory of the integral of functions defined on a compact interval in Part 1, since we think that is the case of greatest interest to the student. In addition, this case does not exhibit some of the technical problems that, in our opinion, only distract and impede the understanding of the reader. In Part 2, we show that this theory can be extended to functions defined on all of the real line. We then develop the theory of Lebesgue measure from the integral, and we make a connection with some of the traditional approaches to the Lebesgue integral.

We believe that the generalized Riemann integral provides a good background for integration theory, since the class of integrable functions is so inclusive. However, there is no doubt that the collection of Lebesgue integrable (i.e., absolutely integrable) functions remains of central importance for many applications. Therefore, we have taken pains to ensure that this class of functions is thoroughly discussed. We have developed the theory sufficiently far that, after reading this book, a reader should be able to continue a study of some of the more specialized (or more general) aspects of the theory of integration, or the applications of the integral to other parts of mathematical analysis.

* * *

Since we believe that one learns best by doing, we have included a large collection of exercises; some are very easy and some are rather demanding. Partial solutions of almost one-third of these exercises are given in the back of the book. A pamphlet, designed for instructors, with partial solutions of all of the exercises can be obtained from the publisher.

In preparing this manuscript, we have obtained useful suggestions from a number of people; we wish to thank Professors Nicolae Dinculeanu, Ivan Dobrakov, Donald R. Sherbert and, especially, Eric Schechter. Two groups of students at Eastern Michigan University worked through the early stages of the initial material and made useful suggestions.

We also wish to thank the staff of the American Mathematical Society for their admirable patience in awaiting the completion of the manuscript and for expeditiously turning it into a published book.

* * *

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September 14, 2000 Robert G. Bartle

Urbana and Ypsilanti

THE GREEK ALPHABET

A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ε	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ	Sigma
H	η	Eta	T	τ	Tau
Θ	θ, ϑ	Theta	Υ	υ	Upsilon
I	ι	Iota	Φ	φ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

Part 1

Integration on Compact Intervals

Ralph Henstock (b. June 2, 1923)
and
Jaroslav Kurzweil (b. May 7, 1926)

Courtesy of Dr. Patrick Muldowney,
University of Ulster, Northern Ireland

Gauges and Integrals

The symbol $\mathbb{R}$ always denotes the real number system, the properties of which we assume to be familiar to the reader. Our principal elementary reference will be the *third* edition of the book of the author and D. R. Sherbert, which will be referred to as [B-S], but there are many other books at that level to which the reader can refer.

Our terminology and notation are standard. If A and B are subsets of a set X, we denote their *union* by $A \cup B$, their *intersection* by $A \cap B$, and the *relative complement* by $A - B$. If X is understood we sometimes denote $X - A$ by A^c.

We denote the **distance** between two real numbers x, y by

$$\operatorname{dist}(x, y) := |x - y|.$$

It is clear that, for all $x, y, z \in \mathbb{R}$, then

(0) $\operatorname{dist}(x, y) \geq 0$;

(1) $\operatorname{dist}(x, y) = 0$ if and only if $x = y$;

(2) $\operatorname{dist}(x, y) = \operatorname{dist}(y, x)$;

(3) $\operatorname{dist}(x, z) \leq \operatorname{dist}(x, y) + \operatorname{dist}(y, z)$.

If $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, where $A \neq \emptyset$, we sometimes write

$$\operatorname{dist}(x, A) := \inf\{\operatorname{dist}(x, y) : y \in A\}$$

for the **distance between x and A**. The **closed neighborhood** of x with **radius** $r > 0$ is the set

$$B[x; r] := \{y \in \mathbb{R} : |x - y| \leq r\},$$

which is also called the **closed ball** with center x and radius r. The **open neighborhood** of x with **radius** $r > 0$ is the set

$$B(x;r) := \{y \in \mathbb{R} : |x - y| < r\},$$

which is also called the **open ball** with center x and radius r.

In Part 1 of this book, we are concerned mainly with **bounded intervals** in $\mathbb{R}$. If $a, b \in \mathbb{R}$ and $a \leq b$, we use the notations

$$\begin{aligned} [a,b] &:= \{x \in \mathbb{R} : a \leq x \leq b\}; \\ (a,b) &:= \{x \in \mathbb{R} : a < x < b\}; \\ [a,b) &:= \{x \in \mathbb{R} : a \leq x < b\}; \\ (a,b] &:= \{x \in \mathbb{R} : a < x \leq b\}. \end{aligned}$$

The point a is called the **left endpoint** and the point b is called the **right endpoint** of each of these intervals. Intervals of the first kind contain both of their endpoints and are called **bounded closed intervals, or compact intervals.** Intervals of the second kind contain neither of their endpoints and are called **bounded open intervals.** Intervals of the third and fourth kinds contain exactly one of their endpoints and are called **bounded closed-open** and **bounded open-closed intervals**, respectively.

We say that an interval in $\mathbb{R}$ is **degenerate** if it contains at most one point, and that it is **nondegenerate** if it contains at least two points, in which case it contains infinitely many points. We say that two intervals in $\mathbb{R}$ are **disjoint** if their intersection is empty; that is, if they have no common points. Similarly, we will say that two intervals in $\mathbb{R}$ are **nonoverlapping** if their intersection is either empty or contains at most one point, which is necessarily an endpoint of both intervals.

If $I := [a,b]$ is a nondegenerate compact interval in $\mathbb{R}$, then a **partition** (or a **division**) of I is a finite collection $\mathcal{P} := \{I_i : i = 1, \cdots, n\} = \{I_i\}_{i=1}^n$ of nonoverlapping compact subintervals I_i such that $I = I_1 \cup \cdots \cup I_n$. It is always possible to arrange the intervals in increasing order: i.e., such that $\max I_i = \min I_{i+1}$ for $i = 1, \cdots, n-1$. If we let $x_0 := a$ and $x_i := \max I_i$ for $i = 1, \cdots, n$, we can write the intervals as

$$I_1 := [x_0, x_1], \quad I_2 := [x_1, x_2], \quad \cdots, \quad I_n := [x_{n-1}, x_n].$$

Alternatively, we can define a partition $\mathcal{P}$ of $I = [a,b]$ by specifying a finite ordered set of points in I:

$$a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b,$$

and defining the subintervals I_i by

$$I_i := [x_{i-1}, x_i] \qquad \text{for} \quad i = 1, \cdots, n.$$

Note that $n+1$ points of I are required to define a partition $\mathcal{P}$ of I into n intervals, that the initial partition point is always the left endpoint of I, and the final partition point is always the right endpoint of I. Furthermore, if the subintervals in $\mathcal{P}$ are written in increasing order, then the left endpoint of I_i is the partition point x_{i-1} and the right endpoint of I_i is the partition point x_i for all $i = 1, \cdots, n$. Ordinarily, we will require that the subintervals in a partition are nondegenerate and that the partition points are distinct, for this can be obtained by simply discarding degenerate subintervals or identical partition points.

In the following we will think of a partition of I as either a collection of nonoverlapping subintervals, or as a finite ordered set of partition points.

Length and Tags

If $I := [a, b]$, with $a \leq b$, we define the **length** of I to be

$$l(I) := b - a.$$

Note that $l(I) \geq 0$, and that $l(I) = 0$ if and only if the endpoints of I coincide. Similarly, the length of any interval having one of the three forms:

$$(a, b), \qquad [a, b), \qquad (a, b]$$

is also defined to be equal to $b - a$. In particular, $l(\emptyset) = 0$.

If $\mathcal{P} = \{I_i : i = 1, \cdots, n\}$ is a partition of an interval $I = [a, b]$ such that for each subinterval I_i there is assigned a point $t_i \in I_i$, then we call t_i a **tag of** I_i. In this case we say that the partition **is tagged** and we often write

$$\dot{\mathcal{P}} := \{(I_i, t_i) : i = 1, \cdots, n\} = \{(I_i, t_i)\}_{i=1}^{n},$$

or merely $\dot{\mathcal{P}} = \{(I_i, t_i)\}$. Thus, a **tagged partition** of I is a set of ordered pairs $\{(I_i, t_i) : i = 1, \cdots, n\}$ consisting of intervals I_i that form a partition of I, and points $t_i \in I_i$ that are tags of the intervals I_i. We write a dot over the symbol for a partition to indicate that it is a tagged partition.

It is evident that a given partition of I can be tagged in *infinitely many* ways by choosing differents points t_i as tags.

Riemann Sums

If a function f is defined on a (nondegenerate) compact interval I with values

in $\mathbb{R}$, we often write $f : I \to \mathbb{R}$. If $\dot{\mathcal{P}} = \{(I_i, t_i)\}$ is any tagged partition of I, then the sum

$$S(f; \dot{\mathcal{P}}) := \sum_{i=1}^{n} f(t_i) l(I_i)$$

is called the **Riemann sum of f corresponding to $\dot{\mathcal{P}}$**. If $I_i = [x_{i-1}, x_i]$ for $i = 1, \cdots, n$, then this Riemann sum has the form

$$S(f; \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

It will be familiar to the reader from calculus courses that if $f(x) \geq 0$ for all $x \in I$, then this Riemann sum is an *approximation* to the "area under the graph of $y = f(x)$".

The Riemann approach to the integral of the function f on I is to define the integral as a "limit" of the Riemann sums as the partitions are taken to be "finer and finer" (in some appropriate sense).

It will be noted that if some of the subintervals I_i in $\dot{\mathcal{P}}$ are degenerate (or if some of the partition points coincide), then the corresponding terms in the Riemann sums $S(f; \dot{\mathcal{P}})$ vanish. Thus, if we discard these degenerate subintervals, then the value of the Riemann sum is not changed. Consequently, we will ordinarily assume that the subintervals appearing in our partitions are nondegenerate.

The Right-Left Procedure

In working with Riemann sums, it is sometimes useful to have some (or all) of the tags be endpoints of the subintervals. This can easily be arranged by using the **right-left procedure**: If $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ and if the tag t_k is an interior point of the subinterval $[x_{k-1}, x_k]$, then we let $\dot{\mathcal{P}}^*$ be obtained from $\dot{\mathcal{P}}$ by adding the new partition point $\xi := t_k$, so that

$$a = x_0 \leq \cdots \leq x_{k-1} < \xi < x_k \leq \cdots \leq x_n = b.$$

We now tag both subintervals $[x_{k-1}, \xi]$ and $[\xi, x_k]$ by using the tag $t_k = \xi$; hence ξ is the right endpoint of the first of these subintervals, and the left endpoint of the second of these subintervals. We observe that since

$$f(t_k)(x_k - x_{k-1}) = f(t_k)(\xi - x_{k-1}) + f(t_k)(x_k - \xi),$$

then the Riemann sums $S(f; \dot{\mathcal{P}})$ and $S(f; \dot{\mathcal{P}}^*)$ give the same value. Of course, it is also possible to reverse this process and consolidate two abutting subintervals that have the same point as tag. When we do this, the tag is no longer an endpoint of the resulting subinterval.

Thus, in dealing with tagged partitions, we may assume that:

(i) all of the tags are endpoints of the subintervals, or

(ii) no tag, except possibly a or b, is an endpoint of the subintervals, or

(iii) no point is the tag of two distinct subintervals.

Sometimes it is convenient to make one of these choices.

Subpartitions

By a **subpartition** we mean a subset of a partition. Similarly, a **tagged subpartition** is a subset of a tagged partition. If $\dot{\mathcal{Q}} = \{([y_{j-1}, y_j], s_j)\}_{j=1}^{m}$ is a tagged subpartition of $[a, b]$, we will also use the notation $S(f; \dot{\mathcal{Q}})$ for $\sum_{j=1}^{m} f(s_j)(y_j - y_{j-1})$.

Various Limiting Processes

The precise type of limiting process that is used to define the integral varies somewhat depending on the textbook. The "traditional Riemann method" is to require that the Riemann sums $S(f; \dot{\mathcal{P}})$ approach a limit as the maximum length of the subintervals in the partition approaches zero. This method has the advantage that it can also be applied to functions that have their values in the complex number system, or in the finite-dimensional space $\mathbb{R}^n$ (or even in a Banach space). This method is discussed in detail in the third edition of [B-S].

A popular alternative method — often attributed solely to Gaston Darboux (1842–1917), although Giulio Ascoli (1843–1896), Henry J. S. Smith (1826–1883) and Karl J. Thomae (1840–1921) employed a similar approach in the same year (1875) — is to introduce "lower" and "upper integrals". The "Darboux method" has certain technical advantages, but it also has at least two disadvantages. One is that it makes heavy use of the order properties of the real number system $\mathbb{R}$, and so extensions to more general values of the function require further treatment. Another disadvantage is that in order to prove that exactly the same class of functions is integrable using the "Darboux approach" as the traditional Riemann approach, it is necessary to prove a rather subtle theorem.

In this book we will not use either the traditional Riemann or the Darboux approach in defining a limit of Riemann sums. Instead we shall employ a limiting process that was recently introduced independently by the Czech mathematician Jaroslav Kurzweil (b. 1926) and the English mathematician Ralph Henstock (b. 1923). This method is *slightly* more complicated than the Riemann process, yet it yields an integral that is *considerably* more general and easier to use than the ordinary Riemann integral. It is more

general in that the class of integrable functions is considerably enlarged, and it is easier to use because it enables one to remove (or at least weaken) certain hypotheses that the Riemann theory requires. Since we get *a lot more* with little additional effort, we regard this approach to be a very significant advance.

In the Riemann approach to the integral, the measure of fineness of a partition is given by the maximum length of the subintervals I_i; this means that the lengths of the subintervals are all less than or equal to a certain number. In the Kurzweil-Henstock approach that we adopt, more variation in the lengths of the subintervals is allowed as long as the subintervals over which the function is "rapidly changing" have "small length". Thus, in Figure 1.1, we make the approximation of the Riemann sums to the area a close one by taking the length of the intervals I_3 and I_4 small, since f is increasing rapidly near the right end of $[a, b]$. There is no particular need to make the lengths of I_1 and I_2 small, since the function is nearly constant over the first part of the interval $[a, b]$.

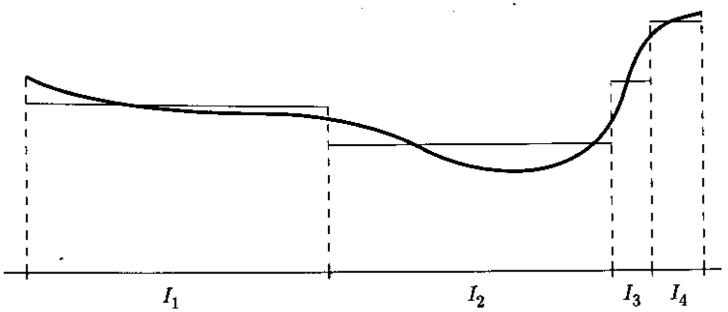


Figure 1.1

Gauges

The Kurzweil–Henstock approach places more attention on the tags than the traditional approach does. In fact, we shall govern the fineness of the tagged partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ by requiring that each subinterval I_i is contained in an interval $B[t_i; \delta_i] := [t_i - \delta_i, t_i + \delta_i]$ that depends on the tag t_i. The following definitions will be used.

1.1 Definition. If $I := [a, b] \subset \mathbb{R}$, then a function $\delta : I \to \mathbb{R}$ is said to be a **gauge** on I if $\delta(t) > 0$ for all $t \in I$. The interval around $t \in I$ **controlled by the gauge** δ is the interval $B[t; \delta(t)] := [t - \delta(t), t + \delta(t)]$.

1.2 Definition. Let $I := [a, b]$ and let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n}$ be a tagged partition. If δ is a gauge on I, then we say that $\dot{\mathcal{P}}$ is **δ-fine** if

$$I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \qquad \text{for all} \quad i = 1, \cdots, n;$$

that is, if each subinterval I_i is contained in the interval $B[t_i; \delta(t_i)]$ controlled by the point t_i. (See Figure 1.2.) Sometimes, when the tagged partition $\dot{\mathcal{P}}$ is δ-fine, we say that $\dot{\mathcal{P}}$ is **subordinate to** δ or write $\dot{\mathcal{P}} \ll \delta$.

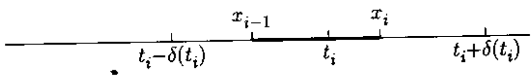


Figure 1.2

Remarks. **(a)** Only a tagged partition can be δ-fine; hence it is not necessary to employ the word "tagged" in referring to δ-fine partitions.

(b) If $I_i := [x_{i-1}, x_i]$, then the partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n}$ is δ-fine if and only if

$$t_i - \delta(t_i) \leq x_{i-1} \leq t_i \leq x_i \leq t_i + \delta(t_i) \qquad \text{for all} \quad i = 1, \cdots, n.$$

(c) The partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n}$ is δ-fine if and only if

$$I_i \subseteq B[t_i; \delta(t_i)] \qquad \text{for all} \quad i = 1, \cdots, n.$$

We now give some examples of gauges that will be instructive and useful.

1.3 Examples. **(a)** If $\delta > 0$ is a positive number, then we can define a gauge $\delta : I \to \mathbb{R}$ by setting $\delta(t) := \delta$ for all $x \in I$.

Such a gauge is called a **constant gauge**. We note that a partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n}$ is δ-fine for this constant gauge if and only if

$$I_i \subseteq [t_i - \delta, t_i + \delta] = B[t_i; \delta] \qquad \text{for all} \quad i = 1, \cdots, n.$$

This is readily seen to imply that $l(I_i) \leq 2\delta$ for all i.

(b) If δ_1 and δ_2 are two gauges on $I := [a, b]$, and if we define

$$\delta(t) := \min\{\delta_1(t), \delta_2(t)\} \qquad \text{for} \quad t \in I,$$

then it is immediate that δ is a gauge on I. Clearly, every partition of I that is δ-fine is both δ_1-fine and δ_2-fine. This construction can be extended to any *finite number* of gauges on I.

(c) It is often convenient to choose a gauge δ that will force a given point to be a tag for any δ-fine partition.

For example, let $I := [0,1]$ and let $\delta(0) := \frac{1}{4}$ and $\delta(t) := \frac{1}{2}t$ for $0 < t \le 1$. Evidently δ is a gauge on I. If $\dot{\mathcal{P}}$ is a δ-fine partition of I, then $0 \in I$ must belong to some subinterval $I_1 = [0, x_1]$ in $\dot{\mathcal{P}}$. We claim that the tag t_1 for I_1 must be 0. Indeed, since $\dot{\mathcal{P}}$ is δ-fine, we must have $[0, x_1] \subseteq [t_1 - \delta(t_1), t_1 + \delta(t_1)]$ which implies that

$$(1.\alpha) \qquad t_1 - \delta(t_1) \le 0.$$

Now, if $t_1 > 0$, then $\delta(t_1) = \frac{1}{2}t_1$ so that $t_1 - \delta(t_1) = t_1 - \frac{1}{2}t_1 > 0$, contradicting the inequality $(1.\alpha)$. Therefore, we must have $t_1 = 0$, as asserted.

We will study this gauge further in the exercises at the end of this section.

(d) Let $a < c < b$ and let δ be a gauge on $[a, b]$. If $\dot{\mathcal{P}}'$ is a partition of $[a, c]$ that is δ-fine and if $\dot{\mathcal{P}}''$ is a partition of $[c, b]$ that is δ-fine, then $\dot{\mathcal{P}}' \cup \dot{\mathcal{P}}''$ is a partition of $[a, b]$ that is δ-fine.

(e) Let $a < c < b$ and let δ' and δ'' be gauges on the intervals $[a, c]$ and $[c, b]$, respectively. If δ is defined on $[a, b]$ by

$$\delta(t) := \begin{cases} \delta'(t) & \text{if} \quad t \in [a, c), \\ \min\{\delta'(c), \delta''(c)\} & \text{if} \quad t = c, \\ \delta''(t) & \text{if} \quad t \in (c, b], \end{cases}$$

then δ is a gauge on $[a, b]$. Moreover, if $\dot{\mathcal{P}}'$ is a partition of $[a, c]$ that is δ'-fine, and $\dot{\mathcal{P}}''$ is a partition of $[c, b]$ that is δ''-fine, then $\dot{\mathcal{P}}' \cup \dot{\mathcal{P}}''$ is a partition of $[a, b]$ that has c as a partition point. However, $\dot{\mathcal{P}}' \cup \dot{\mathcal{P}}''$ may *not* be δ-fine. (Why?)

(f) Let δ' and δ'' be as in (d) and let δ^* be defined on $[a, b]$ by

$$\delta^*(t) := \begin{cases} \min\{\delta'(t), \frac{1}{2}(c - t)\} & \text{if} \quad t \in [a, c), \\ \min\{\delta'(c), \delta''(c)\} & \text{if} \quad t = c, \\ \min\{\delta''(t), \frac{1}{2}(t - c)\} & \text{if} \quad t \in (c, b]. \end{cases}$$

It is clear that δ^* is a gauge on $[a, b]$, and it is easy to show that every δ^*-fine partition $\dot{\mathcal{P}}$ of $[a, b]$ must have c as a tag for any subinterval of $\dot{\mathcal{P}}$ that contains c. Thus, if we use the right-left procedure mentioned above, every δ^*-fine partition $\dot{\mathcal{P}}$ of $[a, b]$ gives rise to a partition of $[a, c]$ that is δ'-fine, and to a partition of $[c, b]$ that is δ''-fine.

Some Intuitive Remarks

If I is a compact interval and δ is a gauge on I, we can think that every point $t \in I$ "controls" (or has some "influence on") every point in the closed interval $B[t;\delta(t)] = [t-\delta(t), t+\delta(t)]$, and hence on every subinterval contained in this interval. We note that some points in I control large intervals, and other points control very small intervals. The question arises whether, for an *arbitrary* gauge δ, one can always find a tagged partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ where each tag t_i controls the corresponding subinterval I_i.

The Existence of δ-Fine Partitions

It will now be shown that if $I := [a,b] \subset \mathbb{R}$ is a nondegenerate compact interval, and if δ is any gauge defined on I, then there always exist tagged partitions of I that are δ-fine. This result was established and used in the space $\mathbb{R}^m$, $m \geq 1$, by Pierre Cousin (1867–1933). It is a reflection of the compactness of I and is sometimes called the "Fineness Theorem".

1.4 Cousin's Theorem. *If $I := [a,b]$ is a nondegenerate compact interval in $\mathbb{R}$ and δ is a gauge on I, then there exists a partition of I that is δ-fine.*

Proof. The proof is by contradiction. We suppose that I does not have a δ-fine partition. Now let $c := \frac{1}{2}(a+b)$ and bisect I into:

$$[a,c], \qquad [c,b].$$

We claim that at least one of these subintervals does not have a δ-fine partition; for, if they both have δ-fine partitions, then the union of these partitions would be a δ-fine partition of $[a,b]$, as was noted in Example 1.3(d). We let $I^1 := [a,c]$ if this subinterval does not have any δ-fine partition; otherwise, let $I^1 := [c,b]$. Relabel I^1 as $[a_1,b_1]$, let $c_1 := \frac{1}{2}(a_1+b_1)$ and bisect I^1 into:

$$[a_1,c_1], \qquad [c_1,b_1].$$

As before, at least one of these subintervals does not have a δ-fine partition. We let $I^2 := [a_1,c_1]$ if it does not have a δ-fine partition; otherwise, let $I^2 := [c_1,b_1]$. Relabel I^2 as $[a_2,b_2]$ and bisect again. In this manner, we obtain a sequence (I^n) of compact subintervals of $I = [a,b]$ that is nested in the sense that

$$[a,b] = I \supset I^1 \supset \cdots \supset I^n \supset I^{n+1} \supset \cdots.$$

The Nested Intervals Property (see [B-S; p. 46]) implies that there is a unique number ξ that lies in all of the intervals I^n. However, since $\delta(\xi) > 0$, the Archimedean Property of $\mathbb{R}$ implies that there exists $p \in \mathbb{N}$ such that

$$l(I^p) = (b-a)/2^p < \delta(\xi),$$

whence $I^p \subset [\xi - \delta(\xi), \xi + \delta(\xi)]$. Therefore the pair (I^p, ξ) is a (trivial) δ-fine partition of I^p. But this is contrary to the construction of the I^n as subintervals of I that have no δ-fine partitions.

This contradiction shows that, for every gauge δ on I, there exists a δ-fine partition of I. Q.E.D.

The Riemann and Generalized Riemann Integrals

We are now prepared to define the integrals. While we will be primarily interested in the (generalized Riemann) integral, we first define the Riemann integral for the purpose of comparison.

1.5 Definition. A function $f : I \to \mathbb{R}$ is said to be **R-integrable** (or **Riemann integrable) on** I if there exists a number $A \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a number $\gamma_\varepsilon > 0$ such that if $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is any tagged partition of I such that $l(I_i) \le \gamma_\varepsilon$ for $i = 1, \cdots, n$, then

$$|S(f; \dot{\mathcal{P}}) - A| \le \varepsilon.$$

The collection of all functions that are R-integrable on an interval I will often be denoted by $\mathcal{R}(I)$.

We will now give *two* definitions of the generalized Riemann integral. The first one differs from Definition 1.5 only in that the constant γ_ε is replaced by a gauge on I; that is, by a *function* $\gamma_\varepsilon : I \to (0, \infty)$.

1.6 Definition. A function $f : I \to \mathbb{R}$ is said to be **generalized Riemann integrable on** I if there exists a number $B \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a gauge γ_ε on I such that if $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is any tagged partition of I such that $l(I_i) \le \gamma_\varepsilon(t_i)$ for $i = 1, \cdots, n$, then

$$|S(f; \dot{\mathcal{P}}) - B| \le \varepsilon. \tag{1.β}$$

In practice, we will use the following definition of the integral, based on the notion of δ-fineness of a partition with respect to a gauge.

1.7 Definition. A function $f : I \to \mathbb{R}$ is said to be **generalized Riemann integrable on** I if there exists a number $C \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a gauge δ_ε on I such that if $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is any tagged partition of I that is δ_ε-fine, then

$$|S(f; \dot{\mathcal{P}}) - C| \le \varepsilon. \tag{1.γ}$$

The collection of all functions that are generalized Riemann integrable on an interval I will be denoted by $\mathcal{R}^*(I)$.

It would be highly inconvenient if Definitions 1.6 and 1.7 led to different collections of integrable functions, or different values for the integral. We will now show that they do not do so.

1.8 Equivalence Theorem. *Definitions* 1.6 *and* 1.7 *lead to equivalent integrals.*

Proof. Suppose that $f : I \to \mathbb{R}$ is integrable in the sense of Definition 1.6, so that there exists a number B such that given $\varepsilon > 0$ there exists a gauge γ_ε as in Definition 1.6. We define $\delta_\varepsilon(t) := \frac{1}{2}\gamma_\varepsilon(t)$ for $t \in I$, so that δ_ε is a gauge on I. If $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$ is a δ_ε-fine partition of I, then

$$I_i \subseteq [t_i - \delta_\varepsilon(t_i), t_i + \delta_\varepsilon(t_i)] = [t_i - \tfrac{1}{2}\gamma_\varepsilon(t_i), t_i + \tfrac{1}{2}\gamma_\varepsilon(t_i)],$$

whence $l(I_i) \leq \gamma_\varepsilon(t_i)$ for all $i = 1, \cdots, n$. Consequently the condition in Definition 1.6 is satisfied and so inequality (1.β) holds. We have shown that if $\dot{\mathcal{P}}$ is any δ_ε-fine partition of I, then $|S(f; \dot{\mathcal{P}}) - B| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, then f is integrable in the sense of Definition 1.7 with $C = B$.

Conversely, suppose that Definition 1.7 is satisfied, so there exists a number C such that given $\varepsilon > 0$ there exists a gauge δ_ε as in Definition 1.7. We define $\gamma_\varepsilon(t) := \delta_\varepsilon(t)$, so that γ_ε is a gauge on I. If the partition $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$ satisfies

$$l(I_i) \leq \gamma_\varepsilon(t_i) = \delta_\varepsilon(t_i),$$

then $I_i \subseteq [t_i - \delta_\varepsilon(t_i), t_i + \delta_\varepsilon(t_i)]$ for all $i = 1, \cdots, n$, so that $\dot{\mathcal{P}}$ is δ_ε-fine. Consequently the condition in Definition 1.7 is satisfied and so inequality (1.γ) holds. We have shown that if $\dot{\mathcal{P}}$ is any partition of I with $l(I_i) \leq \gamma_\varepsilon(t_i)$ for all $i = 1, \cdots, n$, then $|S(f; \dot{\mathcal{P}}) - C| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, then f is integrable in the sense of Definition 1.6 with $B = C$. Q.E.D.

Theorem 1.8 shows that Definitions 1.6 and 1.7 give the same collection of generalized Riemann integrable functions and the same value for the integral. It is important to know that the number C in Definition 1.7 is uniquely determined (when it exists). Because of the importance of this uniqueness result, we give its proof here.

1.9 Uniqueness Theorem. *There is at most one number C that satisfies the property in Definition* 1.7.

Proof. Suppose $C' \neq C''$ and let $\varepsilon := \frac{1}{3}|C' - C''| > 0$. If C' satisfies Definition 1.7, then there exists a gauge δ'_ε on I such that if $\dot{\mathcal{P}}$ is a δ'_ε-fine

partition of I, then $|S(f;\dot{\mathcal{P}}) - C'| \le \varepsilon$. Similarly, if C'' satisfies Definition 1.7, there exists a gauge δ''_ε on I such that if $\dot{\mathcal{P}}$ is a δ''_ε-fine partition of I, then $|S(f;\dot{\mathcal{P}}) - C''| \le \varepsilon$. Now let $\delta_\varepsilon := \min\{\delta'_\varepsilon, \delta''_\varepsilon\}$ so that δ_ε is a gauge on I and let $\dot{\mathcal{P}}$ be a δ_ε-fine partition of I. Then the partition $\dot{\mathcal{P}}$ is both δ'_ε-fine and δ''_ε-fine. Using the Triangle Inequality, we have

$$|C' - C''| \le |C' - S(f;\dot{\mathcal{P}})| + |S(f,\dot{\mathcal{P}}) - C''| \;\le\; \varepsilon + \varepsilon \;<\; |C' - C''|,$$

which is a contradiction. Q.E.D.

The next result is a formal statement of the fact that the Riemann integral is contained in the generalized Riemann integrals of Definitions 1.6 and 1.7.

1.10 Consistency Theorem. *Let $I := [a,b]$ be a compact interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$. If f is R-integrable on I, then f is also integrable on I in the sense of Definitions* 1.6 *and* 1.7, *and the integrals are equal.*

Proof. It is immediate that the Riemann integral is a special case of the integral in Definition 1.6. Since we have seen that the integrals in Definitions 1.6 and 1.7 are equivalent, the assertion follows. Q.E.D.

1.11 Remarks. (a) In the following we will discuss mainly the generalized Riemann integral. To simplify our terminology, *unless there is specific mention to the contrary, the words "integral", "integrable", etc., refer to the generalized Riemann integral of Definitions* 1.6 *and* 1.7. When other notions of the integral are intended, they will be specifically mentioned; in this connection we may refer to the generalized Riemann integral as the R*-integral.

(b) The Consistency Theorem asserts that if a function is R-integrable, then the values of the R-integral and the R*-integral are equal. Thus we may safely denote the value of the integrals of such a function by the same notation. Therefore, we will also denote the R*-integral by one of the symbols:

$$\int_I f \qquad \text{or} \qquad \int_a^b f.$$

In case it is useful to denote the variable, we will employ the notation:

$$\int_a^b f(x)\,dx \qquad \text{or} \qquad \int_a^b f(u)\,du.$$

This "calculus notation" is useful when we are dealing with a function that depends on several parameters. It is also useful in connection with the Substitution Theorems that will be discussed in Section 13. However, its

use in the Substitution Theorem has the danger that one sometimes resorts to a blind "juggling of symbols", rather than a careful application of a theorem.

More on Terminology

The definition of the integral given by the great German mathematician Bernhard Riemann (1826–1866) is essentially Definition 1.5. The Definition 1.7 was discovered independently by Kurzweil and Henstock; therefore it would be entirely appropriate to call this integral the "Kurzweil-Henstock integral" and some authors use this terminology (or some version of it) — others call it the "gauge integral", etc. However, it is a remarkable fact that the integral in Definitions 1.6 and 1.7 also coincides with integrals that were introduced in 1912 by Arnaud Denjoy (1884–1974) and in 1914 by Oskar Perron (1880–1975), although the definitions given by these authors were *very* different. Thus, it would be appropriate to use the term "Denjoy-Perron-Kurzweil-Henstock integral". Since that name is quite unwieldy, we will merely say "the integral", or the "generalized Riemann integral" as we have stated above.

Why do Gauges Work?

Before we get down to a development of the integral, it is appropriate that we attempt to give an answer to the question: Why do nonconstant gauges work better than constant gauges? We have already attempted to suggest a reason by our Figure 1.1 and the accompanying discussion. We will now expand somewhat on that discussion.

(A) A gauge permits one to enclose a finite or countable set of points in a union of intervals that has small total length and so does not contribute much to the Riemann sums.

For example, let $f : [0,1] \to \mathbb{R}$ be Dirichlet's function defined by $f(x) := 1$ if $x \in [0,1]$ is rational, and $f(x) := 0$ if $x \in [0,1]$ is irrational. Although this function is not R-integrable (see [B-S; p. 204]), it will be seen in Example 2.2(b) that f is R*-integrable on $[0,1]$ with integral equal to 0. The proof involves showing that a gauge δ_ε can be constructed that will make the Riemann sums for any δ_ε-fine partition less than ε. This is accomplished by taking δ_ε appropriately small at the rational points in $[0,1]$. It does not matter how we define δ_ε at the irrational numbers, since $f(t_i) = 0$ when t_i is irrational, so these terms make a zero contribution to the Riemann sums.

(B) A gauge can force one to take a particular point as a tag. This can be useful when a particular point is a source of difficulty; by choosing it as a tag, one can sometimes control the difficulty.

For example, let $g(x) := 1/\sqrt{x}$ when $x \in (0,1]$, so that $g(x) \to \infty$ as $x \to 0$. To have the function defined on all of $[0,1]$ we define $g(0) := 0$. Since the function g is not bounded, it cannot be R-integrable. If we use a gauge such as that in Example 1.3(c) that forces the first tag $t_1 = 0$, then the first term in any corresponding Riemann sum will be 0. Hence the function g on the remaining part of the interval $[x_1, 1]$ will be bounded and continuous, and is more easily handled. (See Example 4.6 for more details.)

(C) The use of gauges gives an improved Fundamental Theorem of Calculus for the R*-integral.

Suppose that $F : [a,b] \to \mathbb{R}$ has a derivative $f(t)$ at every point $t \in [a,b]$. Then, by the definition of the derivative at $t \in [a,b]$, given $\varepsilon > 0$ there exists $\delta_\varepsilon(t) > 0$ such that if $0 < |x - t| < \delta_\varepsilon(t)$, $x \in [a,b]$, then

$$\left|\frac{F(x) - F(t)}{x - t} - f(t)\right| \le \varepsilon.$$

Hence the existence of the derivative on I provides the existence of a gauge δ_ε on I. It will be shown in Section 4 that if $\dot{\mathcal{P}}$ is a partition of $[a,b]$ that is δ_ε-fine, then $|F(b) - F(a) - S(f;\dot{\mathcal{P}})| \le \varepsilon(b-a)$. Since $\varepsilon > 0$ is arbitrary, this implies that the derivative $f = F'$ is R*-integrable on $[a,b]$ to the value $F(b) - F(a)$. This argument does *not* require the assumption that f is R- (or R*-)integrable.

The Lebesgue Integral

In fact, there is *one more* theory of integration that we will discuss in this book; namely the one that was introduced in 1902 by the French mathematician Henri Lebesgue (1875–1941). This integral, which we will call the **L-integral** or the **Lebesgue integral**, was introduced to correct certain "defects" in the R-integral and it has been largely, though not totally, successful. Certainly the L-integral is the main integral used in modern mathematical research, so a serious student of mathematics needs to become familiar with it.

However, the L-integral also has certain drawbacks that we believe are largely removed by the R*-integral. Moreover, although there are a number of different approaches to the L-integral, most of them require the investment of a considerable amount of time and effort in developing the notion of the "measure" of certain subsets of $\mathbb{R}$. For that reason, the L-integral is usually regarded as being beyond the reach of most undergraduate students

of mathematics, and it is very largely avoided by almost all physicists and engineers. [However, if one is content to work with the L-integral of a function defined on an abstract measure space, as is done in the theory of probability, then the basic features of the L-integral are relatively elementary (see [B-1]).]

It is a fact that every function that is L-integrable on $[a, b]$ is R*-integrable. Of course, we cannot give a proof of this assertion without either giving a definition of the L-integral or using some of its properties. However, it may be interesting to know that E. J. McShane [McS-2, McS-3] has given a (surprising) definition of the L-integral that makes it clear that the L-integral is a special case of the R*-integral. His modification is to use Definition 1.7 with the only change being that he does *not* require the tags t_i to belong to the subintervals I_i, but only to I; however, he continues to require the intervals I_i to be contained in the intervals controlled by the gauge at t_i. Clearly, if Definition 1.7 is satisfied for all δ_ε-fine Riemann sums when the tags *are not* required to belong to the subintervals, then this definition is also satisfied by these Riemann sums when the tags *are* required to belong to these subintervals. Since it is easier for a function to be R*-integrable than to be L-integrable, *the L-integral is contained in the R*-integral.*

In view of the importance of the L-integral, the question arises whether one can identify the L-integrable functions among the R*-integrable ones. We will show later that the answer is affirmative and that the test is very simple: *A function f is L-integrable if and only if both f and its absolute value $|f|$ are R*-integrable.*

Some References

We wish to cite some references that the reader will find useful. The excellent book of Gordon [G-3] considers the integration theories due to Lebesgue, Denjoy, Perron, McShane and Kurzweil-Henstock on $[a, b]$ in considerable detail. In particular, he discusses the relations between these various integrals and approaches.

The first part of the book by Pfeffer [P-1] develops the McShane and the Kurzweil-Henstock integral on $[a, b]$. The second part of this book is concerned with generalizations of these integrals to the space $\mathbb{R}^m$.

The book of McLeod [McL] discusses many of the topics given here, presented in a pleasant, rather informal style. It is more elementary than either [G-3] or [P-1]. A brief, but very useful, introduction to the theory to be presented here will be found in Chapters 12–17 of DePree and Swartz [DP-S]. The reader is also advised to examine Chapter 24 of Schechter [Sch]. The book of Mawhin [M-1] has much of the material in this book, and is

strongly recommended to the reader, especially for its treatment of $\mathbb{R}^m$. The book of Lee [Le-1] is somewhat condensed, but has much information. It also gives references to the literature. The present author has been given access to the manuscript of the book of Lee and Výborný [L-V], which will be very useful to the reader. The books of Henstock [H-3], [H-6] and Kurzweil [K-2] are appropriate for advanced readers. The third edition of [B-S; Chapter 10] contains an elementary introduction to the generalized Riemann integral; however, the proofs of some of the theorems are omitted.

For interesting discussions of the history of the integral, we suggest the book of Hawkins [Hw-1] and the historical introduction in Phillips [Ph-1].

There are many books dealing with the Lebesgue integral from various points of view. In addition to books already mentioned, the following are recommended: Asplund and Bungart [A-B], Bartle [B-1], Bruckner, Bruckner and Thomson [BBT], Dudley [Dd], Dunford and Schwartz [D-S], Foran [Fo-1], Halmos [Hl], Hewitt and Stromberg [He-St], Lebesgue [L-1], Nielsen [Ni], Saks [S-1], Stromberg [St], and Wheeden and Zygmund [W-Z]. A reader looking at several of these texts will see very different approaches.

Exercises

1.A If δ_1 and δ_2 are gauges on $I := [a, b]$ and if $\delta_1(t) \leq \delta_2(t)$ for all $t \in I$, show that if $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is δ_1-fine, then it is also δ_2-fine.

1.B If $\delta_1, \cdots, \delta_m$ are gauges on $I := [a, b]$, then the function δ^* defined by $\delta^*(t) := \min\{\delta_1(t), \cdots, \delta_m(t)\}$ for $t \in I$ is a gauge on I. Show that a partition $\dot{\mathcal{P}}$ of I is δ^*-fine if and only it is δ_k-fine for each $k = 1, \cdots, m$.

1.C Let δ be the gauge on $[0, 1]$ given in Example 1.3(c).

(a) Show that $\dot{\mathcal{P}}_1 := \{([0, \frac{1}{4}], 0), ([\frac{1}{4}, \frac{1}{2}], \frac{1}{2}), ([\frac{1}{2}, 1], \frac{3}{4})\}$ is δ-fine.

(b) Show that $\dot{\mathcal{P}}_2 := \{([0, \frac{1}{4}], 0), ([\frac{1}{4}, \frac{1}{2}], \frac{1}{2}), ([\frac{1}{2}, 1], \frac{6}{10})\}$ is not δ-fine.

(c) Is $\dot{\mathcal{P}}_3 := \{([0, \frac{1}{4}], 0), ([\frac{1}{4}, \frac{1}{2}], \frac{1}{2}), ([\frac{1}{2}, 1], \frac{1}{2})\}$ also δ-fine?

1.D Suppose that δ_1 is the gauge on $[0, 1]$ defined by $\delta_1(0) := \frac{1}{4}$, $\delta_1(t) := \frac{3}{4}t$ for $t \in (0, 1]$. Are the tagged partitions considered in the preceding exercise δ_1-fine?

1.E Suppose that δ_2 is the gauge on $[0,1]$ defined by $\delta_2(0) := \frac{1}{4}$, $\delta_2(t) := \frac{9}{10}t$ for $t \in (0,1]$. Are the tagged partitions in Exercise 1.C δ_2-fine?

1.F Let δ be defined on $[0,1]$ by $\delta(t) := \frac{1}{2}(1-t)$ for $t \in [0,1)$ and $\delta(1) := \frac{1}{4}$. Show that every δ-fine partition has 1 as a tag for every interval containing this number.

1.G Let η be defined on $[0,1]$ by $\eta(0) := \frac{1}{4}$, $\eta(t) := \frac{1}{2}\operatorname{dist}(t, \{0,1\})$ for $t \in (0,1)$ and $\eta(1) := \frac{1}{4}$. Show that every η-fine partition of $[0,1]$ must have 0 and 1 as tags.

1.H Construct a partition of $[0,1]$ that is η-fine for the gauge in Exercise 1.G.

1.I Let δ be a gauge on $I := [a,b]$. Give a proof of Cousin's Theorem based on the following argument: Let C be the set of all $c \in I$ such that there exists a δ-fine partition of the interval $[a,c]$. Show that $C \neq \emptyset$ and let $s := \sup C$. Now show that $s = b$ and that $b \in C$.

1.J Suppose that $\{J_1, \cdots, J_m\}$ is a set of open intervals $J_k := (a_k, b_k)$ such that $I := [a,b] \subset \bigcup_{k=1}^{m} J_k$. Show that there exists a partition $\mathcal{P} := \{I_1, \cdots, I_n\}$ of I such that each closed interval I_i in $\mathcal{P}$ is contained in one of the open intervals J_k.

1.K Show that one can replace the inequality $(1.\gamma)$ in Definition 1.7 by the strict inequality:

$$(1.\delta) \qquad |S(f;\dot{\mathcal{P}}) - C| < \varepsilon.$$

More precisely: show that $f : I \to \mathbb{R}$ is integrable to $C \in \mathbb{R}$ in the sense of Definition 1.7 if and only if for every $\varepsilon > 0$ there exists a gauge η_ε on I such that if $\dot{\mathcal{P}}$ is any partition of I that is η_ε-fine, then $(1.\delta)$ holds.

1.L Let $f(0) := 1$ and $f(x) := 0$ for $x \in (0,1]$. Given $\varepsilon > 0$, let $\gamma_\varepsilon(t) := \varepsilon$ for all $t \in [0,1]$. Show that for any γ_ε-fine partition $\dot{\mathcal{P}}$ of $[0,1]$, one has $0 \le S(f;\dot{\mathcal{P}}) \le \varepsilon$. Conclude that $f \in \mathcal{R}^*([0,1])$ and that $\int_0^1 f = 0$. (Of course, we already know that f is R-integrable.)

1.M Let f be as in the preceding exercise. Let $\delta_\varepsilon(t) := \frac{1}{2}t$ for $t \in (0,1]$ and $\delta_\varepsilon(0) := \varepsilon$. Show that for any δ_ε-fine partition $\dot{\mathcal{P}}$ of $[0,1]$, one has $0 \le S(f;\dot{\mathcal{P}}) \le \varepsilon$.

1.N Let $g(0) := 3$, $g(x) := 1$ for $x \in (0, 2)$ and $g(2) := 6$. Given $\varepsilon > 0$, find a gauge δ on $[0, 2]$ such that for any δ-fine partition $\dot{\mathcal{P}}$ of $[0, 2]$, one has $|S(g; \dot{\mathcal{P}}) - 2| \le \varepsilon$. Conclude that g is in $\mathcal{R}^*([0, 2])$ and that $\int_0^2 g = 2$. (Of course, we already know that g is R-integrable.)

1.O Let $h(1/n) := n$ when $n = 1, 2, 3, 4$ and $h(x) := 0$ for all other $x \in [0, 1]$. Given $\varepsilon > 0$ find a gauge δ on $[0, 1]$ such that for any δ-fine partition $\dot{\mathcal{P}}$ of $[0, 1]$ one has $0 \le S(h; \dot{\mathcal{P}}) \le \varepsilon$. Conclude that $h \in \mathcal{R}^*([0, 1])$ and that $\int_0^1 h = 0$. [Hint: Consider a gauge such that $\delta(1/n) = K\varepsilon$ for $n = 1, 2, 3, 4$ and $\delta(t) = 1$ elsewhere. Note that each of the points $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ can be tags for at most two subintervals in I.]

1.P Let $k(1/n) := n$ when $n \in \mathbb{N}$ and $k(x) := 0$ for all other $x \in [0, 1]$. Given $\varepsilon > 0$, find a gauge δ on $[0, 1]$ such that for any δ-fine partition $\dot{\mathcal{P}}$ of $[0, 1]$ one has $0 \le S(k; \dot{\mathcal{P}}) \le \varepsilon$. Conclude that $k \in \mathcal{R}^*([0, 1])$ and that $\int_0^1 k = 0$.

1.Q Let $f, g : I \to \mathbb{R}$ be such that $f(x) = g(x)$ for all $x \in I - E := \{x \in I : x \notin E\}$, where E is a finite subset of $I := [a, b]$. Given $\varepsilon > 0$, show that there exists a gauge δ_ε on I such that if $\dot{\mathcal{P}}$ is a δ_ε-fine partition of I, then $|S(f; \dot{\mathcal{P}}) - S(g; \dot{\mathcal{P}})| \le \varepsilon$.

1.R Let $f, g : I \to \mathbb{R}$ be as in the preceding exercise. Show that $f \in \mathcal{R}^*(I)$ if and only if $g \in \mathcal{R}^*(I)$, in which case $\int_I g = \int_I f$.

1.S Let J be a bounded open interval in $\mathbb{R}$ and suppose that $I_1, \cdots, I_m$ are nonoverlapping compact intervals contained in J. Show that $\sum_{j=1}^m l(I_j) \le l(J)$.

1.T A mapping Δ of points $t \in I := [a, b]$ into nondegenerate closed intervals $\Delta(t) := [a(t), b(t)]$ of $\mathbb{R}$ is called an **interval-gauge** on I in case $t \in (a(t), b(t))$ for each $t \in I$.

(a) Show that if $\delta : I \to \mathbb{R}$ is a gauge on I, then $\Delta(t) := [t - \delta(t), t + \delta(t)]$ is an interval-gauge on I.

(b) Show that if $\Delta = [a(t), b(t)], t \in I$, is an interval-gauge on I, then the function δ defined by $\delta(t) := \min\{t - a(t), b(t) - t\}$ is a gauge on I.

1.U If Δ is an interval-gauge on I and $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is a tagged partition of I, we say that $\dot{\mathcal{P}}$ is **Δ-fine** if $I_i \subseteq \Delta(t_i)$ for $i = 1, \cdots, n$.

(a) Let Δ be an interval-gauge on I and define δ as in 1.T(b). Show that any tagged partition $\dot{\mathcal{P}}$ that is δ-fine is also Δ-fine. Conclude that every interval-gauge Δ has a Δ-fine partition.

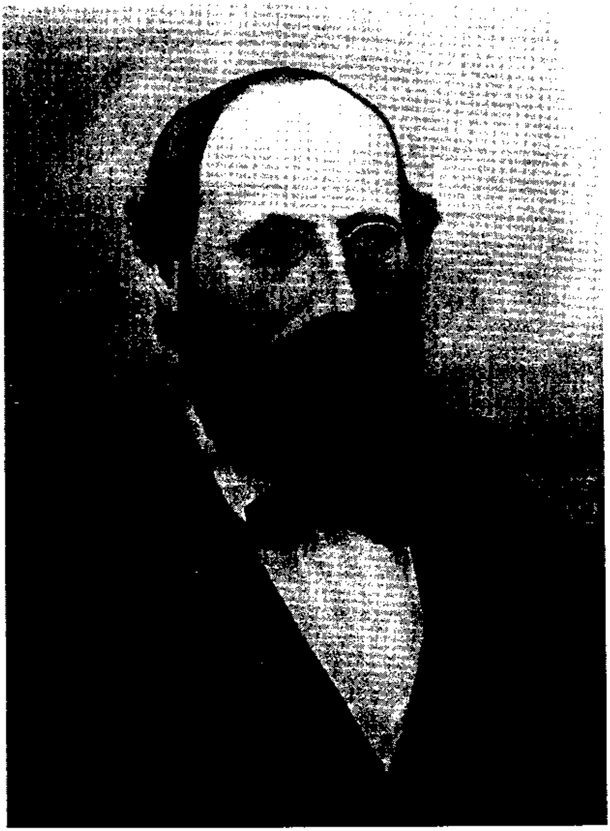
(b) If δ is a gauge on I and $\dot{\mathcal{P}}$ is a δ-fine partition of I, show that $\dot{\mathcal{P}}$ is Δ-fine, where Δ is defined as in 1.T(a).

1.V Show that $f : I \to \mathbb{R}$ is in $\mathcal{R}^*(I)$ if and only if there exists $D \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists an interval-gauge Δ_ε of I such that if $\dot{\mathcal{P}}$ is any Δ_ε-fine partition of I, then $|S(f;\dot{\mathcal{P}}) - D| \leq \varepsilon$.

1.W Let $f : [a,b] \to \mathbb{C}$ (the complex field) and let f_1, f_2 be the real and imaginary parts of f; that is, $f_1(t) := \operatorname{Re} f(t)$, $f_2(t) := \operatorname{Im} f(t)$ for all $t \in I := [a,b]$.

(a) Show that the definition of the integral applies without change to a *complex-valued* function defined on I.

(b) Show that f is integrable if and only if both f_1 and f_2 are integrable, and in this case $\int_I f = \int_I f_1 + i \int_I f_2$. [Hint: If $z \in \mathbb{C}$ has real and imaginary parts x, y, then $|x| \leq |z| \leq |x| + |y|$, and similarly for $|y|$.]



(Georg Friedrich) Bernhard Riemann
(September 17, 1826–July 20, 1866)

Courtesy of the Niedersächsische Staats-
und Universitätsbibliothek, Göttingen, Germany

Some Examples

We now give some examples of functions that are (generalized Riemann) integrable. Some of these functions will be known to be R-integrable and so their integrability is implied by the Consistency Theorem 1.10, but others are not R-integrable. In order to establish the integrability of all of these functions, we will construct appropriate gauges. That will be quite easy to do for the functions in Examples 2.1 and 2.2, somewhat tricky for Examples 2.3, and downright difficult for Examples 2.6 and 2.7. We recommend that the reader study all of the statements carefully. However, it is not necessary to go through all of the details in the latter examples on a first reading.

In subsequent sections, the integrability of functions will ordinarily be established by using certain general theorems, rather than by the construction of gauges. However, in order to prove these general theorems, we often need to construct special gauges. For example, Example 2.7 will later be seen to be an easy consequence of Hake's Theorem 12.8, but the delicate proof of that result requires a careful construction of a gauge, of which the ones constructed in Examples 2.2 are elementary models. So we believe that these easy examples are instructive and help to prepare the way for more difficult functions.

Telescoping Sums

The functions we will consider in Example 2.1 are known to be R-integrable. However, we will present them in order to show how telescoping sums sometimes allow one to calculate Riemann sums. This type of argument will be used in the proof of the Fundamental Theorems 4.5 and 4.7.

2.1 Examples. **(a)** The constant function $f(x) := c$ is integrable on $I := [a, b]$.

Here, if $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of I, then

$$S(f; \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c(b-a).$$

Since all of the Riemann sums are equal to $c(b-a)$, we can choose the gauges quite arbitrarily. For example, we take $\delta_\varepsilon(x) := 1$. If $\dot{\mathcal{P}}$ is a δ_ε-fine partition, then $|S(f; \dot{\mathcal{P}}) - c(b-a)| = 0 < \varepsilon$, whence $f \in \mathcal{R}^*(I)$ and $\int_a^b f = c(b-a)$. We may also write this conclusion in the "calculus form":

$$\int_a^b c\,dx = c(b-a). \tag{2.α}$$

(b) Let $g(x) := x$ for $I := [a, b]$ with $a < b$.

It is convenient to introduce the function $G(x) := \frac{1}{2}x^2$ in order to be able to write Riemann sums of g for a partition $\{[x_{i-1}, x_i]\}_{i=1}^n$ as telescoping sums involving G. It follows from the Mean Value Theorem [B-S; p. 169] and the fact that $G'(x) = g(x) = x$ that there exists $u_i \in [x_{i-1}, x_i]$ such that

$$G(x_i) - G(x_{i-1}) = g(u_i)(x_i - x_{i-1}) = u_i(x_i - x_{i-1}) \qquad \text{for} \quad i = 1, \cdots, n.$$

If we add the above expressions, we obtain the telescoping sum

$$G(b) - G(a) = \sum_{i=1}^{n} \big[G(x_i) - G(x_{i-1})\big] = \sum_{i=1}^{n} u_i(x_i - x_{i-1}).$$

Hence, if $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is *any* tagged partition of I, then we have

$$G(b) - G(a) - S(g; \dot{\mathcal{P}}) = \sum_{i=1}^{n} \Big[u_i - t_i\Big](x_i - x_{i-1}).$$

If δ is a constant gauge on $[a, b]$ and if $\dot{\mathcal{P}}$ is δ-fine, then since $u_i, t_i \in [x_{i-1}, x_i]$, we have $|u_i - t_i| \le 2\delta$. Thus it follows that

$$\begin{aligned} \big|G(b) - G(a) - S(g; \dot{\mathcal{P}})\big| &\le \sum_{i=1}^{n} |u_i - t_i|(x_i - x_{i-1}) \\ &\le \sum_{i=1}^{n} 2\delta(x_i - x_{i-1}) = 2\delta(b-a). \end{aligned}$$

Therefore, if $\varepsilon > 0$ is given, we see that we should choose the constant gauge $\delta_\varepsilon(t) := \varepsilon/2(b-a)$. Since $\varepsilon > 0$ is arbitrary, it follows that $g \in \mathcal{R}^*(I)$ and $\int_a^b g = G(b) - G(a) = \frac{1}{2}(b^2 - a^2)$, which we may also write in the form:

$$\int_a^b x\,dx = \tfrac{1}{2}(b^2 - a^2). \tag{2.β}$$

(c) Let $h(x) := x^2$ on $I := [a, b]$ with $a < b$.

We introduce the function $H(x) := \frac{1}{3}x^3$ on I so that we can write Riemann sums of $h = H'$ as telescoping sums involving H. If $\{[x_{i-1}, x_i]\}_{i=1}^n$ is a partition of I, the Mean Value Theorem implies that there exist $v_i \in [x_{i-1}, x_i]$ such that

$$H(x_i) - H(x_{i-1}) = h(v_i)(x_i - x_{i-1}) = v_i^2(x_i - x_{i-1}) \qquad \text{for} \quad i = 1, \cdots, n.$$

If we add these expressions, we obtain the telescoping sum

$$H(b) - H(a) = \sum_{i=1}^n [H(x_i) - H(x_{i-1})] = \sum_{i=1}^n v_i^2(x_i - x_{i-1}).$$

Hence, if $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is *any* tagged partition of I, then we have

$$H(b) - H(a) - S(h; \dot{\mathcal{P}}) = \sum_{i=1}^n [v_i^2 - t_i^2](x_i - x_{i-1}).$$

If δ is a constant gauge on $[a, b]$ and $\dot{\mathcal{P}}$ is δ-fine, then since $v_i, t_i \in [x_{i-1}, x_i]$, we have $|v_i - t_i| \le 2\delta$ and $|v_i + t_i| \le 2c$ where $c := \max\{|a|, |b|\}$. Consequently,

$$|v_i^2 - t_i^2| = |v_i + t_i| \cdot |v_i - t_i| \le 2c \cdot 2\delta = 4c\delta.$$

Thus it follows that

$$\begin{aligned} \left|H(b) - H(a) - S(h; \dot{\mathcal{P}})\right| &\le \sum_{i=1}^n |v_i^2 - t_i^2|(x_i - x_{i-1}) \\ &\le \sum_{i=1}^n 4c\delta(x_i - x_{i-1}) = 4c\delta(b - a). \end{aligned}$$

Therefore, if $\varepsilon > 0$ is given, we see that we should choose the constant gauge $\delta_\varepsilon(t) := \varepsilon/4c(b - a)$. Since $\varepsilon > 0$ is arbitrary, it follows that $h \in \mathcal{R}^*(I)$ and $\int_a^b h = H(b) - H(a) = \frac{1}{3}(b^3 - a^3)$, which we may also write in the form:

$$\text{(2.}\gamma\text{)} \qquad \int_a^b x^2\, dx = \tfrac{1}{3}(b^3 - a^3).$$

Remark. The reader may feel that the use of the functions G and H in the preceding examples was a "trick" (and we agree). However, it is a standard procedure to evaluate a sum involving one function by manipulating it so that the sum is replaced by a telescoping sum involving a related function.

The reader may also recall that the usual calculus proofs of the integrals in 2.1(b) and 2.1(c) involve using the formulas:

$$\sum_{k=1}^{n} k = \tfrac{1}{2}n(n+1) \qquad \text{and} \qquad \sum_{k=1}^{n} k^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

While these formulas are often *proved* by using mathematical induction, they are most readily *discovered* by constructing telescoping sums based on the observations that

$$2k+1 = (k+1)^2 - k^2 \qquad \text{and} \qquad 3k^2+3k+1 = (k+1)^3 - k^3.$$

Note that if the first terms are added from $k = 1, \cdots, n$, then we get

$$2\sum_{k=1}^{n} k + n = (n+1)^2 - 1,$$

whence the sum $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ follows. Thus the tricks we used are not very different from the traditional ones.

Some Step Functions

We now look at some other integrals where we choose nonconstant gauges. First we will consider certain step functions. Although these functions are known to be R-integrable, we will construct nonconstant gauges that will force very small intervals to be taken near the points of discontinuity. This type of argument will be used later for more complicated functions.

2.2 Examples. (a) Let $I := [a, b]$, let $c \in (a, b)$ and let $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta$. Let $f : I \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} \alpha & \text{if} \quad a \le x < c, \\ \beta & \text{if} \quad c \le x \le b. \end{cases}$$

(See Figure 2.1.) We will show that $f \in \mathcal{R}^*(I)$ and that $\int_a^b f = \alpha(c-a) + \beta(b-c)$.

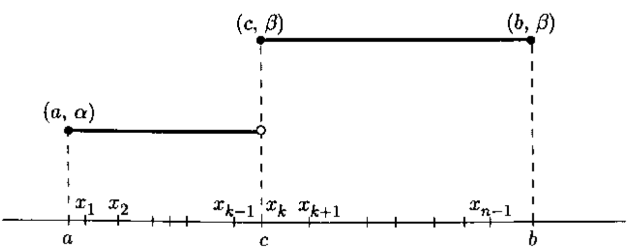


Figure 2.1

We note that f is continuous except at the point $x = c$, so that all the "difficulty" is focused at this point. It will be convenient to force c to be tags of two abutting subintervals of length $\le \delta$, and will then determine exactly how small δ must be.

We can force c to be the tag of a subinterval by choosing δ_ε on I by

$$\delta_\varepsilon(t) := \begin{cases} \frac{1}{2}\operatorname{dist}(t,c) & \text{if} \quad t \neq c, \\ \delta & \text{if} \quad t = c, \end{cases}$$

where δ will be chosen as needed. Now let $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a δ_ε-fine partition of $[a, b]$, which we assume to be ordered: $a = x_0 < x_1 < \cdots < x_n = b$. Our choice of δ_ε forces c to be the tag of any subinterval in $\dot{\mathcal{P}}$ that contains c. Using the right-left procedure, we may assume that c is the tag for the abutting subintervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$, where $x_k = c$. Since $f(t_i) = \alpha$ for $i = 1, \cdots, k-1$, the sum of the first $k - 1$ terms in $S(f; \dot{\mathcal{P}})$ is equal to $\alpha(x_{k-1} - a)$. Since $f(t_i) = \beta$ for $i = k, \cdots, n$, the sum of the remaining terms in $S(f; \dot{\mathcal{P}})$ is $\beta(b - x_{k-1})$. Thus we have

$$S(f; \dot{\mathcal{P}}) = \alpha(x_{k-1} - a) + \beta(b - x_{k-1}).$$

But, since $x_{k-1} - a = (c-a) - (c - x_{k-1})$ and $b - x_{k-1} = (b-c) + (c - x_{k-1})$, we have

$$S(f; \dot{\mathcal{P}}) = \alpha(c-a) + \beta(b-c) + (\beta - \alpha)(c - x_{k-1}).$$

Since $\dot{\mathcal{P}}$ is δ_ε-fine, then $c - \delta \le x_{k-1} < c$ so that $0 < c - x_{k-1} \le \delta$, whence

$$\begin{aligned} \Big| S(f; \dot{\mathcal{P}}) - \big[\alpha(c-a) + \beta(b-c)\big] \Big| &\le |\beta - \alpha|(c - x_{k-1}) \\ &\le |\beta - \alpha|\delta. \end{aligned}$$

We see that it suffices to take $\delta_\varepsilon(c) := \varepsilon/|\beta - \alpha|$ in the definition of δ_ε. Doing that, since $\varepsilon > 0$ is arbitrary, we conclude that $f \in \mathcal{R}^*([a,b])$ and

$$(2.\delta) \qquad \int_a^b f = \alpha(c-a) + \beta(b-c).$$

(b) Let $0 \neq \gamma \in \mathbb{R}$, let $a < c < d < b$, and let $g : [a, b] \to \mathbb{R}$ be defined by

$$g(x) := \begin{cases} \gamma & \text{if} \quad c < x \le d, \\ 0 & \text{otherwise in } [a,b]. \end{cases}$$

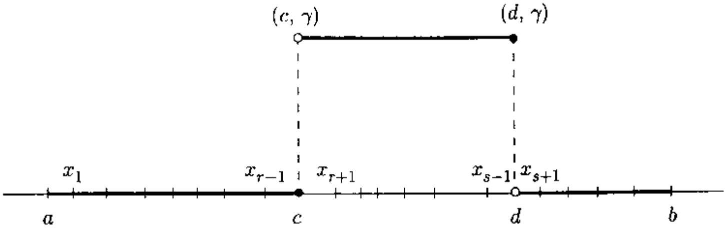


Figure 2.2

(See Figure 2.2.) We will show that $g \in \mathcal{R}^*([a,b])$ and that $\int_a^b g = \gamma(d-c)$.

As in (a), we wish to choose a gauge that will force the points c, d to be tags of those subintervals in any δ_ε-fine partition that contain those points. Some thought (and experimentation) suggests that we define δ_ε on $[a,b]$ by

$$\delta_\varepsilon(t) := \begin{cases} \frac{1}{2}\operatorname{dist}(t, \{c, d\}) & \text{if} \quad t \notin \{c, d\}, \\ \delta & \text{if} \quad t \in \{c, d\}, \end{cases}$$

where $\delta > 0$ is to be determined. Now let $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a δ_ε-fine partition of $[a,b]$. Using the right-left procedure, we may assume that the point c is the tag for the subintervals $[x_{r-1}, x_r]$ and $[x_r, x_{r+1}]$ (whence $x_r = c$) and that d is the tag for $[x_{s-1}, x_s]$ and $[x_s, x_{s+1}]$ (whence $x_s = d$).

Since the tags making a nonzero contribution to $S(g; \dot{\mathcal{P}})$ are $t_{r+1}, \cdots, t_{s+1}$, we have $S(g; \dot{\mathcal{P}}) = \gamma(x_{s+1} - x_{r+1})$.

But $x_{s+1} = d + (x_{s+1} - d)$ and $x_{r+1} = c + (x_{r+1} - c)$, so that

$$S(g; \dot{\mathcal{P}}) = \gamma(d-c) + \gamma(x_{s+1} - d) - \gamma(x_{r+1} - c).$$

Since $\dot{\mathcal{P}}$ is δ_ε-fine, then $|x_{s+1} - d| \le \delta$ and $|x_{r+1} - c| \le \delta$, whence it follows that

$$\left|S(g; \dot{\mathcal{P}}) - \gamma(d-c)\right| \le 2|\gamma|\delta.$$

But since $\gamma \neq 0$, we see that we should take $\delta \le \varepsilon/2|\gamma|$ in the definition of δ_ε at the points $x = c, d$. Having done that, and noting that $\varepsilon > 0$ is arbitrary, we conclude that $g \in \mathcal{R}^*([a,b])$ and that

$$\text{(2.}\varepsilon\text{)} \qquad \int_a^b g = \gamma(d-c).$$

In the exercises, it will be seen that the values of f in Example 2.2(a) at the point c, and of g in Example 2.2(b) at the points c, d can be changed

without affecting the integrability of these functions or the values of their integrals.

The functions in Example 2.2 are R-integrable, so it would have been possible to use constant gauges. We will now consider some functions that are *not* R-integrable and for which nonconstant gauges are essential.

2.3 Examples. (a) We consider the discontinuous function introduced in 1829 by Peter G. L. Dirichlet (1805–1859), namely the function f on $[0,1]$ defined by

$$f(x) := \begin{cases} 1 & \text{if} \quad x \in [0,1] \text{ is rational,} \\ 0 & \text{if} \quad x \in [0,1] \text{ is irrational.} \end{cases}$$

It is well known (see [B-S; pp. 122, 204]) that f is discontinuous at every point of $[0,1]$ and that it is *not* R-integrable. However, we will now show that Dirichlet's function is in $\mathcal{R}^*([0,1])$ with integral 0.

If $\{r_k : k \in \mathbb{N}\}$ is an enumeration of the rational numbers in $[0,1]$ and $\varepsilon > 0$, we define the gauge

$$\delta_\varepsilon(t) := \begin{cases} \varepsilon/2^{k+1} & \text{if} \quad t = r_k, \\ 1 & \text{if} \quad t \text{ is irrational.} \end{cases}$$

Now let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ be a δ_ε-fine partition of $[0,1]$. If the tag $t_i \in I_i$ is irrational, then $f(t_i) = 0$ and the contribution of this subinterval to the Riemann sum is 0.

If the tag t_i is rational, then $f(t_i) = 1$ but the length $l(I_i) = x_i - x_{i-1}$ is small since $\dot{\mathcal{P}} \ll \delta_\varepsilon$. More precisely, if the kth rational number r_k is the tag for the interval I_i, then $I_i \subseteq [r_k - \delta_\varepsilon(r_k), r_k + \delta_\varepsilon(r_k)]$, so that $l(I_i) \le 2\delta_\varepsilon(r_k) = \varepsilon/2^k$. Further, if r_k is the tag for two consecutive subintervals in $\dot{\mathcal{P}}$, the sum of the lengths of these two nonoverlapping subintervals is $\le \varepsilon/2^k$. We conclude that the rational r_k can contribute at most $\varepsilon/2^k$ to the Riemann sum $S(f;\dot{\mathcal{P}})$. Since only rational tags make a nonzero contribution to $S(f;\dot{\mathcal{P}})$, we have

$$|S(f;\dot{\mathcal{P}})| \le \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, Dirichlet's function is integrable and $\int_0^1 f = 0$.

(b) We now consider a function that is a slight modification of one introduced in 1875 by Karl J. Thomae (1840–1921). We define $g : [0,1] \to \mathbb{R}$ by $g(x) := q$ when $x = p/q$ is a nonzero rational number and the positive integers p, q, have no common integer factors except 1, and $g(x) := 0$ when x is an irrational number in $[0,1]$ or $x = 0$.

It is an exercise to show that g is not continuous at any point of I and is *unbounded* on any nondegenerate subinterval of I. So g is not R-integrable; however, we claim that it is in $\mathcal{R}^*([0,1])$ with integral 0. To show this, let $\{r_k = p_k/q_k : k \in \mathbb{N}\}$ be an enumeration of the nonzero rational numbers in $[0,1]$ and define the gauge $\delta_\varepsilon(p_k/q_k) := \varepsilon/q_k 2^{k+1}$, and $\delta_\varepsilon(t) := 1$ if $t \in I$ is irrational or 0. A slight modification of the argument given in 2.3(a) shows that $g \in \mathcal{R}^*([0,1])$ with integral 0. We leave the details to the reader.

Null Sets, Null Functions and Exceptional Sets

The notions of a null set and a null function will be very important for us in the following. We also want to allow exceptional behavior on certain small sets.

WARNING. Some people use the term "null set" as a synonym for the terms "empty set" or "void set" referring to $\emptyset$ (= the set that has no elements). However, we will always use the term "null set" in conformity with Definition 2.4, as is customary in the theory of integration.

2.4 Definition. **(a)** A set $Z \subset \mathbb{R}$ is said to be a **null set** (or a **set of measure zero**) if for every $\varepsilon > 0$ there exists a countable collection $\{J_k\}_{k=1}^{\infty}$ of open intervals such that

$$Z \subseteq \bigcup_{k=1}^{\infty} J_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} l(J_k) \leq \varepsilon.$$

(b) If $A \subseteq \mathbb{R}$, then a function $f : A \to \mathbb{R}$ is said to be a **null function** if the set $\{x \in A : f(x) \neq 0\}$ is a null set.

(c) If $Q(x)$ is a statement about the point $x \in I$ and if $E \subset I$, we say that E is an **exceptional set** for Q if the statement $Q(x)$ holds for all $x \in I - E$.

(d) In part (c), if $E \subset I$ is a null set, we say that $Q(x)$ holds **almost everywhere** on I, and we often write:

$$Q(x) \text{ holds } \textbf{a.e.} \text{ on } I.$$

(e) In part (c), if E is a countable [respectively, finite] set, we say that $Q(x)$ holds **with countably** [respectively, **finitely**] **many exceptions**. In these cases we write:

$$Q(x) \text{ holds } \textbf{c.e.} \text{ [respectively, } \textbf{f.e.}\text{] on } I.$$

We now give some examples of null sets.

2.5 Examples. **(a)** Any subset of a null set in $\mathbb{R}$ is a null set.

(b) Any singleton set $\{p\}$ is a null set.
For, given $\varepsilon > 0$ we take $J_1 := (p - \frac{1}{2}\varepsilon, p + \frac{1}{2}\varepsilon)$ and $J_2 = J_3 = \cdots = \emptyset$.

(c) Any countable set in $\mathbb{R}$ is a null set.
For, if $Z := \{p_1, p_2, \cdots\}$ is an enumeration of Z and if $\varepsilon > 0$, we take

$$J_k := (p_k - \varepsilon/2^{k+1}, p_k + \varepsilon/2^{k+1}) \qquad \text{for} \quad k = 1, 2, \cdots.$$

Since $l(J_k) = \varepsilon/2^k$, it follows that $\sum_{k=1}^{\infty} l(J_k) \le \varepsilon$.

(d) A countable union of null sets is a null set.

For, let $\{Z_m\}_{m=1}^{\infty}$ be a countable collection of null sets in $\mathbb{R}$. If $\varepsilon > 0$ and $m \in \mathbb{N}$, let $\{J_{m,k}\}_{k=1}^{\infty}$ be a countable collection of open intervals such that

$$Z_m \subseteq \bigcup_{k=1}^{\infty} J_{m,k} \qquad \text{and} \qquad \sum_{k=1}^{\infty} l(J_{m,k}) \le \frac{\varepsilon}{2^m}.$$

Then the collection $\{J_{m,k}\}_{m,k=1}^{\infty}$ is countable and it is seen that

$$Z \subseteq \bigcup_{m,k=1}^{\infty} J_{m,k} \qquad \text{and} \qquad \sum_{m,k=1}^{\infty} l(J_{m,k}) \le \varepsilon.$$

It will be shown later that there exist uncountable null sets in $\mathbb{R}$; for example, the Cantor set $\Gamma \subset [0, 1]$ (see Theorem 4.16).

(e) The functions in Example 2.3(a,b) are null functions, since they are equal to 0 a.e.

(f) It is trivial that if $Q_1(x)$ holds f.e. on I, then it also holds c.e. on I. Also, if $Q_2(x)$ holds c.e. on I, then it holds a.e. on I. In general, neither converse statement is true.

(g) The interval $I := [0, 1]$ is not a null set.

For, if I is a null set, there is a countable collection $\{J_k\}_{k=1}^{\infty}$ of open intervals satisfying Definition 2.4(a) with $\varepsilon < 1$. The compactness of I implies that a finite number $\{J_1, \cdots, J_m\}$ of these intervals will cover I. Exercise 1.J implies that there exists a partition $\{I_1, \cdots, I_n\}$ of I into nonoverlapping closed intervals such that each I_i is contained in some interval J_k. Exercise 1.S can now be used to show that $1 = \sum_{i=1}^{n} l(I_i) \le \sum_{k=1}^{m} l(J_k) < 1$, a contradiction. We leave the details as an exercise.

The Integrability of Null Functions

We will now establish the very important fact that: *Every null function is in $\mathcal{R}^*(I)$ and has integral equal to* 0. However, we first consider a simpler situation in 2.6(a) before addressing the general situation, which is rather delicate.

2.6 Examples. **(a)** Let Z be any null set contained in the interval $I := [a, b]$ and let $\varphi : I \to \mathbb{R}$ be the null function defined by $\varphi(x) := 1$ if $x \in Z$ and $\varphi(x) := 0$ if $x \in I - Z$. We will show that $\varphi \in \mathcal{R}^*(I)$ with $\int_I \varphi = 0$.

Let $\varepsilon > 0$ and let $\{J_k\}_{k=1}^{\infty}$ be a countable collection of open intervals as in Definition 2.4(a). We now define a gauge on I. If $t \in I - Z$, we let $\delta_\varepsilon(t) := 1$; and if $t \in Z$, we let $k(t)$ be the smallest index k such that $t \in J_k$ and choose $\delta_\varepsilon(t) > 0$ such that $[t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)] \subset J_{k(t)}$.

Now let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n}$ be a δ_ε-fine partition of I. If $t_i \in I - Z$, then $\varphi(t_i) = 0$ so the sum of the terms in $S(\varphi; \dot{\mathcal{P}})$ with tags in $I - Z$ equals 0. For each $k \in \mathbb{N}$, the (nonoverlapping) intervals I_i with tags in $Z \cap J_k$ have total length $\le l(J_k)$. (See Exercise 1.S.) Consequently, the sum of the terms in $S(\varphi; \dot{\mathcal{P}})$ with tags in J_k is $\le l(J_k)$. Therefore

$$0 \le S(\varphi; \dot{\mathcal{P}}) < \sum_{k=1}^{\infty} l(J_k) \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\varphi \in \mathcal{R}^*(I)$ and that $\int_I \varphi = 0$.

(b) Let ψ be any null function on I. Then $\psi \in \mathcal{R}^*(I)$ with integral 0.

Let $Z := \{x \in I : \psi(x) \ne 0\}$ so that Z is a null set. For each $m \in \mathbb{N}$ we let $Z_m := \{x \in Z : m - 1 \le |\psi(x)| < m\}$. Since $Z_m \subseteq Z$, it is clear that each set Z_m is a null set. Hence, given $\varepsilon > 0$ and $m \in \mathbb{N}$, let $\{J_{m,k}\}_{k=1}^{\infty}$ be a collection of open intervals such that

$$Z_m \subseteq \bigcup_{k=1}^{\infty} J_{m,k} \qquad \text{and} \qquad \sum_{k=1}^{\infty} l(J_{m,k}) \le \frac{\varepsilon}{m2^m}.$$

We now define a gauge on I. If $t \in I - Z$, we set $\delta_\varepsilon(t) := 1$. If $t \in Z$, then since the $\{Z_m\}_{m=1}^{\infty}$ are pairwise disjoint, there exists a unique $m(t) \in \mathbb{N}$ such that $t \in Z_{m(t)}$. Further, we let $k(t)$ be the smallest index k such that $t \in J_{m(t),k}$. We then choose $\delta_\varepsilon(t) > 0$ such that $[t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)] \subset J_{m(t),k(t)}$.

Now let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n}$ be a δ_ε-fine partition of I. If $t_i \in I - Z$, then $\psi(t_i) = 0$ so the sum of the terms in $S(\psi; \dot{\mathcal{P}})$ with tags in $I - Z$ equals 0.

We now show that the sum of the terms in $|S(\psi;\dot{\mathcal{P}})|$ with tags in Z is $\leq \varepsilon$. Fix $m \in \mathbb{N}$; then for each $k \in \mathbb{N}$, the (nonoverlapping) intervals I_i with tags in $Z_m \cap J_{m,k}$ are contained in $J_{m,k}$, and so have total length $\leq l(J_{m,k})$. But since $|\psi(t_i)| < m$ for $t_i \in Z_m$, the sum of the terms in $|S(\psi;\dot{\mathcal{P}})|$ with tags in Z_m is $\leq m(\varepsilon/m2^m) = \varepsilon/2^m$.

Since the sets $\{Z_m\}_{m=1}^{\infty}$ are pairwise disjoint and have union Z, the sum of the terms with tags in Z is $\leq \sum_{m=1}^{\infty} \varepsilon/2^m = \varepsilon$. Therefore, $|S(\psi;\dot{\mathcal{P}})| \leq \varepsilon$ and, since $\varepsilon > 0$ is arbitrary, we conclude that $\psi \in \mathcal{R}^*(I)$ and $\int_I \psi = 0$.

Nonabsolutely Convergent Integrals

We will conclude this section with some very instructive examples of integrable functions. Our first example shows that *every convergent series* gives rise to a (generalized Riemann) integrable function. This result shows the far-reaching nature of the generalized Riemann integral.

In view of the generality of this result, it should not be not surprising that its proof is quite delicate. While a short proof can be based on Hake's Theorem 12.8, we will give the direct proof here. However, the reader may wish to defer a detailed reading of this proof to a later time.

2.7 Example. Let $\sum_{k=1}^{\infty} a_k$ be any convergent series in $\mathbb{R}$, and let A be its limit. Let $c_n := 1 - 1/2^n$ for $n = 0, 1, 2, \cdots$, so that $c_0 = 0, c_1 = 1/2, c_2 = 3/4, c_3 = 7/8, \cdots$. We define a function $h : [0,1] \to \mathbb{R}$ by

$$h(x) := \begin{cases} 2^k a_k & \text{for} \quad x \in [c_{k-1}, c_k),\ k \in \mathbb{N}, \\ 0 & \text{for} \quad x = 1. \end{cases}$$

(See Figure 2.3 on the next page.) We will show that $h \in \mathcal{R}^*([0,1])$ and that $\int_0^1 h = A = \sum_{k=1}^{\infty} a_k$.

Indeed, the length of the interval $[c_{k-1}, c_k]$ is seen to be $1/2^k$, so a preliminary estimation of the "area" between the x-axis and the graph of h suggests that *if the integral exists*, it is likely to equal

$$\sum_{k=1}^{\infty} (2^k a_k) \cdot (1/2^k) = \sum_{k=1}^{\infty} a_k = A.$$

To show that h is integrable on $[0,1]$ with integral A, we need to choose an appropriate gauge. The basic strategy is to choose a gauge that forces the points 1 and c_k for sufficently small $k \in \mathbb{N}$ to be tags, since these points are the "difficult points" at which the function jumps. How to treat the jumps at the points c_k is suggested by an examination of Examples 2.2(a,b).

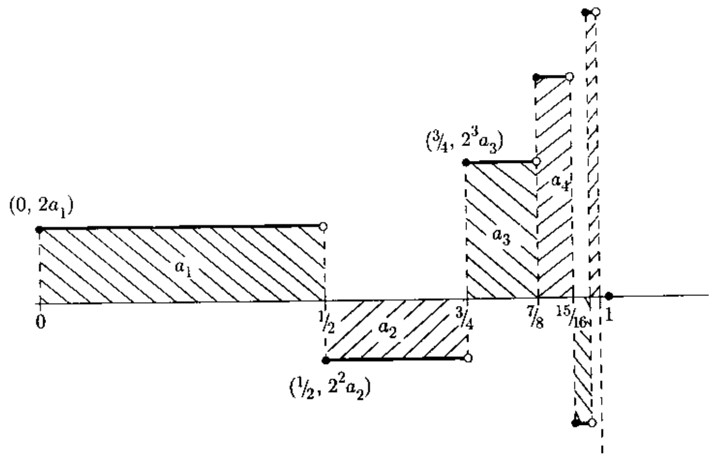


Figure 2.3 Graph of h.

We let $M \geq \sup\{|a_k| : k \in \mathbb{N}\}$ and $M \geq 1$, and given $\varepsilon > 0$ with $\varepsilon \leq 1$, we let $m(\varepsilon) \in \mathbb{N}$ be such that if $m \geq m(\varepsilon)$ then

$$|a_m| \leq \varepsilon \qquad \text{and} \qquad \Big|\sum_{k=m}^{\infty} a_k\Big| \leq \varepsilon.$$

We let $E := \{c_k : k \in \mathbb{N}\} \cup \{1\}$ and define the gauge δ_ε on $[0,1]$ by

$$\delta_\varepsilon(t) := \begin{cases} \frac{1}{2}\operatorname{dist}(t, E) & \text{for} \quad t \in [0,1] - E, \\ \varepsilon/4^{k+1}M & \text{for} \quad t = c_k,\ k \in \mathbb{N}, \\ 1/2^{m(\varepsilon)} & \text{for} \quad t = 1. \end{cases}$$

Now let $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a δ_ε-fine partition of $[0,1]$. We may assume that $c_1 = \frac{1}{2} \leq x_{n-1} < 1$. It is clear that the point $t = 1$ is the tag for the final subinterval $[x_{n-1}, 1]$ in $\dot{\mathcal{P}}$. We let $\mu := \inf\{k \in \mathbb{N} : x_{n-1} \leq c_k\}$ so that $c_k < x_{n-1}$ for $k = 0, 1, \cdots, \mu - 1$. Now the δ_ε-fineness of $\dot{\mathcal{P}}$ implies that

$$1 - 1/2^{m(\varepsilon)} = 1 - \delta_\varepsilon(1) \leq x_{n-1} \leq c_\mu = 1 - 1/2^\mu,$$

whence we have $m(\varepsilon) \leq \mu$.

It follows from the definition of δ_ε that each c_k in $[0, x_{n-1}] \subseteq [0, c_\mu]$ is a tag for any subinterval in $\dot{\mathcal{P}}$ that contains this point; we may also suppose that each such point c_k is a tag for two consecutive subintervals in $\dot{\mathcal{P}}$.

There are two cases that need to be considered.

Case 1: $x_{n-1} = c_\mu$.

Here, for each $k = 1, \cdots, \mu$ we consider the contribution T_k to $S(h; \dot{\mathcal{P}})$ corresponding to the subintervals

$$[c_{k-1}, x_r], \cdots, [x_s, c_k].$$

The last of these subintervals has tag at c_k, where $h(c_k) = 2^{k+1}a_{k+1}$. All of the other tags $t_r, \cdots, t_{s-1}$ yield the value $h(t_i) = 2^k a_k$. Thus we have

$$T_k = 2^k a_k(x_s - c_{k-1}) + 2^{k+1}a_{k+1}(c_k - x_s).$$

Since $x_s - c_{k-1} = (x_s - c_k) + (c_k - c_{k-1}) = (x_s - c_k) + 1/2^k$, we obtain

$$T_k = 2^k a_k(1/2^k) + (2^{k+1}a_{k+1} - 2^k a_k)(c_k - x_s),$$

whence it follows that

$$T_k - a_k = (2^{k+1}a_{k+1} - 2^k a_k)(c_k - x_s),$$

so that

$$|T_k - a_k| \le 2^k \cdot 3M \cdot |c_k - x_s|.$$

By the δ_ε-fineness of $\dot{\mathcal{P}}$, we have $|c_k - x_s| \le \delta_\varepsilon(c_k) = \varepsilon/4^{k+1}M$, whence

$$|T_k - a_k| \le 2^k \cdot 3M \cdot \frac{\varepsilon}{4^{k+1}M} < \frac{\varepsilon}{2^k}.$$

Since the contribution to $S(h; \dot{\mathcal{P}})$ due to $[x_{n-1}, 1]$ is $h(1)(1 - x_{n-1}) = 0$, we have $S(h; \dot{\mathcal{P}}) = \sum_{k=1}^{\mu} T_k$, whence

$$\begin{aligned} |S(h; \dot{\mathcal{P}}) - A| &\le \Big|\sum_{k=1}^{\mu} T_k - \sum_{k=1}^{\mu} a_k\Big| + \Big|\sum_{k=\mu+1}^{\infty} a_k\Big| \\ &\le \sum_{k=1}^{\mu} |T_k - a_k| + \varepsilon \le \sum_{k=1}^{\mu} \frac{\varepsilon}{2^k} + \varepsilon \le 2\varepsilon. \end{aligned}$$

Therefore, we conclude that if $\dot{\mathcal{P}} \ll \delta_\varepsilon$ and if $x_{n-1} = c_\mu = 1/2^\mu$, then

$$(2.\zeta) \qquad |S(h; \dot{\mathcal{P}}) - A| \le 2\varepsilon.$$

Case 2: $x_{n-1} < c_\mu$.

Here the subintervals in $\dot{\mathcal{P}}$ immediately preceding $[x_{n-1}, 1]$ have the form

$$[c_{\mu-1}, x_r], \cdots, [x_{n-2}, x_{n-1}].$$

Since the value of h at all of the tags for these intervals is $2^\mu a_\mu$, the contribution T_μ to $S(h;\dot{\mathcal{P}})$ from these intervals is

$$T_\mu = 2^\mu a_\mu (x_{n-1} - c_{\mu-1}).$$

But since $c_{\mu-1} < x_{n-1} < c_\mu$, then $0 < x_{n-1} - c_{\mu-1} < c_\mu - c_{\mu-1} = 1/2^\mu$, so that

$$|T_\mu| \le 2^\mu |a_\mu| \cdot \frac{1}{2^\mu} = |a_\mu| \le \varepsilon.$$

In this case we have $S(h;\dot{\mathcal{P}}) = \sum_{k=1}^{\mu-1} T_k + T_\mu + 0$, so that

$$(2.\eta) \qquad \left|S(h;\dot{\mathcal{P}}) - A\right| \le \Big|\sum_{k=1}^{\mu-1} T_k - \sum_{k=1}^{\mu-1} a_k\Big| + \left|T_\mu\right| + \Big|\sum_{k=\mu}^{\infty} a_k\Big| \le 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows from $(2.\zeta)$ and $(2.\eta)$ that $h \in \mathcal{R}^*([0,1])$ and that $\int_0^1 h = A$.

We now use Example 2.7 to obtain a nonabsolutely convergent integral.

2.8 Examples. (a) Let $\kappa : [0,1] \to \mathbb{R}$ be defined by

$$\kappa(x) := \begin{cases} (-1)^{k+1} 2^k/k & \text{for} \quad x \in [c_{k-1}, c_k),\ k \in \mathbb{N}, \\ 0 & \text{for} \quad x = 1, \end{cases}$$

where $c_k := 1 - 1/2^k, k = 0, 1, 2, \cdots$. Since the alternating harmonic series $\sum_{k=1}^\infty (-1)^{k+1}/k$ converges [B-S; p. 92], it follows from Example 2.7 that κ belongs to $\mathcal{R}^*([0,1])$ and that

$$\int_0^1 \kappa = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}.$$

(b) Let $\lambda : [0,1] \to \mathbb{R}$ be defined by (see Figure 2.4)

$$\lambda(x) := \begin{cases} 2^k/k & \text{for} \quad x \in [c_{k-1}, c_k),\ k \in \mathbb{N}, \\ 0 & \text{for} \quad x = 1. \end{cases}$$

Note that $\lambda(x) = |\kappa(x)|$ for all $x \in [0,1]$, where κ is the function in 2.8(a). We claim that although $\kappa \in \mathcal{R}^*([0,1])$, the function λ does *not* belong to $\mathcal{R}^*([0,1])$. Indeed, let $n \in \mathbb{N}$ be arbitrary and let $\lambda_n(x) := \lambda(x)$ for $x \in [0, c_n)$ and $\lambda_n(x) := 0$ for $x \in [c_n, 1]$. It follows from Example 2.7 that λ_n is in $\mathcal{R}^*([0,1])$ and that

$$\int_0^1 \lambda_n = \sum_{k=1}^n \frac{1}{k}.$$

But since $0 \le \lambda_n(x) \le \lambda(x)$ for all $x \in [0,1]$, it is geometrically clear (and

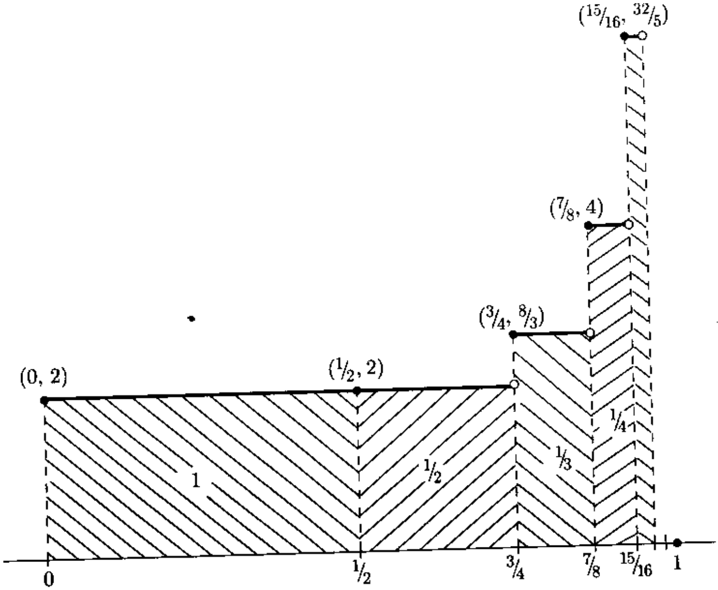


Figure 2.4 Graph of λ.

will be proved in Corollary 3.3) that *if λ is integrable*, then we must have

$$\sum_{k=1}^{n} \frac{1}{k} = \int_0^1 \lambda_n \leq \int_0^1 \lambda.$$

Since the terms on the left side of this inequality are the partial sums in the harmonic series, they are not bounded. Thus λ does not have (a finite) integral, so $\lambda \notin \mathcal{R}^*([0,1])$.

Example 2.8 shows that: *The absolute value of a function in $\mathcal{R}^*(I)$ is not necessarily in $\mathcal{R}^*(I)$.* This is in contrast to the situation for the R-integral (and the L-integral) where the absolute value of an integrable function is always integrable. This example shows that we must exercise care in taking the absolute value of integrable functions, and that the generalized Riemann integral permits the integration of certain functions that are not "absolutely integrable".

Exercises

2.A If $0 \leq x_{i-1} \leq x_i$ and if $v_i \geq 0$ is such that $v_i^2 := \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)$, show that $x_{i-1} \leq v_i \leq x_i$ and that $v_i^2(x_i - x_{i-1}) = \frac{1}{3}(x_i^3 - x_{i-1}^3)$.

2.B Use the relations $3k^2+3k+1=(k+1)^3-k^3$ and $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ to show that $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

2.C Use the relation $4k^3+6k^2+4k+1=(k+1)^4-k^4$ to obtain the summation formula $\sum_{k=1}^{n} k^3 = \left[\frac{1}{2}n(n+1)\right]^2$.

2.D Let $a<b$, let $Q(x) := \frac{1}{4}x^4$ and $q(x) := x^3$ for $x \in I := [a,b]$ and let $c := \max\{|a|,|b|\}$. Apply the argument in Example 2.1(a,b) to show that if $\dot{\mathcal{P}} := \{([x_{i-1},x_i],t_i)\}_{i=1}^{n}$ is a tagged partition of I, then

$$Q(b)-Q(a)-S(q;\dot{\mathcal{P}}) = \sum_{i=1}^{n}(w_i^3-t_i^3)(x_i-x_{i-1}),$$

where $w_i, t_i \in [x_{i-1},x_i]$. Show that if $\dot{\mathcal{P}}$ is δ-fine, where δ is a constant gauge, then

$$|Q(b)-Q(a)-S(q;\dot{\mathcal{P}})| \le 3c^2 2\delta(b-a).$$

2.E Use the preceding exercise to show that if $a<b$, then $\int_a^b x^3\,dx = \frac{1}{4}(b^4-a^4)$.

2.F Let f be the function in Example 2.2(a), let $\dot{\mathcal{P}} = \{([x_{i-1},x_i],t_i)\}$ be a tagged partition of $[a,b]$, and let k be such that $x_{k-1} < c \le x_k$. Show that

$$\left|S(f;\dot{\mathcal{P}}) - [\alpha(c-a)+\beta(b-c)]\right| \le 2\gamma(x_k-x_{k-1}),$$

where $\gamma := \max\{|\alpha|,|\beta|\}$. If $\eta_\varepsilon(t) := \varepsilon/4\gamma$ and $\dot{\mathcal{P}}$ is η_ε-fine, show that we have $\left|S(f;\dot{\mathcal{P}}) - [\alpha(c-a)+\beta(b-c)]\right| \le \varepsilon$.

2.G Let $f_1(x) := \alpha$ for $a \le x \le c$ and $f_1(x) := \beta$ for $c < x \le b$. By using the gauge in Example 2.2(a), show that f_1 is integrable on $[a,b]$ and that $\int_a^b f_1 = \alpha(c-a)+\beta(b-c)$.

2.H Show that the function f_1 in Exercise 2.G is integrable by using the constant gauge $\eta_\varepsilon(x) := \varepsilon/4\gamma$, where $\gamma := \max\{|\alpha|,|\beta|\}$.

2.I Show that the function f_1 in Exercise 2.G is integrable by using Exercise 1.R and Example 2.2(a).

2.J Let g be as in Example 2.2(b) and let γ_1, γ_2 be arbitrary real numbers. Let $g_1(x) := g(x)$ for $x \in I - \{c,d\}$ and let $g_1(c) := \gamma_1$ and $g_1(d) := \gamma_2$. Show that $g_1 \in \mathcal{R}^*([a,b])$ and that $\int_a^b g_1 = \gamma(d-c)$.

2.K Show that the function g in Example 2.3(b) is unbounded on every nondegenerate subinterval in $[0,1]$. Write out the details of the proof that $g \in \mathcal{R}^*([0,1])$.

2.L Let $\lambda(x) := 0$ if x is a rational number in $[0,1]$, and $\lambda(x) := 1$ if x is an irrational number in $[0,1]$. Show that $\lambda \in \mathcal{R}^*([0,1])$ and evaluate the integral $\int_0^1 \lambda$.

2.M Show that a set $Z \subset \mathbb{R}$ is a null set in the sense of Definition 2.4(a) if and only if for every $\varepsilon > 0$ there is a countable collection $\{K_k\}_{k=1}^{\infty}$ of *closed* intervals such that $Z \subseteq \bigcup_{k=1}^{\infty} K_k$ and $\sum_{k=1}^{\infty} l(K_k) \leq \varepsilon$.

2.N Let $A := \{\sqrt{3}/2^n + \sqrt{2}/3^m : m, n \in \mathbb{N}\}$. Is the set A a null set in $\mathbb{R}$? Why?

2.O If $a < b$, write out the details to show that the interval $[a, b]$ is not a null set.

2.P Let $C \subset I := [a, b]$ be a countable set and let $h : I \to \mathbb{R}$ be such that $h(x) = 0$ for $x \in I - C$ and $|h(x)| \leq M$ for all $x \in C$. Modify the gauges in Examples 2.3(a,b) to show that $h \in \mathcal{R}^*(I)$ and that $\int_I h = 0$.

2.Q Let $Z \subset I := [a, b]$ be a null set and let $k : I \to \mathbb{R}$ be such that $k(x) = 0$ for $x \in I - Z$ and $|k(x)| \leq M$ for all $x \in Z$. Modify the gauge in Examples 2.6(a) to show that $k \in \mathcal{R}^*(I)$ and that $\int_I k = 0$.

2.R Show that the set $\mathbb{Q}$ of rational numbers is a null set in $\mathbb{R}$. Show that the set $\mathbb{R} - \mathbb{Q}$ of irrational numbers is *not* a null set in $\mathbb{R}$.

2.S Let $\sum_{k=1}^{\infty} a_k$ be a convergent series in $\mathbb{R}$ and let h be the function in Example 2.7. For each $n \in \mathbb{N}$, let $h_n(x) := h(x)$ for $x \in [0, c_n)$ and $h_n(x) := 0$ for $x \in [c_n, 1]$. Given $\varepsilon > 0$, show that the gauge δ_ε, given in 2.7, can be used *for all* $n \in \mathbb{N}$.

2.T Let $\sum_{k=0}^{\infty} a_k$ be a series of *complex* numbers that converges to $A \in \mathbb{C}$. Show that a complex-valued function h can be defined as in Example 2.7, that h is integrable on $[0,1]$, and that $\int_a^b h = \sum_{k=1}^{\infty} a_k$.

Henri Lebesgue
(June 28, 1875–July 26, 1941)

Courtesy of the London Mathematical Society,
London, England

Basic Properties of the Integral

We will now establish the most important elementary properties of the (generalized Riemann) integral. Since they are formally the same as for the R-integral, the reader will find them quite familiar. Even the proofs are only slightly different from those for the R-integral, so the reader should find most of this section easy reading.

In this section $I := [a, b]$ denotes any compact interval in $\mathbb{R}$. However, it will be shown in Part 2 that most (but not all) of the results presented here remain true for infinite intervals having one of the forms $[a, \infty], [-\infty, b]$, or $[-\infty, \infty]$. For the sake of future convenience, we will mark those theorems that remain valid with *no change* in statement by the symbol $\bullet$, and those that require only a *minor change* in their statements by $\diamond$. However, sometimes a change or supplementary argument is needed in the proof for these infinite intervals.

Although we will be considering functions with values in $\mathbb{R}$, some of the exercises will consider functions with values in the complex field $\mathbb{C}$. However, it is convenient *not* to permit the functions to take on the extended real values $-\infty$ and ∞.

$\bullet$ **3.1 Theorem. (a)** *If f and g are integrable on I to $\mathbb{R}$, then their sum $f + g$ is also integrable on I and*

$$\int_I (f+g) = \int_I f + \int_I g. \tag{3.α}$$

(b) *If f is integrable on I and $c \in \mathbb{R}$, then cf is integrable on I and*

$$\int_I cf = c \int_I f. \tag{3.β}$$

Proof. (a) Let A, B denote the integrals of f, g, respectively. Given $\varepsilon > 0$, let $\delta'_\varepsilon, \delta''_\varepsilon$ be gauges on I such that if the partition $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is δ'_ε-fine, then

$$|S(f; \dot{\mathcal{P}}) - A| \le \tfrac{1}{2}\varepsilon,$$

and if $\dot{\mathcal{P}}$ is δ''_ε-fine, then

$$|S(g; \dot{\mathcal{P}}) - B| \le \tfrac{1}{2}\varepsilon.$$

Now let $\delta_\varepsilon(t) := \min\{\delta'_\varepsilon(t), \delta''_\varepsilon(t)\}$ so that if a partition $\dot{\mathcal{P}}$ is δ_ε-fine, then it is both δ'-fine and δ''-fine. Since it is easily seen that

$$\begin{aligned} S(f+g; \dot{\mathcal{P}}) &= \sum_{i=1}^n (f+g)(t_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) + \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) \\ &= S(f; \dot{\mathcal{P}}) + S(g; \dot{\mathcal{P}}), \end{aligned}$$

it is clear that

$$\begin{aligned} \big|S(f+g; \dot{\mathcal{P}}) - [A+B]\big| &= \big|[S(f; \dot{\mathcal{P}}) - A] + [S(g; \dot{\mathcal{P}}) - B]\big| \\ &\le |S(f; \dot{\mathcal{P}}) - A| + |S(g; \dot{\mathcal{P}}) - B| \\ &\le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then $f + g \in \mathcal{R}^*(I)$ with integral $A + B$.

(b) We leave the proof of this assertion to the reader. Q.E.D.

By using mathematical induction, we can extend Theorem 3.1 to the case of a linear combination of functions in $\mathcal{R}^*(I)$. (See Exercise 3.B.)

• **3.2 Theorem.** *If $f \in \mathcal{R}^*(I)$ and $f(x) \ge 0$ for all $x \in I$, then*

$$\int_I f \ge 0. \tag{3.γ}$$

Proof. Let δ_ε be a gauge on I such that for any partition $\dot{\mathcal{P}} \ll \delta_\varepsilon$, we have $|S(f; \dot{\mathcal{P}}) - \int_I f| \le \varepsilon$. Since $f(x) \ge 0$ for all $x \in I$, then

$$S(f; \dot{\mathcal{P}}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) \ge 0.$$

Therefore $0 \leq S(f;\dot{\mathcal{P}}) \leq \int_I f + \varepsilon$, and since $\varepsilon > 0$ is arbitrary, it follows that $\int_I f \geq 0$. Q.E.D.

• **3.3 Corollary.** *If $f, g \in \mathcal{R}^*(I)$ and $f(x) \leq g(x)$ for all $x \in I$, then*

$$\int_I f \leq \int_I g. \tag{3.δ}$$

Proof. Let $h := g - f$ so that h is integrable by Theorem 3.1. Since the hypothesis implies that $h(x) \geq 0$ for all $x \in I$, it follows from Theorem 3.2 that $\int_I h \geq 0$. Since $\int_I g - \int_I f = \int_I h$, the conclusion is immediate. Q.E.D.

3.4 Corollary. *If $f \in \mathcal{R}^*(I)$ and if m, M are such that $m \leq f(x) \leq M$ for all $x \in I := [a, b]$, then*

$$m(b-a) \leq \int_a^b f \leq M(b-a). \tag{3.ε}$$

Proof. Since $f(x) - m \geq 0$ for all $x \in I$, it follows from Theorems 3.1 and 3.2 that

$$\int_a^b f - \int_a^b m = \int_a^b (f - m) \geq 0.$$

Hence $\int_a^b f \geq m(b-a)$. The other inequality is similar. Q.E.D.

• **3.5 Corollary.** *If f and $|f|$ are both integrable on I, then*

$$\left| \int_I f \right| \leq \int_I |f|. \tag{3.ζ}$$

Proof. Since $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in I$, if we apply Corollary 3.3, then $-\int_I |f| \leq \int_I f \leq \int_I |f|$ whence the result follows. Q.E.D.

The Cauchy Criterion

The next result is useful when there is no particular value that can be predicted to be the integral of a function. It is also useful in many other circumstances, as we will see later.

• **3.6 Cauchy Criterion.** *A function $f : I \to \mathbb{R}$ is integrable if and only if for any $\varepsilon > 0$ there exists a gauge η_ε on I such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are any partitions that are η_ε-fine, then*

$$|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| \leq \varepsilon. \tag{3.η}$$

Proof. $(\Rightarrow)$ If $f \in \mathcal{R}^*(I)$ with integral A, let $\eta_\varepsilon := \delta_{\varepsilon/2} > 0$ be a gauge on I such that if $\dot{\mathcal{P}}, \dot{\mathcal{Q}} \ll \eta_\varepsilon$, then

$$|S(f;\dot{\mathcal{P}}) - A| \le \tfrac{1}{2}\varepsilon \qquad \text{and} \qquad |S(f;\dot{\mathcal{Q}}) - A| \le \tfrac{1}{2}\varepsilon.$$

Consequently, we conclude that for such partitions $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$, then

$$|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| \le |S(f;\dot{\mathcal{P}}) - A| + |S(f;\dot{\mathcal{Q}}) - A| \le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

$(\Leftarrow)$ For each $n \in \mathbb{N}$, let δ_n be a gauge on I such that if $\dot{\mathcal{P}}, \dot{\mathcal{Q}} \ll \delta_n$, then

$$|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| \le 1/n.$$

Evidently we may assume that these gauges satisfy $\delta_n(t) \ge \delta_{n+1}(t)$ for $t \in I$, $n \in N$; otherwise, we replace δ_n by the gauge $\delta_n'(t) := \min\{\delta_1(t), \cdots, \delta_n(t)\}$.

For each $n \in \mathbb{N}$, let $\dot{\mathcal{P}}_n \ll \delta_n$. Clearly, if $m > n$, then both $\dot{\mathcal{P}}_m$ and $\dot{\mathcal{P}}_n$ are δ_n-fine partitions; hence

$$|S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{P}}_m)| \le 1/n \qquad \text{for } m > n.$$

Consequently, the sequence $\big(S(f;\dot{\mathcal{P}}_m)\big)_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$; therefore (see [B-S; p. 82]) this sequence converges in $\mathbb{R}$ and we let $A := \lim_m S(f;\dot{\mathcal{P}}_m)$. Passing to the limit as $m \to \infty$ in the above inequality, we have

$$|S(f;\dot{\mathcal{P}}_n) - A| \le 1/n \qquad \text{for all } n \in \mathbb{N}.$$

We now show that A is the integral of f. Indeed, given $\varepsilon > 0$, let $K \in \mathbb{N}$ with $K > 2/\varepsilon$. If $\dot{\mathcal{Q}}$ is an arbitrary δ_K-fine partition, then

$$\begin{aligned} |S(f;\dot{\mathcal{Q}}) - A| &\le |S(f;\dot{\mathcal{Q}}) - S(f;\dot{\mathcal{P}}_K)| + |S(f;\dot{\mathcal{P}}_K) - A| \\ &\le 1/K + 1/K < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the function f is integrable to A. Q.E.D.

The Integral as a Function of Intervals

We will now show that if a function is integrable over an interval, then it is also integrable over any closed *subinterval* of that interval. In addition, the integral is "additive" in the sense of the next theorem.

• **3.7 Additivity Theorem.** *Let $f : [a,b] \to \mathbb{R}$ and let $c \in (a,b)$. Then f is integrable on $[a,b]$ if and only if its restrictions to $[a,c]$ and $[c,b]$ are both integrable. In this case we have*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{3.6}$$

Proof. ($\Leftarrow$) Suppose that the restriction f_1 of f to the interval $I_1 := [a,c]$, and the restriction f_2 of f to $I_2 := [c,b]$ are integrable to A_1 and A_2, respectively. Then, given $\varepsilon > 0$, there is a gauge δ'_ε on I_1 and a gauge δ''_ε on I_2 such that if $\dot{\mathcal{P}}_1$ is a δ'_ε-fine partition of I_1 and $\dot{\mathcal{P}}_2$ is a δ''_ε-fine partition of I_2, then

$$|S(f_1;\dot{\mathcal{P}}_1) - A_1| \le \tfrac{1}{2}\varepsilon \qquad \text{and} \qquad |S(f_2;\dot{\mathcal{P}}_2) - A_2| \le \tfrac{1}{2}\varepsilon.$$

We define a gauge δ_ε on $[a,b]$ by:

$$\delta_\varepsilon(t) := \begin{cases} \min\{\delta'_\varepsilon(t), \frac{1}{2}(c-t)\} & \text{if} \quad t \in [a,c), \\ \min\{\delta'_\varepsilon(c), \delta''_\varepsilon(c)\} & \text{if} \quad t = c, \\ \min\{\delta''_\varepsilon(t), \frac{1}{2}(t-c)\} & \text{if} \quad t \in (c,b]. \end{cases}$$

Let $\dot{\mathcal{P}}$ be a partition of $I := [a,b]$ that is δ_ε-fine; then the point c must be a tag of at least one subinterval in $\dot{\mathcal{P}}$, and we may use the right-left procedure to arrange that it is in two subintervals, and hence is a partition point of $\dot{\mathcal{P}}$. Let $\dot{\mathcal{P}}_1$ be the partition of I_1 consisting of the partition points $\dot{\mathcal{P}} \cap I_1$, and let $\dot{\mathcal{P}}_2$ be the partition of I_2 consisting of the partition points $\dot{\mathcal{P}} \cap I_2$, so that

$$S(f;\dot{\mathcal{P}}) = S(f_1;\dot{\mathcal{P}}_1) + S(f_2;\dot{\mathcal{P}}_2).$$

Since $\dot{\mathcal{P}}_1$ is δ'_ε-fine and $\dot{\mathcal{P}}_2$ is δ''_ε-fine, we conclude that

$$|S(f;\dot{\mathcal{P}}) - (A_1 + A_2)| \le |S(f_1;\dot{\mathcal{P}}_1) - A_1| + |S(f_2;\dot{\mathcal{P}}_2) - A_2| \le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then f is integrable on I and $(3.\theta)$ holds.

($\Rightarrow$) Conversely, suppose that f is integrable on I and, for each $\varepsilon > 0$, let η_ε be a gauge satisfying the Cauchy Criterion. As above, let f_1 denote the restriction of f to I_1. Let η'_ε be the restriction of η_ε to I_1, and let $\dot{\mathcal{P}}_1, \dot{\mathcal{Q}}_1$ be partitions of I_1 that are η'_ε-fine. By adjoining additional partition points and tags from I_2, we can extend $\dot{\mathcal{P}}_1$ and $\dot{\mathcal{Q}}_1$ to partitions $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ of I that are η_ε-fine. If we use the *same* additional points and tags in I_2 for both $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$, it is easy to see that

$$S(f_1;\dot{\mathcal{P}}_1) - S(f_1;\dot{\mathcal{Q}}_1) = S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}}).$$

But since $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are η_ε-fine, we conclude that $|S(f_1;\dot{\mathcal{P}}_1) - S(f_1;\dot{\mathcal{Q}}_1)| \le \varepsilon$. Therefore, the Cauchy Criterion shows that the restriction f_1 of f to I_1 is integrable on I_1. In the same way, the restriction f_2 of f to I_2 is integrable. The equality $(3.\theta)$ now follows from the first part of the theorem. Q.E.D.

The next result is an important one; it will be seen later that the restriction of a generalized Riemann integrable function to an *arbitrary set* is not necessarily integrable.

• **3.8 Corollary.** *If $f \in \mathcal{R}^*([a,b])$ and if $[c,d] \subseteq [a,b]$, then the restriction of f to $[c,d]$ is integrable.*

Proof. Indeed, since f is integrable on $[a,b]$ and $c \in [a,b]$, then it follows from the theorem that the restriction of f to $[c,b]$ is integrable. But if $d \in [c,b]$, another application of the theorem shows that the restriction of f to $[c,d]$ is integrable. (We have used an obvious fact about restrictions of restrictions here; see Exercise 3.I.) Q.E.D.

• **3.9 Corollary.** *If $f \in \mathcal{R}^*([a,b])$ and if $a = c_0 < c_1 < \cdots < c_n = b$, then the restrictions of f to each of the subintervals $[c_{i-1}, c_i]$ are integrable and*

$$\int_a^b f = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} f. \tag{3.ι}$$

Proof. The assertion was proved for the case $n = 2$ in Theorem 3.7. The general case follows by using mathematical induction. Q.E.D.

If $f \in \mathcal{R}^*([a,b])$ and $\alpha, \beta \in [a,b]$ with $\alpha < \beta$, we have defined $\int_\alpha^\beta f$ to be the integral of the restriction of f to the subinterval $[\alpha, \beta]$. It is also convenient to define this integral for arbitrary values of $\alpha, \beta \in [a,b]$.

• **3.10 Definition.** If $f \in \mathcal{R}^*([a,b])$ and $\alpha, \beta \in [a,b]$, $\alpha < \beta$, we define

$$\int_\beta^\alpha f := -\int_\alpha^\beta f \qquad \text{and} \qquad \int_\alpha^\alpha f := 0.$$

• **3.11 Theorem.** *If $f \in \mathcal{R}[a,b]$ and if α, β, γ are any numbers in $[a,b]$, then*

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f, \tag{3.κ}$$

in the sense that the existence of any two of these integrals implies the existence of the third integral and the equality (3.κ).

Proof. If any two of the numbers α, β, γ are equal, then (3.κ) holds. Thus we may suppose that all three of these numbers are distinct.

For the sake of symmetry, we introduce the expression

$$L(\alpha, \beta, \gamma) := \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f.$$

It is clear that (3.κ) holds if and only if $L(\alpha. \beta, \gamma) = 0$. Therefore, to establish the assertion, we need to show that $L = 0$ for all six permutations of the arguments α, β and γ.

We note that the Additivity Theorem 3.7 implies that $L(\alpha,\beta,\gamma) = 0$ when $\alpha < \gamma < \beta$. But it is easily seen that both $L(\beta,\gamma,\alpha)$ and $L(\gamma,\alpha,\beta)$ equal $L(\alpha,\beta,\gamma)$. Moreover, the numbers

$$L(\beta,\alpha,\gamma), \qquad L(\alpha,\gamma,\beta) \qquad \text{and} \qquad L(\gamma,\beta,\alpha)$$

are all equal to $-L(\alpha,\beta,\gamma)$. Therefore L vanishes for all possible configurations of these three points. Q.E.D.

The Squeeze Theorem

The next result, a consequence of the Cauchy Criterion, is often useful in showing that a function is integrable.

• **3.12 Squeeze Theorem.** *A function f belongs to $\mathcal{R}^*(I)$ if and only if for every $\varepsilon > 0$ there exist functions φ_ε and ψ_ε in $\mathcal{R}^*(I)$ with $\varphi_\varepsilon(x) \le f(x) \le \psi_\varepsilon(x)$ for all $x \in I$, and such that*

$$\int_I (\psi_\varepsilon - \varphi_\varepsilon) \le \varepsilon. \tag{3.λ}$$

Proof. $(\Rightarrow)$ If $f \in \mathcal{R}^*(I)$ and $\varepsilon > 0$ is given, we can take $\varphi_\varepsilon := \psi_\varepsilon := f$.

$(\Leftarrow)$ Let $\varepsilon > 0$ be given; then if $\varphi_\varepsilon \le f \le \psi_\varepsilon$, it follows that for any tagged partition $\dot{\mathcal{P}}$ of I we have

$$S(\varphi_\varepsilon;\dot{\mathcal{P}}) \le S(f;\dot{\mathcal{P}}) \le S(\psi_\varepsilon;\dot{\mathcal{P}}).$$

Since $\varphi_\varepsilon \in \mathcal{R}^*(I)$, there exists a gauge $\delta'_\varepsilon > 0$ on I such that if $\dot{\mathcal{P}} \ll \delta'_\varepsilon$, then $|S(\varphi_\varepsilon;\dot{\mathcal{P}}) - \int_I \varphi_\varepsilon| \le \varepsilon$, whence it follows that $\int_I \varphi_\varepsilon - \varepsilon \le S(\varphi_\varepsilon;\dot{\mathcal{P}})$. Similarly there exists a gauge $\delta''_\varepsilon > 0$ such that if $\dot{\mathcal{P}} \ll \delta''_\varepsilon$, then $S(\psi_\varepsilon;\dot{\mathcal{P}}) \le \int_I \psi_\varepsilon + \varepsilon$. Now let $\delta_\varepsilon := \min\{\delta'_\varepsilon, \delta''_\varepsilon\}$, so that if $\dot{\mathcal{P}} \ll \delta_\varepsilon$, then

$$\int_I \varphi_\varepsilon - \varepsilon \le S(f;\dot{\mathcal{P}}) \le \int_I \psi_\varepsilon + \varepsilon,$$

and if $\dot{\mathcal{Q}} \ll \delta_\varepsilon$, then

$$-\int_I \psi_\varepsilon - \varepsilon \le -S(f;\dot{\mathcal{Q}}) \le -\int_I \varphi_\varepsilon + \varepsilon.$$

Adding these inequalities, we obtain

$$-\int_I (\psi_\varepsilon - \varphi_\varepsilon) - 2\varepsilon \le S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}}) \le \int_I (\psi_\varepsilon - \varphi_\varepsilon) + 2\varepsilon,$$

whence we conclude from (3.λ) that

$$|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| \le \int_I (\psi_\varepsilon - \varphi_\varepsilon) + 2\varepsilon \le 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, f satisfies the Cauchy Criterion. Therefore f is integrable on I. Q.E.D.

Step Functions

We now establish the integrability of certain important classes of functions. First, we will discuss step functions, and then turn to more complicated classes.

◇ **3.13 Definition.** A function $s : I \to \mathbb{R}$ is said to be a **step function** on $I := [a, b]$ if there exists a partition $\{[c_{i-1}, c_i]\}_{i=1}^n$ of I and real numbers $\{\alpha_i\}_{i=1}^n$ such that

$$(3.\mu) \qquad s(x) = \alpha_i \qquad \text{for} \quad x \in (c_{i-1}, c_i),\ i = 1, \cdots, n.$$

Remark. The step function s also has values at the partition points c_i, which may differ from the values α_i. For the purposes of integration, these values at c_i are totally unimportant, as is seen in Exercises 1.R or 3.C.

• **3.14 Theorem.** *Every step function on $I := [a, b]$ is integrable. In fact, if s is the step function given by $(3.\mu)$, then*

$$(3.\nu) \qquad \int_a^b s = \sum_{i=1}^n \alpha_i (c_i - c_{i-1}).$$

Proof. Define s_i on I by $s_i(x) := \alpha_i$ for $x \in (c_{i-1}, c_i)$ and $s_i(x) := 0$ elsewhere on I. We have seen in Example 2.2(b) and Exercise 2.J that $s_i \in \mathcal{R}^*(I)$ with integral $\alpha_i(c_i - c_{i-1})$. Now apply Theorem 3.1 and induction. Q.E.D.

Regulated Functions

We now introduce an important (and a quite inclusive) class of functions on the interval $I := [a, b]$ that will be seen to be integrable. It will be shown that continuous and monotone functions are contained in this class.

• **3.15 Definition.** A function $f : I \to \mathbb{R}$ is said to be **regulated** on $I := [a, b]$ if for every $\varepsilon > 0$ there exists a step function $s_\varepsilon : I \to \mathbb{R}$ such that

$$(3.\xi) \qquad |f(x) - s_\varepsilon(x)| \le \varepsilon \qquad \text{for all} \quad x \in I.$$

Remark. By letting $\varepsilon = 1/n$ $(n \in \mathbb{N})$, it is clear that a function f is regulated if and only if there is a sequence $(s_n)_{n=1}^\infty$ of step functions on $I \to \mathbb{R}$ that converges **uniformly** to f on I (see [B-S; p. 229] or Definition 8.2 below).

3.16 Integrability of regulated functions. *If $f : I \to \mathbb{R}$ is a regulated function on $I := [a, b]$, then $f \in \mathcal{R}^*(I)$.*

Proof. Given $\varepsilon > 0$, let $s_\varepsilon : I \to \mathbb{R}$ be a step function such that (3.ξ) holds. Therefore, we have

$$s_\varepsilon(x) - \varepsilon \le f(x) \le s_\varepsilon(x) + \varepsilon \qquad \text{for} \quad x \in [a, b].$$

If we let $\varphi_\varepsilon(x) := s_\varepsilon(x) - \varepsilon$ and $\psi_\varepsilon(x) := s_\varepsilon(x) + \varepsilon$ for $x \in I$, then the step functions φ_ε and ψ_ε are integrable and $\varphi_\varepsilon(x) \le f(x) \le \psi_\varepsilon(x)$ for $x \in I$. Moreover, since

$$\int_a^b (\psi_\varepsilon - \varphi_\varepsilon) = \int_a^b 2\varepsilon = 2(b-a)\varepsilon,$$

it follows from the Squeeze Theorem 3.12 that $f \in \mathcal{R}^*(I)$. Q.E.D.

The following characterization of regulated functions will be useful.

3.17 Characterization of regulated functions. *A function $f : I \to \mathbb{R}$ is a regulated function if and only if it has all of its one-sided limits at every point of the interval $I := [a, b]$.*

Proof. ($\Rightarrow$) First we note that every step function has one-sided limits at each point. To prove that a regulated function f has the same property, let $c \in [a, b)$; we will prove that f has a right hand limit at c. To do so, let $\varepsilon > 0$ be given and let $s_\varepsilon : I \to \mathbb{R}$ be a step function such that (3.ξ) holds. Since s_ε is a step function and $\lim_{x\to c+} s_\varepsilon(x)$ exists, there exists $\delta_\varepsilon(c) > 0$ such that if $x, y \in (c, c + \delta_\varepsilon(c))$, then $s_\varepsilon(x) = s_\varepsilon(y)$. Therefore, if $x, y \in (c, c + \delta_\varepsilon(c))$, then

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - s_\varepsilon(x)| + |s_\varepsilon(x) - s_\varepsilon(y)| + |s_\varepsilon(y) - f(y)| \\ &\le \varepsilon + 0 + \varepsilon = 2\varepsilon. \end{aligned}$$

But since $\varepsilon > 0$ is arbitrary, the Cauchy Criterion implies that the right hand limit $\lim_{x\to c+} f(x)$ exists.

The existence of left hand limits at $c \in (a, b]$ is proved in the same way.

($\Leftarrow$) Suppose f has all one-sided limits at every point of I. The Cauchy Criterion for the existence of the one-sided limits guarantees that, given $\varepsilon > 0$, there is a gauge δ_ε on I such that if $t \in I$ and y_1, y_2 are both in $[t - \delta_\varepsilon(t), t)$, or are both in $(t, t + \delta_\varepsilon(t)]$, then $|f(y_1) - f(y_2)| \le \varepsilon$.

Now let $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a δ_ε-fine partition of I. We define $s_\varepsilon(z) := f(z)$ if z is one of the numbers

$$a = x_0 \le t_1 \le \cdots \le x_{i-1} \le t_i \le x_i \le \cdots \le t_n \le x_n = b.$$

On the interval $(x_{i-1}, t_i) \subseteq [t_i - \delta_\varepsilon(t_i), t_i)$, we define $s_\varepsilon(x) := f\big(\tfrac{1}{2}(x_{i-1} + t_i)\big)$ so that

$$|f(x) - s_\varepsilon(x)| = \big|f(x) - f\big(\tfrac{1}{2}(x_{i-1} + t_i)\big)\big| \le \varepsilon.$$

Similarly, on the interval $(t_i, x_i) \subseteq (t_i, t_i + \delta_\varepsilon(t_i)]$, we define $s_\varepsilon(x) := f\left(\frac{1}{2}(t_i + x_i)\right)$, so that

$$|f(x) - s_\varepsilon(x)| = \left|f(x) - f\left(\tfrac{1}{2}(t_i + x_i)\right)\right| \leq \varepsilon.$$

Hence the step function s_ε satisfies $|f(x) - s_\varepsilon(x)| \leq \varepsilon$ for all $x \in I$. But since $\varepsilon > 0$ is arbitrary, we conclude that f is a regulated function. Q.E.D.

3.18 Integrability of continuous functions. *If $f : I \to \mathbb{R}$ is continuous on $I := [a, b]$, then $f \in \mathcal{R}^*(I)$.*

Proof. Since a continuous function on I has a limit at every point of I, Theorem 3.17 implies that a continuous function is a regulated function. Hence, by Theorem 3.16, a continuous function is integrable on I. Q.E.D.

We recall that a function $f : I \to \mathbb{R}$ is said to be **increasing** (or nondecreasing) on I if $x, y \in I$, $x \leq y$, imply that $f(x) \leq f(y)$. Similarly, f is said to be **decreasing** (or nonincreasing) on I if $x, y \in I$, $x \leq y$, imply that $f(x) \leq f(y)$. A function is said to be **monotone** on I if it is either increasing on I or decreasing on I.

3.19 Integrability of monotone functions. *If $f : I \to \mathbb{R}$ is monotone on $I := [a, b]$, then f is regulated and $f \in \mathcal{R}^*(I)$.*

Proof. It is known (see [B-S; p. 149]) that a monotone function on I has one-sided limits at every point of I. It follows from Theorem 3.17 that a monotone function is a regulated function. Therefore, by Theorem 3.16, a monotone function is integrable on I. Q.E.D.

The next result about regulated functions will be used in Section 4.

• **3.20 Theorem.** *The set of points of discontinuity of a regulated function $f : I \to \mathbb{R}$ is a countable subset of $I := [a, b]$.*

Proof. For each $n \in \mathbb{N}$, let $s_n : I \to \mathbb{R}$ be a step function such that

$$|f(x) - s_n(x)| \leq 1/n \qquad \text{for} \quad x \in I.$$

Since the step function s_n has a finite set D_n of points of discontinuity, the set $D := \bigcup_{n=1}^{\infty} D_n$ is a countable set in I. We will show that if $c \in I - D$, then c is a point of continuity of f. Indeed, given $\varepsilon > 0$, choose $N > 1/\varepsilon$, so that $|f(x) - s_N(x)| \leq 1/N < \varepsilon$ for all $x \in I$. Since s_N is continuous at c, there exists $\gamma > 0$ such that if $|x - c| < \gamma$, $x \in I$, then $|s_N(x) - s_N(c)| \leq \varepsilon$. Combining these estimates, we conclude that if $|x - c| < \gamma$, $x \in I$, then

$$\begin{aligned} |f(x) - f(c)| &\leq |f(x) - s_N(x)| + |s_N(x) - s_N(c)| + |s_N(c) - f(c)| \\ &\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then f is continuous at $c \in I - D$. Q.E.D.

It is easy to see that the *product* of two integrable functions is not necessarily integrable. However, the following partial result is sometimes useful; a stronger theorem will be given in 10.12.

• **3.21 Theorem.** *Let $f \in \mathcal{R}^*([a,b])$ be bounded below and let g be a regulated function on $[a,b]$. Then the product $f \cdot g$ belongs to $\mathcal{R}^*([a,b])$.*

Proof. It is evidently enough to consider the case where $f(x) \geq 0$ for $x \in I := [a,b]$. It is also clear that if s is a step function, then $f \cdot s$ is integrable. Let $A > \int_a^{b} f \geq 0$ and let $\varepsilon > 0$. If g is a regulated function, let s_ε be a step function on I such that $|g(x) - s_\varepsilon(x)| \leq \varepsilon/2A$ for all $x \in I$. If we define $\varphi_\varepsilon(x) := f(x)\big(s_\varepsilon(x) - \varepsilon/2A\big)$ and $\psi_\varepsilon(x) := f(x)\big(s_\varepsilon(x) + \varepsilon/2A\big)$ for all $x \in I$, then $\varphi_\varepsilon, \psi_\varepsilon \in \mathcal{R}^*(I)$ and it follows that

$$\varphi_\varepsilon(x) \leq f(x) \cdot g(x) \leq \psi_\varepsilon(x) \qquad \text{for all} \quad x \in I,$$

and that

$$\int_I (\psi_\varepsilon - \varphi_\varepsilon) = (\varepsilon/A) \int_I f \leq \varepsilon.$$

Therefore the Squeeze Theorem 3.12 implies that $f \cdot g \in \mathcal{R}^*(I)$. Q.E.D.

Translations

We will close this section with a theorem showing that the translation (either additive or multiplicative) of an $\mathcal{R}^*$-integrable function is $\mathcal{R}^*$-integrable. These results are special instances of the Substitution Theorem that will be discussed in Section 13.

Let $I := [a,b]$ be a compact interval in $\mathbb{R}$ and let $r \in \mathbb{R}$. We define the r-**additive translate** of I to be the interval $I_r := [a+r, b+r]$, and the r-**additive translate** of f to be the function $f_r(y) := f(y-r)$ for all $y \in I_r$. Similarly, if $r > 0$, we define the r-**multiplicative translate** of I to be the interval $I_{(r)} := [ar, br]$, and the r-**multiplicative translate** of f to be the function $f_{(r)}(z) := f(z/r)$ for all $z \in I_{(r)}$. (If $r < 0$, then the multiplicative translates can also be defined, except that the order of the points in the interval is reversed.)

• **3.22 Theorem.** **(a)** *If $f \in \mathcal{R}^*(I)$, then $f_r \in \mathcal{R}^*(I_r)$ and*

$$\int_{I_r} f_r = \int_I f. \tag{3.o}$$

(b) *If $f \in \mathcal{R}^*(I)$ and $r > 0$, then $f_{(r)} \in \mathcal{R}^*(I_{(r)})$ and*

$$\int_{I_{(r)}} f_{(r)} = r \int_I f. \tag{3.π}$$

Proof. (a) Given $\varepsilon > 0$, let δ_ε be a gauge on I such that if $\dot{\mathcal{P}}_1$ is any δ_ε-fine partition of I, then $|S(f;\dot{\mathcal{P}}_1) - \int_I f| \le \varepsilon$. Now define $\eta_\varepsilon(s) := \delta_\varepsilon(s-r)$ for all $s \in I_r$ so that η_ε is a gauge on the interval I_r.

Suppose that $\dot{\mathcal{Q}} := \{([y_{i-1}, y_i], s_i)\}_{i=1}^n$ is an η_ε-fine partition of I_r, whence

$$s_i - \eta_\varepsilon(s_i) \le y_{i-1} \le s_i \le y_i \le s_i + \eta_\varepsilon(s_i).$$

If we let $x_i := y_i - r$ and $t_i := s_i - r$, then $\eta_\varepsilon(s_i) = \delta_\varepsilon(s_i - r) = \delta_\varepsilon(t_i)$, so that

$$t_i - \delta_\varepsilon(t_i) \le x_{i-1} \le t_i \le x_i \le t_i + \delta_\varepsilon(t_i),$$

whence $\dot{\mathcal{P}} := \{[x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a δ_ε-fine partition of I. Since we readily see that $S(f_r;\dot{\mathcal{Q}}) = S(f;\dot{\mathcal{P}})$, we infer that

$$\left|S(f_r;\dot{\mathcal{Q}}) - \int_I f\right| = \left|S(f;\dot{\mathcal{P}}) - \int_I f\right| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f_r \in \mathcal{R}^*(I_r)$ and that (3.o) holds.

(b) Given $\varepsilon > 0$, let δ_ε be a gauge on I such that if $\dot{\mathcal{P}}_1$ is any δ_ε-fine partition of I, then $|S(f;\dot{\mathcal{P}}_1) - \int_I f| \le \varepsilon/r$. Now define $\zeta_\varepsilon(z) := r\delta_\varepsilon(z/r)$ for $z \in I_{(r)}$ so that ζ_ε is a gauge on $I_{(r)}$.

Suppose that $\dot{\mathcal{U}} := \{([z_{i-1}, z_i], u_i)\}_{i=1}^n$ is a ζ_ε-fine partition of $I_{(r)}$, whence

$$u_i - \zeta_\varepsilon(u_i) \le z_{i-1} \le u_i \le z_i \le u_i + \zeta_\varepsilon(u_i).$$

If we let $x_i := z_i/r$ and $t_i := u_i/r$, then $\zeta_\varepsilon(u_i) = \zeta_\varepsilon(rt_i) = r\delta_\varepsilon(t_i)$, so that

$$t_i - \delta_\varepsilon(t_i) \le x_{i-1} \le t_i \le x_i \le t_i + \delta_\varepsilon(t_i),$$

whence $\dot{\mathcal{P}} := \{[x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a δ_ε-fine partition of I. Since we readily see that $S(f_{(r)};\dot{\mathcal{U}}) = rS(f;\dot{\mathcal{P}})$, we infer that

$$\left|S(f_{(r)};\dot{\mathcal{U}}) - r\int_I f\right| = r \cdot \left|S(f;\dot{\mathcal{P}}) - \int_I f\right| \le r \cdot (\varepsilon/r) = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $f_{(r)} \in \mathcal{R}^*(I_{(r)})$ and (3.π) holds. Q.E.D.

Exercises

3.A Write out the details of the proof of Theorem 3.1(b).

3.B Use induction to show that if $f_1, \cdots, f_n$ are in $\mathcal{R}^*(I)$ and if $c_1, \cdots, c_n$ are in $\mathbb{R}$, then the linear combination $f := \sum_{k=1}^{n} c_k f_k$ belongs to $\mathcal{R}^*(I)$ and $\int_I f = \sum_{k=1}^{n} c_k \int_I f_k$.

3.C Suppose that $f \in \mathcal{R}^*(I)$, that $g : I \to \mathbb{R}$, and that $g(x) = f(x)$ a.e. on I. Show that $g \in \mathcal{R}^*(I)$ and that $\int_I g = \int_I f$. [Hint: Consider $h := g - f$.]

3.D Suppose that $f, g \in \mathcal{R}^*(I)$ and that $f(x) \leq g(x)$ a.e. on I. Show that $\int_I f \leq \int_I g$.

3.E If $f, g \in \mathcal{R}^*(I)$ and $|f(x)| \leq g(x)$ a.e. on I, show that $|\int_I f| \leq \int_I g$.

3.F Suppose that $f, g : I \to \mathbb{R}$, that $g \in \mathcal{R}^*(I)$ with $\int_I g = 0$, and that $|f(x)| \leq g(x)$ for all $x \in I$. Show that f and $|f|$ are integrable and that $\int_I f = 0 = \int_I |f|$.

3.G Show that the conclusion in Exercise 3.F remains true under the hypothesis that $|f(x)| \leq g(x)$ a.e. on I.

3.H Write out a proof of the equality $S(f; \dot{\mathcal{P}}) = S(f_1; \dot{\mathcal{P}}_1) + S(f_2; \dot{\mathcal{P}}_2)$, used in the proof of Theorem 3.7. Also, establish the relation $S(f_1; \dot{\mathcal{P}}_1) - S(f_1; \dot{\mathcal{Q}}_1) = S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})$ that was used in the second part of that proof.

3.I Let $f : [a, b] \to \mathbb{R}$ and let $[c, d] \subseteq [a, b]$. Let f_1 be the restriction $f|[c, b]$ of f to $[c, b]$, and let f_2 be the restriction $f_1|[c, d]$. Show that $f_2 = f|[c, d]$.

3.J Let $f \in \mathcal{R}^*([a, b])$ and let $c \in (a, b)$. If $g(x) := 0$ for $x \in [a, c)$ and if $g(x) := f(x)$ for $x \in [c, b]$, show that $g \in \mathcal{R}^*([a, b])$ and that $\int_a^b g = \int_c^b f$.

3.K Suppose that (f_n) is a sequence in $\mathcal{R}^*(I)$ such that $0 \leq f_n(x) \leq f(x)$ for all $x \in I$, and $\int_I f_n \geq n$ for all $n \in \mathbb{N}$. Show that $f \notin \mathcal{R}^*(I)$.

3.L Show that the function $f(x) := 1/x$ for $x \in (0, 1]$ and $f(0) := 0$ is not in $\mathcal{R}^*([0, 1])$. [Hint: Construct step functions f_n with $f_n \leq f$.]

3.M Let $f, g \in \mathcal{R}^*(I)$. Show that the function $h := \max\{f, g\}$ may not be integrable on I. [Hint: Recall Example 2.8(b).]

3.N Let $f \in \mathcal{R}^*(I)$ and let $E \subset I$. Let $f_E(x) := 0$ for $x \in I - E$ and $f_E(x) := f(x)$ for $x \in E$. Give an example where $f_E \notin \mathcal{R}^*(I)$. [Hint: The set E cannot be a closed subinterval.]

3.O Give an example of functions $f, g \in \mathcal{R}^*(I)$ with g bounded on I such that the product $f \cdot g \notin \mathcal{R}^*(I)$.

3.P Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that there exists a point $c \in [a, b]$ such that $(b-a)f(c) = \int_a^b f$. [This is sometimes called the *Mean Value Theorem for Integrals*.]

3.Q Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, $a < b$, and let $f(x) > 0$ for all $x \in [a, b]$. Prove that $\int_a^b f > 0$.

3.R Show that any regulated function on $[a, b]$ is bounded.

3.S The function $g(x) := 1/\sqrt{x}$ for $x \in (0, 1]$ and $g(0) := 0$ will be seen in Example 4.6 to be in $\mathcal{R}^*([0, 1])$. Show that g is not regulated.

3.T Let $I := [a, b]$ with $a < b$, and let $r < 0$. If $f \in \mathcal{R}^*(I)$, let $f_{(r)}(z) := f(z/r)$ for all $z \in [br, ar]$. Modify the proof of Theorem 3.22(b) to show that $f_{(r)} \in \mathcal{R}^*([br, ar])$ and that

$$\int_{br}^{ar} f_{(r)} = -r \int_a^b f.$$

3.U Show that Theorem 3.1 holds when f and g are integrable functions on I to $\mathbb{C}$ (the complex numbers) and when $c \in \mathbb{C}$.

3.V (a) Show that any complex-valued regulated function f on $I := [a, b]$ can be written in the form $f = g + ih$, where $g, h : I \to \mathbb{R}$ are regulated real-valued functions.

(b) Use part (a) to show that every complex-valued regulated function is integrable.

3.W Show that any continuous function on $[a, b]$ to $\mathbb{C}$ is integrable.

3.X If $f : I \to \mathbb{C}$ and $|f| : I \to \mathbb{R}$ are integrable on I, show that $|\int_I f| \le \int_I |f|$. [Hint: Write $|\int_I f| = \alpha \int_I f$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.]

The Fundamental Theorems of Calculus

There are two aspects to what is traditionally called the "Fundamental Theorem of Calculus": one part is concerned with the integration of derivatives, and the other part with the differentiation of integrals. Both aspects will be discussed in this section.

The reader first encountered the Fundamental Theorem in a course on calculus and learned to evaluate the integral of a function f known to be R-integrable by finding a function F on $[a, b]$ with $F'(x) = f(x)$ for all $x \in [a, b]$, and then evaluating $F(b) - F(a)$. This method — which is essentially due to Newton and Leibniz — is the one used to evaluate virtually all elementary integrals (although we will later obtain some convergence theorems that can be also useful). In any case, we seldom evaluate integrals directly using the definition, or by calculating Riemann sums. The reader may imagine that, since the usual Newton-Leibniz formula enables one to evaluate R-integrals, then a more complicated method will be required to evaluate generalized Riemann integrals. In fact, as we will see, the rule for evaluating R*-integrals is the same as for R-integrals. Moreover, since the derivative of a function is always generalized Riemann integrable (but not always R-integrable or even L-integrable), the situation is actually *simpler* for the R*-integral than for the R-integral.

In this section, we will employ a "spiral" approach and establish weaker or more elementary results first, and then establish stronger results that require slightly more complicated proofs.

Primitives and Indefinite Integrals

Before we state our main theorems, it is convenient to introduce some terminology. In the following material, it will often be convenient to consider three types of *exceptional sets*: finite sets, countable sets, and null sets. In doing so, we will use the prefixes f-, c-, and a- (for almost everywhere).

$\diamond$ **4.1 Definition.** Let $I := [a, b] \subset \mathbb{R}$ and let $F, f : I \to \mathbb{R}$.

(a) We say that F is a **primitive** (or an **antiderivative**) **of f on I** if the derivative $F'(x)$ exists and $F'(x) = f(x)$ *for all* $x \in I$.

(b) We say that F is an **a-primitive** [respectively, **c-primitive, f-primitive**] of f on I if F is continuous on I, and there exists a null [respectively, countable, finite] set E of points $x \in I$ where either $F'(x)$ does not exist, or does not equal $f(x)$. The set E is called the **exceptional set** of f.

(c) If $f \in \mathcal{R}^*(I)$ and $u \in I$, then the function $F_u : I \to \mathbb{R}$ defined by

$$F_u(x) := \int_u^x f \qquad (4.\alpha)$$

is called the **indefinite integral of f with base point u**. If the base point is the left endpoint (or is well understood) we usually do not write the subscript. Any function that differs from F_a by a constant is called an **indefinite integral of f**.

WARNING. Some authors use the words "antiderivative", "primitive", and "indefinite integral" as synonyms, or make distinctions that are slightly different from the ones used here.

4.2 Remarks. (a) In Definition 4.1(a) we did not need to assume that F is continuous on I, since the existence of its derivative on I guarantees this to be the case. But, in 4.1(b), it is important to assume that F is continuous on I.

(b) For c in an exceptional set E the derivative $F'(c)$ may not exist, or $F'(c)$ may exist but not equal $f(c)$. In fact, sometimes the function f is not even defined at certain points in E; in this case we extend f to be equal to 0 at such points.

(c) If f is integrable on $[a, b]$, then it follows from Corollary 3.8 and Definition 3.10 that the integral in $(4.\alpha)$ is defined, so Definition 4.1(c) makes sense. Note that we always have $F_u(u) = 0$.

(d) If F_u is the indefinite integral of f with base point u, then since $F_u(x) = F_a(x) - \int_a^u f$ for $x \in I$, it follows that F_u is an indefinite integral of f.

4.3 Examples. **(a)** If $f(x) := x^n$ for $x \in [a, b]$, $n \in \mathbb{N}$, then $F(x) := x^{n+1}/(n+1)$ is a primitive of f on any interval $[a, b] \subset \mathbb{R}$. It will be seen later that F is also the indefinite integral of f with base point 0.

(b) If $g(x) := 1/\sqrt{x}$ for $x \in (0, 1]$, then g is not defined at $x = 0$ so we define $g(0) := 0$. Further, g is not bounded on $[0, 1]$, so its R-*integral does not exist*. Nevertheless, we will see in Example 4.6 that g is R*-integrable. In any case, it is true that the function $G(x) := 2\sqrt{x}$ for $x \in [0, 1]$ is an f-primitive of g on $[0, 1]$ with the (finite) exceptional set $E = \{0\}$. It will be seen later that G is also the indefinite integral of g with base point 0.

(c) If sgn is the **signum function** on $\mathbb{R}$ defined by

$$\operatorname{sgn}(x) := \begin{cases} -1 & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x = 0, \\ +1 & \text{if} \quad x > 0, \end{cases}$$

then sgn is integrable over any interval $[a, b] \subset \mathbb{R}$, since its restriction to this interval is a step function. It is easy to see that the function $H(x) := |x|$ is an f-primitive of sgn on any interval $[a, b]$ with exceptional set $E = \{0\}$. It is an exercise to show that H is the indefinite integral of sgn with base point 0.

(d) Let f be the Dirichlet function introduced in Example 2.3(a), which is everywhere discontinuous, yet R*-integrable on $I := [0, 1]$. If we let $F(x) := 0$ for all $x \in I$, then F is a c-primitive of f on I since $F'(x) = 0 = f(x)$ for all irrational numbers $x \in I$. Here the exceptional set E is the set of all rational numbers in I, which is a countable set. Also, it is easily seen that the zero function F is the indefinite integral of f with base point 0.

The Straddle Lemma

We will discuss primitives and indefinite integrals later in this section. In order to prove the Fundamental Theorem I, we need a lemma that is a direct consequence of the definition of the derivative. The reader should observe that the points u, v "straddle" the point t; that explains the name given the lemma.

4.4 Straddle Lemma. *Let $F : I \to \mathbb{R}$ be differentiable at a point $t \in I$. Given $\varepsilon > 0$ there exists $\delta_\varepsilon(t) > 0$ such that if $u, v \in I$ satisfy*

$$t - \delta_\varepsilon(t) \le u \le t \le v \le t + \delta_\varepsilon(t),$$

then we have

$$|F(v) - F(u) - F'(t)(v - u)| \le \varepsilon(v - u). \tag{4.β}$$

Proof. By definition of the derivative $F'(t)$ at the point $t \in I$, given $\varepsilon > 0$ there exists $\delta_\varepsilon(t) > 0$ such that if $0 < |z - t| \le \delta_\varepsilon(t)$, $z \in I$, then

$$\left| \frac{F(z) - F(t)}{z - t} - F'(t) \right| \le \varepsilon,$$

whence it follows that

$$|F(z) - F(t) - F'(t)(z-t)| \le \varepsilon |z - t|$$

for all $z \in I$ such that $|z - t| \le \delta_\varepsilon(t)$. In particular, if we pick $u \le t$ and $v \ge t$ in this $\delta_\varepsilon(t)$-interval around t and note that $v - t \ge 0$ and $t - u \ge 0$, then on subtracting and adding the term $F(t) - F'(t)t$, we have

$$\begin{aligned} &\big|F(v) - F(u) - F'(t)(v-u)\big| \\ &= \Big|\big[F(v) - F(t) - F'(t)(v-t)\big] - \big[F(u) - F(t) - F'(t)(u-t)\big]\Big| \\ &\le \Big|F(v) - F(t) - F'(t)(v-t)\Big| + \Big|F(u) - F(t) - F'(t)(u-t)\Big| \\ &\le \varepsilon(v-t) + \varepsilon(t-u) = \varepsilon(v-u). \end{aligned}$$

Thus inequality (4.β) is proved. Q.E.D.

Integrating Derivatives

We now establish the first of several versions of the Fundamental Theorem that guarantees that the *derivative* of any function on an interval $I := [a, b]$ is always R*-integrable, without imposing further hypotheses on this derivative. It was in order to obtain this result that Denjoy and Perron developed their theories of integration. (Stronger results will be obtained in Theorems 4.7 and 5.12.)

• **4.5 Fundamental Theorem I.** *If $f : [a, b] \to \mathbb{R}$ has a primitive F on $[a, b]$, then $f \in \mathcal{R}^*([a, b])$ and*

$$\int_a^b f = F(b) - F(a). \tag{4.γ}$$

Proof. Given $\varepsilon > 0$, let the gauge δ_ε be as in the Straddle Lemma, and let $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a δ_ε-fine partition of $[a, b]$. Since x_{i-1} and x_i straddle the tag t_i, it follows from (4.β) that

$$|F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})| \le \varepsilon(x_i - x_{i-1}). \tag{4.δ}$$

We wish to estimate the quantity $F(b) - F(a) - S(f;\dot{\mathcal{P}})$. To do so we make use of the telescoping sum $F(b) - F(a) = \sum_{i=1}^{n}[F(x_i) - F(x_{i-1})]$ to obtain

$$F(b) - F(a) - S(f;\dot{\mathcal{P}}) = \sum_{i=1}^{n} \Big[F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})\Big].$$

We use the Triangle Inequality, to obtain the inequality

$$\Big|F(b) - F(a) - S(f;\dot{\mathcal{P}})\Big| \leq \sum_{i=1}^{n} \Big|F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})\Big|.$$

It follows from (4.δ) that the last term is dominated by the telescoping sum

$$\sum_{i=1}^{n} \varepsilon(x_i - x_{i-1}) = \varepsilon(b - a).$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f \in \mathcal{R}^*([a,b])$ with integral equal to $F(b) - F(a)$. **Q.E.D.**

We can restate the Fundamental Theorem 4.5 as: *If $F : [a,b] \to \mathbb{R}$ is differentiable at every point of $[a,b]$, then $F' \in \mathcal{R}^*([a,b])$ and we have $\int_a^b F' = F(b) - F(a)$.*

It is an exercise to show that if F is a primitive on $[a,b]$ of a function f and $u \in [a,b]$, then $F - F(u)$ is the indefinite integral of f with base point u.

In the next example, we show that the proof of the Fundamental Theorem 4.5 can be modified to permit *one* point of nondifferentiability. These ideas will enable us to strengthen 4.5.

4.6 Example. Let $g(x) := 1/\sqrt{x}$ for $x \in (0,1]$ and $g(0) := 0$, so that g is not bounded on $[0,1]$. If $G(x) := 2\sqrt{x}$ for $x \in [0,1]$, then G is continuous on $[0,1]$ and $G'(x) = g(x)$ for all $x \in (0,1]$, but $G'(0)$ does *not* exist. Hence G is an f-primitive (but not a primitive) of g on $[0,1]$ with exceptional set $E = \{0\}$.

As in the proof of the Straddle Lemma, if $t \in (0,1]$ and $\varepsilon > 0$, let $\delta_\varepsilon(t) > 0$ be such that the inequality (4.β) holds for G. We define $\delta_\varepsilon(0) := \varepsilon^2/4$ so that if $0 \leq v \leq \delta_\varepsilon(0)$, then $G(v) - G(0) = 2\sqrt{v} \leq \varepsilon$. Now let $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ be a tagged partition of I that is δ_ε-fine. If all of the tags belong to $(0,1]$, then the proof of the Fundamental Theorem 4.5 applies without change. However, if the first tag $t_1 = 0$, then the first term

in the Riemann sum $S(g;\dot{\mathcal{P}})$ is equal to $g(0)(x_1 - x_0) = 0$; moreover, we have

$$G(x_1) - G(x_0) - g(0)(x_1 - x_0) = G(x_1) = 2\sqrt{x_1} \le \varepsilon.$$

If we apply the argument given in the proof of the Fundamental Theorem 4.5 to the remaining terms, we obtain

$$\Big|\sum_{i=2}^{n} \big[G(x_i) - G(x_{i-1}) - g(t_i)(x_i - x_{i-1})\big]\Big| \le \varepsilon(x_n - x_1) \le \varepsilon.$$

Therefore, on adding these terms we have

$$\big|G(1) - G(0) - S(g;\dot{\mathcal{P}})\big| \le \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that g is R*-integrable on $[0,1]$ and that $\int_0^1 g = G(1) - G(0) = 2$. We may write this in the form $\int_0^1 (1/\sqrt{x})\,dx = 2$, with the understanding that the integrand is given the value 0 at the point where it is not defined.

Clearly, the argument in Example 4.6 can be extended to any integrand f such that there exists a continuous function F on $[a,b]$ such that $f(x) = F'(x)$ for all but a *finite number* of points. We now show that, in fact, a *countable* number of exceptional points is permitted. This yields a significant improvement over Theorem 4.5.

• **4.7 Fundamental Theorem I*.** *If $f : [a,b] \to \mathbb{R}$ has a c-primitive F on $[a,b]$, then $f \in \mathcal{R}^*([a,b])$ and*

$$\int_a^b f = F(b) - F(a). \tag{4.ε}$$

Proof. Let $E = \{c_k\}_{k=1}^{\infty}$ be the exceptional set for the c-primitive F. Since E is countable, it is a null set. In view of Exercise 3.C, we may suppose that $f(c_k) = 0$.

We now define a gauge on $I := [a,b]$. If $\varepsilon > 0$ and $t \in I - E$, let $\delta_\varepsilon(t)$ be as in the Straddle Lemma. If $t \in E$, then $t = c_k$ for some $k \in \mathbb{N}$; from the continuity of F at c_k, we choose $\delta_\varepsilon(c_k) > 0$ such that $|F(z) - F(c_k)| \le \varepsilon/2^{k+2}$ for all $z \in I$ that satisfy $|z - c_k| \le \delta_\varepsilon(c_k)$. Thus, a gauge δ_ε is defined on I.

Now let $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ be a δ_ε-fine partition of I. If none of the tags belongs to E, then the proof given in Theorem 4.5 applies without change. However, if $c_k \in E$ is the tag of a subinterval $[x_{i-1}, x_i]$, then

$$\begin{aligned} &\big|F(x_i) - F(x_{i-1}) - f(c_k)(x_i - x_{i-1})\big| \\ &\le |F(x_i) - F(c_k)| + |F(c_k) - F(x_{i-1})| + |f(c_k)(x_i - x_{i-1})| \\ &\le \varepsilon/2^{k+2} + \varepsilon/2^{k+2} + 0 = \varepsilon/2^{k+1}. \end{aligned}$$

Now each point of E can be the tag of at most two subintervals in $\dot{\mathcal{P}}$; therefore the sum of the terms with $t_i \in E$ satisfies

$$\sum_{t_i \in E} \left| F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1}) \right| \le \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon.$$

Also, by the Straddle Lemma, the sum of the terms with $t_i \notin E$ satisfies

$$\sum_{t_i \notin E} \left| F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1}) \right| \le \varepsilon \sum_{t_i \notin E} (x_i - x_{i-1}) \le \varepsilon(b-a).$$

Consequently, when $\dot{\mathcal{P}} \ll \delta_\varepsilon$, we have

$$|F(b) - F(a) - S(f;\dot{\mathcal{P}})| \le \varepsilon(1 + b - a).$$

Since $\varepsilon > 0$ is arbitrary, then $f \in \mathcal{R}^*(I)$ with integral $F(b) - F(a)$. Q.E.D.

Theorem 4.7 can be stated: *If F is a c-primitive of f on $[a,b]$, then $f \in \mathcal{R}^*([a,b])$ and F is an indefinite integral of f.*

Differentiating Integrals

We now turn to the part of the Fundamental Theorem that discusses the differentiation of an indefinite integral. Here the situation is not as definitive as in the first part of the Fundamental Theorem. However, it is true that an indefinite integral of an R*-integrable function is continuous on I. (In Exercise 4.K one shows that this is true for a bounded function, and the general case will be proved in Section 5.) Moreover, it will be proved in Section 5 that an indefinite integral F is an a-primitive of f; that is, $F'(x) = f(x)$ almost everywhere on the interval.

For the moment we will focus our attention on the differentiation of an indefinite integral at a specific point $c \in [a,b]$. We will see that (one-sided) continuity of f at a point c implies (one-sided) differentiability of any indefinite integral at c. We recall that saying f has a **right hand limit** A at $c \in [a,b)$ means that given $\varepsilon > 0$, there exists $\eta > 0$ with $\eta \le b - c$ such that if x belongs to $(c, c+\eta)$, then

$$A - \varepsilon \le f(x) \le A + \varepsilon. \tag{4.ζ}$$

We leave it to the reader to formulate the definition of a **left hand limit** of a function, and to state and prove a "left hand version" of the following result.

• **4.8 Fundamental Theorem II.** *Let $f \in \mathcal{R}^*([a,b])$ and let f have a right hand limit A at a point $c \in [a,b)$. Then the indefinite integral*

$$F_u(x) := \int_u^x f$$

has a right hand derivative at c equal to A.

Proof. We will consider the case $u = a$ and denote F_a by F, leaving the general case as an exercise for the reader.

Let h satisfy $0 < h < \eta$. Since f is integrable on the intervals $[a,c]$, $[a,c+h]$, and $[c,c+h]$ (by Corollary 3.8), we have $F(c+h) - F(c) = \int_c^{c+h} f$. Now on the interval $(c, c+h]$ the function f satisfies (4.ζ), so that (see Corollary 3.4 and Exercise 3.D) we have $(A - \varepsilon)h \le \int_c^{c+h} f \le (A+\varepsilon)h$. It follows that

$$\left| \frac{F(c+h) - F(c)}{h} - A \right| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{h \to 0+} \frac{F(c+h) - F(c)}{h} = A;$$

this means that F has a right hand derivative $F'_+(c)$ at c which is equal to A. Q.E.D.

• **4.9 Corollary.** *Let $f \in \mathcal{R}^*([a,b])$ be continuous at $c \in [a,b]$. Then the indefinite integral F_u of f is differentiable at c and $F'_u(c) = f(c)$.*

Proof. Let $c \in (a,b)$. If f is continuous at c, then both the left and right limits of f equal $f(c)$. Consequently, both the left and right derivatives of F_u at c exist and equal $f(c)$. Similarly for the endpoints. Q.E.D.

We recall from Definition 4.1(c) that a function F is said to be an indefinite integral of $f \in \mathcal{R}^*(I)$ in case $F - F_a$ is a constant function. The preceding corollary implies that if f is continuous at a point $c \in I$, then $F'(c) = F'_a(c) = f(c)$. We can reformulate that corollary in the following statement.

• **4.10 Corollary.** *Let f be continuous on $I := [a,b]$. Then any indefinite integral F of f is differentiable on I and $F'(x) = f(x)$ for all $x \in I$.*

Proof. Apply the preceding corollary to each point of I. Q.E.D.

We can restate Corollary 4.10 in the form: *If f is continuous on $[a,b]$, then any indefinite integral of f is a primitive of f on $[a,b]$.*

We now state a much deeper theorem about the differentiation of the integral; its proof is delicate and will be given in Section 5, where we will give a complete characterization of indefinite integrals of functions in $\mathcal{R}^*([a,b])$.

• **4.11 Fundamental Theorem II*.** *If $f \in \mathcal{R}^*(I)$ where $I := [a,b]$, then any indefinite integral F is continuous on I and is an a-primitive of f on $[a,b]$. Thus, there exists a null set $Z \subset I$ such that*

$$F'(x) = f(x) \qquad \text{for all} \quad x \in I - Z. \tag{4.η}$$

Unfortunately, the preceding theorem *does not* assert that an indefinite integral of a function $f \in \mathcal{R}^*([a,b])$ is a c-primitive of f; see Example 4.18(c). The next result is a useful one; it establishes the existence of a c-primitive for a large and important class of functions.

4.12 Theorem. *If $f : [a,b] \to \mathbb{R}$ is a regulated function, then any indefinite integral of f is a c-primitive of f on $[a,b]$.*

Proof. We saw in Theorem 3.20 that if f is a regulated function, then there exists a countable set D such that f is continuous at every point of $I - D$. It follows from 4.11 that the indefinite integral $F_u(x) = \int_u^x f$ is continuous on I, and from Corollary 4.9 that it is differentiable at every point $c \in I - D$ and that $F_u'(c) = f(c)$. Therefore, F_u is a c-primitive of f on I. Q.E.D.

Some Remarks

We now offer two sets of remarks that are intended to clarify the rather subtle distinction between c-primitives and indefinite integrals.

4.13 Remarks. (a) An R*-integrable function always has indefinite integrals, and every indefinite integral of a function in $\mathcal{R}^*(I)$ is an a-primitive.

(b) An R*-integrable function does not always have a c-primitive; see Example 4.18(c).

However, every continuous function has a primitive (Corollary 4.10) and every regulated function has a c-primitive (Theorem 4.12).

(c) If F is a c-primitive of $f : I \to \mathbb{R}$, then $f \in \mathcal{R}^*(I)$ and F is an indefinite integral of f.

(d) If F is an a-primitive of $f \in \mathcal{R}^*(I)$, then F need not be an indefinite integral of f; see Example 4.18(a).

4.14 Remarks. In discussing c-primitives, the exceptional set (where $F'(x) = f(x)$ does not hold) is a *countable set*, while in Theorem 4.11 the

exceptional set Z is a *null set*. Now every countable set is a null set, but the converse is not true, as we will see in Theorem 4.16. It is natural to ask whether this gap between a countable set and a null set of exceptional points can be bridged. There are two parts to this question:

(a) Can we replace Theorem 4.7 by the assertion: If F is a continuous function on $I := [a,b]$ and there exists a null set Z such that $F'(x) = f(x)$ for all $x \in I - Z$, then $f \in \mathcal{R}^*([a,b])$ and $\int_a^b f = F(b) - F(a)$? [That is, if F is an a-primitive of $f : [a,b] \to \mathbb{R}$, then is $f \in \mathcal{R}^*([a,b])$ and $\int_a^b f = F(b) - F(a)$?]

(b) Can we replace Theorem 4.11 by the assertion: If F is an indefinite integral of $f \in \mathcal{R}^*([a,b])$, then there exists a countable set C such that $F'(x) = f(x)$ for all $x \in I - C$? [In other words, if F is an indefinite integral of f, then is F a c-primitive of f ?]

The answer to both of these questions is: No. However, in order to establish this claim, we will construct the Cantor set and the Cantor-Lebesgue singular function, both of which will be useful later.

The Cantor Set

We will construct a subset of $I := [0,1]$ by a process of successively removing open middle thirds. We start with I and obtain the set Γ_1 by removing the open interval $(\frac{1}{3}, \frac{2}{3})$ to obtain

$$\Gamma_1 := [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1].$$

Next we remove the open middle thirds of the two intervals in Γ_1 to obtain

$$\Gamma_2 := [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$

The set Γ_3 is obtained by removing the middle thirds of each of the 2^2 sets in Γ_2; thus Γ_3 consists of 2^3 closed intervals, each having length $1/3^3$. Continuing in this way, we obtain Γ_n as the union of 2^n intervals of the form $[k/3^n, (k+1)/3^n]$. Note the first few stages of this construction, as indicated in Figure 4.1.

4.15 Definition. The **Cantor set** Γ is the intersection of the decreasing sequence of sets $\Gamma_n, n \in \mathbb{N}$, obtained in this way.

Historical note. Recently, K. Hannabuss [*Math. Intelligencer* **18** (1996), no. 3, 28–31] has pointed out that what is universally called the Cantor set appeared in an 1875 paper by H. J. S. Smith, some eight years before Cantor mentioned it. See also Hawkins [Hw-1; p. 38].

Figure 4.1 Construction of the Cantor set.

4.16 Theorem. *The Cantor set Γ is an uncountable null set.*

Proof. In fact, the set Γ_n is the union of 2^n closed intervals, each of which has length $1/3^n$. If $\varepsilon > 0$ is given, let n_0 be such that $(2/3)^{n_0} < \varepsilon$. Since $\Gamma \subset \Gamma_{n_0}$, then Γ is contained in the union of a finite number of closed intervals with total length $< \varepsilon$. It follows from Exercise 2.M that Γ is a null set.

Assume that Γ is a countable set and let $\{x_n : n \in \mathbb{N}\}$ be an enumeration of it. Let I_1 be the closed interval of length $1/3$ in Γ_1 such that $x_1 \notin I_1$. If $n \geq 2$, let I_n be the first interval in $I_{n-1} \cap \Gamma_n$ having length $1/3^n$ such that $x_n \notin I_n$. In this way, we obtain a nested sequence (I_n) of compact intervals; invoking the Nested Intervals Theorem [B-S; p. 46], we obtain a point $z \in \bigcap_{k=1}^{\infty} I_k$ such that $z \in \Gamma$. Since $x_k \notin I_k$, we conclude that $z \neq x_k$ for all $k \in \mathbb{N}$. Therefore the above enumeration does not exhaust Γ, and this set is not countable. Q.E.D.

We will now construct a function $\Lambda : [0,1] \to \mathbb{R}$ that is often useful in constructing examples and counterexamples. First let Λ_1 be the piecewise linear function with $\Lambda_1(0) := 0, \Lambda_1(x) := 1/2$ for $x \in [\frac{1}{3}, \frac{2}{3}]$ and $\Lambda_1(1) := 1$. Next, let Λ_2 be the piecewise linear function with $\Lambda_2(0) := 0$, taking the values $1/4, 1/2$, and $3/4$ on the intervals

$$[\tfrac{1}{9}, \tfrac{2}{9}], \qquad [\tfrac{1}{3}, \tfrac{2}{3}], \qquad [\tfrac{7}{9}, \tfrac{8}{9}],$$

respectively, and $\Lambda_2(1) := 1$. In general, Λ_n is the piecewise linear function with $\Lambda_n(0) := 0$, taking the values $1/2^n, 2/2^n, \cdots, (2^n - 1)/2^n$ on the closed intervals corresponding to the intervals that were removed to construct Γ_n, and with $\Lambda_n(1) := 1$. By definition, each Λ_n is an increasing (i.e., = non-decreasing) continuous function. We claim that this sequence of functions converges on $[0,1]$ to a limit function, which we call the **Cantor-Lebesgue singular function** and denote by Λ. (Sometimes the graph of Λ is called "the Devil's staircase".)

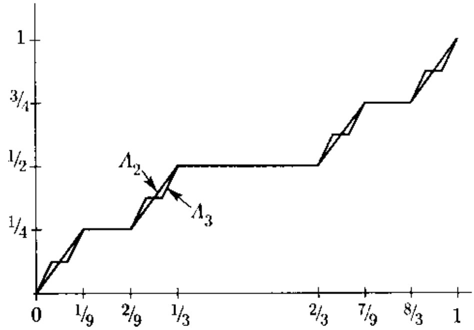


Figure 4.2 Construction of the Cantor-Lebesgue singular function.

4.17 Theorem. *The Cantor-Lebesgue singular function* $\Lambda : [0,1] \to \mathbb{R}$ *is continuous and increasing on* $[0,1]$ *and its derivative* $\Lambda'(x) = 0$ *for all points* $x \in [0,1] - \Gamma$.

Proof. Since the graphs of Λ_n and Λ_{n+1} either coincide or lie in the same horizontal strips with thickness $1/2^n$, we have $|\Lambda_n(x) - \Lambda_{n+1}(x)| \le 1/2^n$ for all $n \in \mathbb{N}$, $x \in [0,1]$. Therefore, if $m > n$ then

$$|\Lambda_n(x) - \Lambda_m(x)| \le \sum_{k=n}^{m-1} |\Lambda_k(x) - \Lambda_{k+1}(x)| \le \sum_{k=n}^{m-1} \frac{1}{2^k} \le \frac{1}{2^{n-1}}.$$

This implies that the limit

$$\Lambda(x) := \lim_{n\to\infty} \Lambda_n(x)$$

exists and that $|\Lambda_n(x) - \Lambda(x)| \le 1/2^{n-1}$ for all $x \in [0,1]$. Therefore the sequence (Λ_n) converges uniformly on $[0,1]$ to Λ. Since each Λ_n is continuous, it follows (see [B-S; p. 234]) that Λ is continuous on $[0,1]$. Since each Λ_n is increasing, we also conclude that Λ is increasing on $[0,1]$.

If $x \in [0,1] - \Gamma$, then there is an open interval containing x on which all of the functions Λ_n are constant (and equal) for sufficiently large n. Therefore Λ is constant on this open interval and $\Lambda'(x) = 0$. Q.E.D.

4.18 Examples. (a) We return to the question raised in 4.14(a). We have seen that the Cantor-Lebesgue singular function Λ is a continuous function on $[0,1]$ and that $\Lambda'(x) = 0$ for all $x \in [0,1] - \Gamma$. Since Γ is a null set, then Λ is an a-primitive of the 0-function on $[0,1]$. However, $\int_0^1 \Lambda' = 0 \neq 1 = \Lambda(1) - \Lambda(0)$, showing that the answer to 4.14(a) is: No.

(b) We return to the question raised in 4.14(b). We let $\varphi(x) := 1$ for $x \in \Gamma$ and $\varphi(x) := 0$ for $x \in [0,1]-\Gamma$. Since the Cantor set Γ is a null set, the function φ is a null function. and Example 2.6(a) implies that $\varphi \in \mathcal{R}^*([0,x])$ and

$$\Phi(x) := \int_0^x \varphi = 0$$

for all $x \in [0,1]$. Consequently, $\Phi'(x) = 0$ for all $x \in [0,1]$. However, $\varphi(x) = 1$ for $x \in \Gamma$, so that $\Phi'(x) \neq \varphi(x)$ on an uncountable null set.

(c) The example in (b) shows that, while an integrable function does have an a-primitive, it does *not* always have a c-primitive.

Indeed, if Ψ were a c-primitive of φ, then $\Psi - \Psi(0) = \Phi$, so $\Psi'(x) = 0$ for all $x \in [0,1]$. But the hypothesis that Ψ is a c-primitive of φ implies that $\varphi(x) \neq 0$ only on a *countable* set, contrary to the fact that the Cantor set Γ is *not countable*, as was seen in Theorem 4.16.

A Characterization of Indefinite Integrals

The remarks just made show that there is some delicacy in the identification of indefinite integrals of functions in $\mathcal{R}^*(I)$. However, in Section 5 we will give a *complete characterization* for indefinite integrals of (generalized Riemann) integrable functions. We will see that a function F is the indefinite integral of a function in $\mathcal{R}^*(I)$ if and only if F is differentiable a.e., and that on the null set Z where F is not differentiable, the function F satisfies an additional condition which is automatically satisfied if Z is a countable set.

Integration by Parts

We now give a weak (but often useful) form of the Integration by Parts formula. Stronger forms of this result will be given in Section 12.

4.19 Theorem. *Let F and G be differentiable on $I := [a,b]$. Then $F'G$ belongs to $\mathcal{R}^*(I)$ if and only if FG' belongs to $\mathcal{R}^*(I)$. In this case, we have*

$$\int_a^b F'G = FG\Big|_a^b - \int_a^b FG'. \tag{4.θ}$$

Proof. The Product Theorem from calculus asserts that $(FG)'$ exists on I and that

$$(FG)' = F'G + FG'. \tag{4.ι}$$

The Fundamental Theorem 4.5 implies that $(FG)' \in \mathcal{R}^*(I)$, and it follows from equation (4.ι) that $F'G \in \mathcal{R}^*(I)$ if and only if $FG' \in \mathcal{R}^*(I)$. Formula (4.θ) now follows immediately. Q.E.D.

Some Examples

We will conclude this section by giving three examples of functions that have a c-primitive, but not an f-primitive.

4.20 Examples. **(a)** Let f be the Dirichlet function defined on $[0,1]$ by $f(x) := 0$ if x is irrational, and $f(x) := 1$ if $x \in \mathbb{Q} \cap [0,1]$.

It was seen in Example 2.3(a) that $f \in \mathcal{R}^*([0,1])$. It is an exercise to show that the zero function $F(x) := 0$ for all $x \in [0,1]$ is a c-primitive of f.

(b) As in Example 2.7, let $c_k := 1 - 1/2^k$ for $k = 0, 1, \cdots$, so that $c_0 = 0$, $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{4}, \cdots$. We let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 1 & \text{if} \quad x \in [c_{2k}, c_{2k+1}), k = 0, 1, \cdots, \\ 0 & \text{if} \quad x \in [c_{2k+1}, c_{2k+2}), k = 0, 1, \cdots, \\ 0 & \text{if} \quad x = 1. \end{cases}$$

See Figure 4.3 for a graph of f.

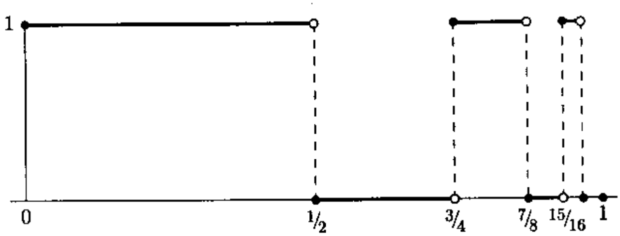


Figure 4.3 Graph of f.

We will show that $f \in \mathcal{R}^*([0,1])$ by exhibiting a c-primitive of f. In fact, we define $F : [0,1] \to \mathbb{R}$ by

$$F(x) := \begin{cases} 0 & \text{if} \quad x = 0, \\ x - c_{2k} + F(c_{2k}) & \text{if} \quad x \in (c_{2k}, c_{2k+1}], k = 0, 1, \cdots, \\ F(c_{2k+1}) & \text{if} \quad x \in (c_{2k+1}, c_{2k+2}], k = 0, 1, \cdots, \\ \frac{2}{3} & \text{if} \quad x = 1. \end{cases}$$

Thus we have $F(x) = x$ for $x \in [0, \frac{1}{2}], F(x) = \frac{1}{2}$ for $x \in [\frac{1}{2}, \frac{3}{4}], F(x) = x - \frac{3}{4} + \frac{1}{2} = x - \frac{1}{4}$ for $x \in [\frac{3}{4}, \frac{7}{8}], F(x) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$ for $x \in [\frac{7}{8}, \frac{15}{16}]$, etc. See Figure 4.4 for a graph of F.

An elementary induction argument shows that

$$F(c_{2k+1}) = F(c_{2k}) + \frac{1}{2^{2k+1}} = F(c_{2k-1}) + \frac{1}{2^{2k+1}},$$

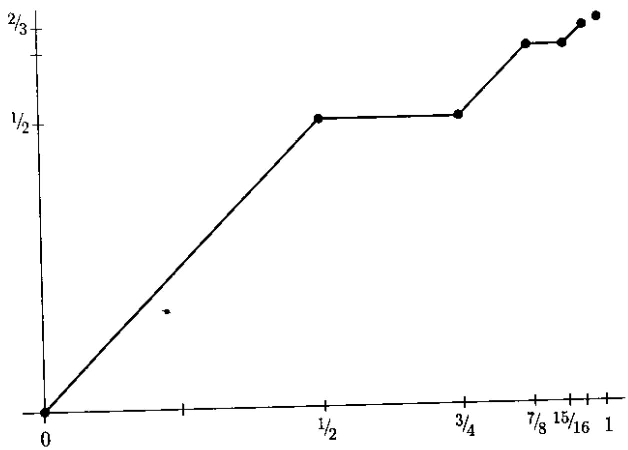


Figure 4.4 Graph of F.

so that we have

$$F(c_{2k+1}) = \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2k+1}} = \frac{2}{3}\left(1 - \frac{1}{4^k}\right).$$

We claim that F is continuous on $[0, 1)$. Indeed, this is obvious at every point $x \neq c_k, k \in \mathbb{N}$. Further, by considering the cases c_{2k} and c_{2k+1} separately, one determines that both the left and right hand limits exist at these points and equal $F(c_{2k})$ and $F(c_{2k+1})$, respectively. To see that F is continuous from the left at $x = 1$, one can show that F is increasing on $[0, 1]$ and that $\lim F(c_{2k+1}) = 2/3 = F(1)$.

From the definition of F it is evident that the derivative $F'(x) = 1$ when $x \in (c_{2k}, c_{2k+1})$, and that $F'(x) = 0$ when $x \in (c_{2k+1}, c_{2k+2})$. It is also clear that F does not have a two-sided derivative at any of the points $c_k, k = 0, 1, \cdots$. Therefore, F is a c-primitive of f, but it is not an f-primitive. It follows from the Fundamental Theorem 4.7 that $f \in \mathcal{R}^*([0, 1])$ and that

$$\int_0^1 f = F(1) - F(0) = \tfrac{2}{3}.$$

(c) We now consider the function $\Psi(x) := x|\cos(\pi/x)|$ for $x \in (0, 1]$ and $\Psi(0) := 0$. It is clear that Ψ is continuous on $[0, 1]$. Moreover, $\Psi(a) = 0$ if and only if $a \in E := \{0\} \cup \{2/(2k + 1) : k \in \mathbb{N}\}$. For a graph of Ψ, see Figure 4.5 on the next page.

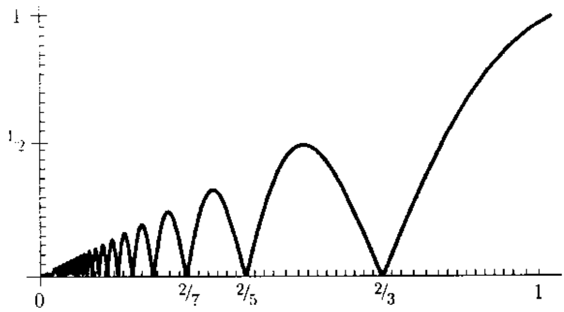


Figure 4.5 Graph of Ψ.

Direct calculation shows that the derivative $\Psi'(0)$ does not exist, since $\Psi(1/k) = 1/k$, while $\Psi(2/(2k+1)) = 0$. To investigate the existence of $\Psi'(x)$ elsewhere on $(0,1]$, we recall that the absolute value function $x \mapsto |x|$ has a derivative equal to $\operatorname{sgn} x$ when $x \neq 0$. Therefore the Chain Rule implies that the function $x \mapsto |\cos(\pi/x)|$ has a derivative when x is not a zero of $\cos(\pi/x)$; that is, when $x \neq 2/(2k+1)$. The Product Rule for differentiation then shows that Ψ has a derivative for $x \notin E$. Moreover, it can be shown that Ψ does not have a derivative when $x \in E$. (For a related discussion, see [B-S; p. 163].)

Thus, if we let $\psi(x) := \Psi'(x)$ for $x \notin E$ and $\psi(x) := 0$ for $x \in E$, then Ψ is a c-primitive of ψ, whence it follows that $\psi \in \mathcal{R}^*([0,1])$.

Exercises

You may make free use of differentiation and other formulas learned in calculus.

4.A (a) Show that the signum function does not have a primitive on any compact interval I containing 0. [Hint: Use the Darboux Intermediate Value Theorem [B-S; p. 174].]

(b) Show that $H(x) := |x|$ is an f-primitive of sgn on I with exceptional set $\{0\}$.

(c) Show that H is the indefinite integral of sgn with base point 0.

(d) Find the indefinite integrals of sgn with base points -1 and 2.

4.B (a) Show that the restriction of $\ln(x + \sqrt{x^2 + a^2})$ is a primitive of the function $(x^2 + a^2)^{-1/2}$ on any compact interval $I \subset \mathbb{R}$ when $a \neq 0$.

(b) Show that $F(x) := x \ln x - x$ for $x \in (0, \infty)$ and $F(0) := 0$ is an f-primitive of $f(x) := \ln x$ for $x \in (0, \infty)$ and $f(0) := 0$ on any interval $I \subset [0, \infty)$.

(c) Evaluate $\int_0^1 \ln x \, dx$ and $\int_0^2 \ln x \, dx$.

4.C Let $g(x) := x^{-1/2}$ for $x \in (0, 1]$ and $g(0) := 0$.

(a) Show that $G(x) := 2x^{1/2}$ is an f-primitive of g on $[0, 1]$.

(b) Show that g^2 does not have a c-primitive on $[0, 1]$.

4.D If f and g have primitives [resp., c-primitives] on $[a, b]$, show that cf and $f + g$ have primitives [resp., c-primitives] on $[a, b]$.

4.E (a) Show that the function Arctan ($= \tan^{-1}$), inverse to the restriction of tan to the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, is a primitive of $t(x) := (1 + x^2)^{-1}$ on any compact interval $I \subset \mathbb{R}$.

(b) Show that the function Arcsin ($= \sin^{-1}$), inverse to the restriction of sin to $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, is an f-primitive of the function $s(x) := (1 - x^2)^{-1/2}$ for $x \in (-1, 1)$ and $s(-1) := s(1) := 0$.

(c) Show that $\int_{-1}^{1} (1 - x^2)^{-1/2} \, dx = \pi$.

4.F Let f be Dirichlet's function on $[0, 1]$ (see Example 2.3(a)).

(a) Show that $F(x) := 0$ for $x \in [0, 1]$ is a c-primitive of f on $[0, 1]$. Also show that F is the indefinite integral of f with base point 0.

(b) Show that $\Psi(x) := 1$ for $x \in [0, 1]$ is a c-primitive of f on $[0, 1]$. While Ψ is an indefinite integral of f, it has no base point.

4.G Let h be Thomae's function defined on $[0, 1]$ by $h(x) := 0$ if x is irrational in $[0, 1]$ or $x = 0$, and $h(x) := 1/q$ if $x = p/q$ where $p, q \in \mathbb{N}$ have no common integer factors except 1.

(a) Show that h is continuous at every irrational point and is integrable on $[0, 1]$.

(b) Show that h does not have a primitive on $[0, 1]$, but it does have a c-primitive.

(c) Show that every indefinite integral H of h is differentiable at every point of $[0, 1]$, but that $H'(x) \neq h(x)$ for all nonzero rational numbers.

4.H (a) Let $F(x) := x^2 \cos(\pi/x)$ for $x \in (0,1]$ and $F(0) := 0$. Show that F' exists at every point of $[0,1]$ and is bounded. Is $f := F'$ integrable on $[0,1]$?

(b) Let $G(x) := x^2 \cos(\pi/x^2)$ for $x \in (0,1]$ and $G(0) := 0$. Show that G' exists at every point of $[0,1]$ and is unbounded but integrable.

4.I Let G be the function in Exercise 4.H(b). Show that inequality (4.β) does not hold for $t = 0$ for certain points u, v near $t = 0$.

4.J Show that a regulated function has a primitive on $[a,b]$ if and only if it is continuous on $[a,b]$.

4.K Suppose f is integrable and $|f(x)| \leq K$ on $[a,b]$. If F is an indefinite integral of f, show that $|F(x) - F(y)| \leq K|x-y|$ for all $x, y \in [a,b]$.

4.L Suppose that $f \in \mathcal{R}^*([a,b])$ and is bounded on some neighborhood of a point c. Show that any indefinite integral F of f is continuous at c. However, F may not be differentiable at c.

4.M Show that any indefinite integral of the signum function on $[-1,1]$ has both one-sided derivatives at $x = 0$, but not a two-sided derivative there.

4.N Give an example of an integrable function that is not continuous at a point c, but has an indefinite integral that is differentiable at c.

4.O Show that two indefinite integrals of an integrable function differ by a constant function. For example, if F_u, F_v are indefinite integrals with base points u, v, then $F_u - F_v = F_u(v) = -F_v(u)$.

4.P Let $f \in \mathcal{R}^*([a,b])$ and let F be a c-primitive of f. Exhibit the indefinite integral of f with base point $u \in [a,b]$ in terms of F.

4.Q If F, G are continuous on $I := [a,b]$ and there exists a countable set $C \subset I$ such that $F'(x) = G'(x)$ for all $x \in I - C$, show that there exists a constant function K such that $F - G = K$. Thus, two c-primitives of $f \in \mathcal{R}^*(I)$ differ by a constant.

4.R Let $\kappa : [0,1] \to \mathbb{R}$ be the function in Example 2.8(a) and let $K(x) := \int_0^x \kappa$ be its indefinite integral with base point 0. Show that K is a piecewise linear function on $[0,b]$ for $0 < b < 1$ and such that

$K(0) = 0$ and $K(c_n) = \sum_{k=1}^{n}(-1)^{k+1}/k$ for $n \in \mathbb{N}$. Show that $K(1) = \sum_{k=1}^{\infty}(-1)^{k+1}/k$ $(= \ln 2)$.

4.S Let $f(x) := \lfloor x \rfloor$ (the greatest integer function) for $x \in \mathbb{R}$. Show that f has a c-primitive on $\mathbb{R}$ given by $F(x) := nx - \frac{1}{2}n(n+1)$ for $x \in [n, n+1)$. Show that F has right and left hand derivatives at the point $n \in \mathbb{Z}$, but not a (two-sided) derivative there.

4.T If f is continuous on $[a, b]$, define $G(x) := \int_x^b f$. Show that G is differentiable on $[a, b]$ and calculate its derivative.

4.U (a) Show that if (x_n) is a convergent sequence in Γ, then its limit also belongs to Γ.

(b) If $x \in \Gamma$, then there exist sequences (y_n) in Γ and (z_n) in $[0, 1] - \Gamma$ that converge to x.

(c) Show that Γ does not contain any nonempty open interval.

(d) Show that if $x \in \Gamma$, then there exists a unique sequence $(a_n)_{n=1}^{\infty}$ with $a_n \in \{0, 2\}$, such that $x = \sum_{n=1}^{\infty} 3^{-n} a_n$. (This is called *the ternary expansion of* x.)

4.V (a) Show that each nonhorizontal line in the graph of Λ_n has slope $(3/2)^n$.

(b) If $x \in \Gamma$ has the ternary expansion given in 4.U(d), show that $\Lambda(x)$ has the binary expansion $\sum_{n=1}^{\infty} 2^{-n}(a_n/2)$.

4.W Suppose that $f \in \mathcal{R}^*(I)$ has a c-primitive on I. Show that there exists $g : I \to \mathbb{R}$ with $f(x) = g(x)$ a.e., but such that g does not have a c-primitive.

4.X Show that Theorems 4.7 and 4.11 remain true if f has values in $\mathbb{C}$ (the complex field).

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Courtesy of the Archives of the
Academy of Sciences, Paris, France

The Saks-Henstock Lemma

The Saks-Henstock Lemma, sometimes called the "Henstock Lemma", is of fundamental importance in proving the deeper properties of the (generalized Riemann) integral. Henstock [H-5; p. 197] attributes this result to Saks [S-1; p. 214], but its use for the general integral is certainly due to Henstock. Our first use of this lemma is to prove the continuity of the indefinite integral. We will then establish the differentiability almost everywhere of the indefinite integral, announced in Theorem 4.11; however, in order to prove this result we also require the important Vitali Covering Theorem. Next we give a characterization of null functions. Finally, we present a necessary and sufficient condition for a function to be an indefinite integral of a function in $\mathcal{R}^*([a,b])$.

This section contains some rather subtle arguments. The reader may want to look over the results and defer a detailed reading until a later time.

The Saks-Henstock Lemma

The definition of the integral of a function f on $I := [a,b]$ requires that, given $\varepsilon > 0$ there exists a gauge δ_ε on I such that if $\dot{\mathcal{P}}$ is any δ_ε-fine partition of I, then the Riemann sum $S(f;\dot{\mathcal{P}})$ satisfies the inequality $|S(f;\dot{\mathcal{P}}) - \int_a^b f| \leq \varepsilon$. The Saks-Henstock Lemma asserts that the same degree of approximation is valid for the difference between *any* subset of terms from this Riemann sum and the sum of the integrals of f over the corresponding subintervals. This fact may not seem so surprising when the subintervals in the subset of $\dot{\mathcal{P}}$ consist of abutting intervals. However, it is not at all obvious that the

result remains true for an arbitrary collection of subintervals. Even more surprising is that we can replace the absolute value of the sum of these differences by the sum of the absolute values and still have essentially the same degree of approximation. This, despite the fact that the existence of the integral may depend on the subtraction of terms in the Riemann sums.

5.1 Definition. Let $I := [a, b]$ be a nondegenerate compact interval.

(a) A **subpartition** of I is a collection $\{J_j\}_{j=1}^{s}$ of nonoverlapping closed intervals in I.

(b) A **tagged subpartition** of I is a collection $\dot{\mathcal{P}}_0 := \{(J_j, t_j)\}_{j=1}^{s}$ of ordered pairs, consisting of intervals $\{J_j\}_{j=1}^{s}$ that form a subpartition of I, and tags $t_j \in J_j$ for $j = 1, \cdots, s$.

(c) If δ is a gauge on I, we say that the tagged subpartition $\dot{\mathcal{P}}_0$ is **δ-fine** if $J_j \subseteq [t_j - \delta(t_j), t_j + \delta(t_j)]$ for $j = 1, \cdots, s$.

(d) If δ is a gauge on a subset $E \subseteq I$, we say that the tagged subpartition $\dot{\mathcal{P}}_0$ is **(δ, E)-fine** if all tags $t_j \in E$ and $J_j \subseteq [t_j - \delta(t_j), t_j + \delta(t_j)]$ for $j = 1, \cdots, s$.

5.2 Remarks. (a) Any subset of a partition of I is a subpartition of I. Conversely, it is an exercise to show that if Π_0 is a subpartition of I, then there exists a partition of I of which Π_0 is a subset.

(b) If $\dot{\mathcal{P}}_0$ is a subpartition of I that is δ-fine, then it is an exercise to show that there exists a δ-fine partition of I of which $\dot{\mathcal{P}}_0$ is a subset.

(c) Definition 5.1(d) only requires that δ be defined on E, but one can set $\delta(x) := 1$ for $x \in I - E$ and obtain a gauge on all of I.

If $\dot{\mathcal{P}}_0 = \{(J_j, t_j) : j = 1, \cdots, s\}$ is a tagged subpartition of I, then we let $U(\dot{\mathcal{P}}_0) := \bigcup_{j=1}^{s} J_j$. If $f \in \mathcal{R}^*(I)$, we define

$$S(f; \dot{\mathcal{P}}_0) := \sum_{j=1}^{s} f(t_j) l(J_j) \qquad \text{and} \qquad \int_{U(\dot{\mathcal{P}}_0)} f := \sum_{j=1}^{s} \int_{J_j} f,$$

where $l(J)$ denotes the length of the interval J.

• **5.3 Saks-Henstock Lemma.** *Let $f \in \mathcal{R}^*([a, b])$ and for $\varepsilon > 0$ let δ_ε be a gauge on I such that if $\dot{\mathcal{P}} \ll \delta_\varepsilon$, then*

$$(5.\alpha) \qquad \left|S(f; \dot{\mathcal{P}}) - \int_I f\right| \leq \varepsilon.$$

If $\dot{\mathcal{P}}_0 = \{(J_j, t_j) : j = 1, \cdots, s\}$ is any δ_ε-fine subpartition of I, then

$$(5.\beta) \qquad \left|\sum_{j=1}^{s}\left\{f(t_j)l(J_j) - \int_{J_j} f\right\}\right| = \left|S(f;\dot{\mathcal{P}}_0) - \int_{U(P_0)} f\right| \le \varepsilon.$$

Proof. Let $K_1, \cdots, K_m$ denote closed subintervals in I such that $\{J_j\} \cup \{K_k\}$ form a partition of I. Our basic strategy is to use the fact that f is integrable on each of the intervals $K_1, \cdots, K_m$ and obtain partitions of these intervals that are so fine that the sum of their contributions is arbitrarily small.

Now let $\alpha > 0$ be arbitrary. Since (by Corollary 3.8) the restriction of f to each subinterval K_k $(k = 1, \cdots, m)$ is integrable, there exists a gauge $\delta_{\alpha,k}$ on K_k such that if $\dot{\mathcal{Q}}_k$ is a $\delta_{\alpha,k}$-fine partition of K_k, then

$$(5.\gamma) \qquad \left|S(f;\dot{\mathcal{Q}}_k) - \int_{K_k} f\right| \le \alpha/m.$$

Clearly we may assume that $\delta_{\alpha,k}(x) \le \delta_\varepsilon(x)$ for all $x \in K_k$. Now let $\dot{\mathcal{P}}^*$ denote the tagged partition $\dot{\mathcal{P}}^* := \dot{\mathcal{P}}_0 \cup \dot{\mathcal{Q}}_1 \cup \cdots \cup \dot{\mathcal{Q}}_m$ of I. Evidently $\dot{\mathcal{P}}^*$ is δ_ε-fine, so that inequality $(5.\alpha)$ holds for $\dot{\mathcal{P}}^*$. Further, it is clear that

$$\begin{aligned} S(f;\dot{\mathcal{P}}^*) &= S(f;\dot{\mathcal{P}}_0) + S(f;\dot{\mathcal{Q}}_1) + \cdots + S(f;\dot{\mathcal{Q}}_m), \\ \int_I f &= \int_{U(\dot{\mathcal{P}}_0)} f + \int_{K_1} f + \cdots + \int_{K_m} f. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} &\left|S(f;\dot{\mathcal{P}}_0) - \int_{U(\dot{\mathcal{P}}_0)} f\right| \\ &\qquad = \left|\left\{S(f;\dot{\mathcal{P}}^*) - \sum_{k=1}^{m} S(f;\dot{\mathcal{Q}}_k)\right\} - \left\{\int_I f - \sum_{k=1}^{m}\int_{K_k} f\right\}\right| \\ &\qquad\qquad \le \left|S(f;\dot{\mathcal{P}}^*) - \int_I f\right| + \sum_{k=1}^{m}\left|S(f;\dot{\mathcal{Q}}_k) - \int_{K_k} f\right|. \end{aligned}$$

If we use inequalities $(5.\alpha)$ and $(5.\gamma)$, this last sum is dominated by

$$\varepsilon + m(\alpha/m) = \varepsilon + \alpha.$$

Since $\alpha > 0$ is arbitrary, then $|S(f;\dot{\mathcal{P}}_0) - \int_{U(\dot{\mathcal{P}}_0)} f| \le \varepsilon$, as claimed. Q.E.D.

We now show that we can interchange the absolute values and the sum in $(5.\beta)$ if we double the possible error.

• **5.4 Corollary.** *With the hypotheses of Lemma* 5.3, *we have*

$$(5.\delta) \qquad \sum_{j=1}^{s} \left| f(t_j)l(J_j) - \int_{J_j} f \right| \leq 2\varepsilon.$$

Proof. Let $\dot{\mathcal{P}}_0^+$ be those pairs in $\dot{\mathcal{P}}_0$ for which $f(t_j)l(J_j) - \int_{J_j} f \geq 0$, and let $\dot{\mathcal{P}}_0^-$ be those pairs for which these terms are < 0. Now apply the Saks-Henstock Lemma to both $\dot{\mathcal{P}}_0^+$ and $\dot{\mathcal{P}}_0^-$. We obtain the inequalities

$$\sum_{\dot{\mathcal{P}}_0^+} \left| f(t_j)l(J_j) - \int_{J_j} f \right| = \sum_{\dot{\mathcal{P}}_0^+} \left\{ f(t_j)l(J_j) - \int_{J_j} f \right\} \leq \varepsilon,$$
$$\sum_{\dot{\mathcal{P}}_0^-} \left| f(t_j)l(J_j) - \int_{J_j} f \right| = -\sum_{\dot{\mathcal{P}}_0^-} \left\{ f(t_j)l(J_j) - \int_{J_j} f \right\} \leq \varepsilon.$$

If we add these two terms, then we obtain $(5.\delta)$. Q.E.D.

The following result will be used in Section 7.

• **5.5 Corollary.** *With the hypotheses of the Saks-Henstock Lemma* 5.3, *we have*

$$(5.\varepsilon) \qquad \left| \sum_{j=1}^{s} |f(t_j)|l(J_j) - \sum_{j=1}^{s} \left| \int_{J_j} f \right| \right| \leq 2\varepsilon.$$

Proof. One consequence of the Triangle Inequality is that

$$-|A - B| \leq |A| - |B| \leq |A - B|.$$

If we take $A := f(t_j)l(J_j)$ and $B := \int_{J_j} f$, sum from $j = 1, \cdots, s$, and apply inequality $(5.\delta)$, then we obtain $(5.\varepsilon)$. Q.E.D.

Continuity of the Indefinite Integrals

We now give an important application of the Saks-Henstock Lemma: we will establish the continuity of the indefinite integrals of an integrable function (stated without proof in 4.11). For simplicity we consider here only the indefinite integral with the left endpoint a as base point, since any other indefinite integral differs from this one by a constant function.

• **5.6 Theorem.** *If f belongs to $\mathcal{R}^*([a,b])$, then the indefinite integral $F(x) := \int_a^x f$ is continuous on $[a,b]$.*

Proof. Let $c \in [a, b)$; we will show that F is continuous from the right at c. If $\varepsilon > 0$, let the gauge δ_ε on $I := [a, b]$ be as in the hypothesis of the Saks-Henstock Lemma 5.3. We now define a gauge by

$$\delta'_\varepsilon(t) := \begin{cases} \min\{\delta_\varepsilon(t), \frac{1}{2}|t-c|\} & \text{if} \quad t \in I,\ t \neq c, \\ \min\{\delta_\varepsilon(c), \varepsilon/(|f(c)|+1)\} & \text{if} \quad t = c. \end{cases}$$

Now let $0 < h < \delta'_\varepsilon(c)$ and let $\dot{\mathcal{P}}_0$ be the δ'_ε-fine subpartition consisting of the single pair $([c, c+h], c)$. If we apply the Saks-Henstock Lemma to $\dot{\mathcal{P}}_0$, we infer that

$$\left| f(c)h - \int_c^{c+h} f \right| \leq \varepsilon.$$

Hence it follows from the fact that $h \leq \varepsilon/(|f(c)|+1)$ that

$$|F(c+h) - F(c)| = \left| \int_c^{c+h} f \right| \ \leq \ |f(c)|h + \varepsilon \ < \ \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then F is continuous from the right at c. In the same way we show that F is continuous from the left at any point in $(a, b]$. Q.E.D.

The Vitali Covering Theorem

In order to give a proof of the differentiation part of the Fundamental Theorem 4.11 we require a version of the Vitali Covering Theorem, which we do *not* assume to be known to the reader. Therefore a slight detour will be needed.

5.7 Definition. Let $E \subseteq [a, b]$ and let $\mathcal{F}$ be a collection of nondegenerate closed subintervals in $[a-1, b+1]$. We say that $\mathcal{F}$ is a **Vitali covering for** E if for every $x \in E$ and every $s > 0$ there exists an interval $J \in \mathcal{F}$ such that $x \in J$ and $0 < l(J) < s$.

It is clear that if $\mathcal{F}$ is a Vitali covering for E, then every point $x \in E$ belongs to infinitely many intervals in $\mathcal{F}$. As an example of a countable Vitali covering for the interval $I := [0, 1]$, consider the collection of all closed balls $B[r; 1/n]$, where $r \in I \cap \mathbb{Q}$ and $n \in \mathbb{N}$.

5.8 Vitali Covering Theorem. *Let $E \subseteq [a, b]$ and let $\mathcal{F}$ be a Vitali covering for E. Then, given $\varepsilon > 0$ there exist disjoint intervals $I_1, \cdots, I_p$ from $\mathcal{F}$ and a countable collection of closed intervals $\{J_i : i = p+1, \cdots\}$ in $\mathbb{R}$ with*

$$(5.\zeta) \qquad E - \bigcup_{i=1}^{p} I_i \subseteq \bigcup_{i=p+1}^{\infty} J_i \qquad \text{and} \qquad \sum_{i=p+1}^{\infty} l(J_i) \leq \varepsilon.$$

Therefore, it follows that

$$(5.\eta) \qquad E \subseteq \bigcup_{i=1}^{p} I_i \cup \bigcup_{i=p+1}^{\infty} J_i.$$

Proof. We choose $I_1 \in \mathcal{F}$ arbitrarily and suppose that disjoint intervals $I_1, \cdots, I_r$ from $\mathcal{F}$ have already been chosen. If $E \subseteq \bigcup_{i=1}^{r} I_i$, we can take $J_i = \emptyset$ for $i \geq r+1$ and the proof is complete. Otherwise, we let $\mathcal{F}_r$ be the collection of all intervals $I \in \mathcal{F}$ that contain points of E and are disjoint from each of the intervals $I_1, \cdots, I_r$. We let λ_r ($\leq b-a+2$) be the supremum of the lengths of all such intervals $I \in \mathcal{F}_r$, and we choose $I_{r+1} \in \mathcal{F}_r$ such that $l(I_{r+1}) > \frac{1}{2}\lambda_r$. This construction gives an infinite sequence of intervals (I_i), unless E is contained in the union of some finite number of these closed intervals.

Suppose that we obtain an infinite sequence (I_i). Since the intervals I_i are pairwise disjoint and are contained in the interval $[a-1, b+1]$, we must have $\sum_{i=1}^{\infty} l(I_i) \leq b-a+2$ (see Exercise 5.J). Therefore, given $\varepsilon > 0$, there exists $p \in \mathbb{N}$ such that we have $\sum_{i=p+1}^{\infty} l(I_i) \leq \varepsilon/5$. Now let $D_p := E - \bigcup_{i=1}^{p} I_i$ and let $x \in D_p$ be arbitrary. Since $\mathcal{F}$ is a Vitali covering for E there exists an interval $I_x \in \mathcal{F}$ such that $x \in I_x$ and $I_x \cap I_i = \emptyset$ for all $i = 1, \cdots, p$; therefore $I_x \in \mathcal{F}_p$. We claim that the interval I_x must intersect at least one interval I_n with $n > p$. For, if $I_x \cap I_i = \emptyset$ for $i = 1, \cdots, n$, then $I_x \in \mathcal{F}_n$ and we have $0 < l(I_x) \leq \lambda_n$. But $0 \leq \lambda_n < 2l(I_{n+1})$, so that $\lim_n \lambda_n = 0$; therefore $0 < l(I_x) \leq \lambda_n$ cannot hold for all $n \in \mathbb{N}$. Hence let $n(x) \in \mathbb{N}$ be the smallest integer n such that $I_x \cap I_n \neq \emptyset$, so that $n(x) > p$ and, since $I_x \in \mathcal{F}_{n(x)-1}$, we have $l(I_x) \leq \lambda_{n(x)-1} < 2l(I_{n(x)})$. Since I_x contains the point $x \in D_p$ and has a point in $I_{n(x)}$, the distance from x to the midpoint $x_{n(x)}$ of $I_{n(x)}$ is $\leq l(I_x) + \frac{1}{2}l(I_{n(x)}) < \frac{5}{2}l(I_{n(x)})$. Therefore, x belongs to the interval $J_{n(x)}$ with the same midpoint $x_{n(x)}$ as $I_{n(x)}$ and 5 times its length. For $i \geq p+1$, let J_i be formed from I_i in this way. Since $x \in D_p$ is arbitrary, the argument just given implies that

$$(5.\theta) \qquad E - \bigcup_{i=1}^{p} I_i = D_p \subseteq \bigcup_{i=p+1}^{\infty} J_i.$$

Also, since $l(J_i) = 5l(I_i)$ for $i > p$, we have $\sum_{i=p+1}^{\infty} l(J_i) \leq \varepsilon$. **Q.E.D.**

The Differentiation Theorem

We are finally prepared to prove the difficult part of Theorem 4.11.

• **5.9 Differentiation Theorem.** *Let f be integrable on $I := [a, b]$ and let F be an indefinite integral of f. Then there exists a null set $Z \subset I$ such*

that if $x \in I - Z$ then $F'(x)$ exists and equals $f(x)$; thus, F is an a-primitive of f.

Proof. As usual, it is enough to handle the indefinite integral F of f with base point a.

We let E be the set of points $x \in [a, b)$ such that the right hand derivative $F'_+(x)$ of F either does not exist at x or does not equal $f(x)$. We will show that E is a null set, and a similar argument shows that the set of points in $(a, b]$ where F does not have a left hand derivative equal to $f(x)$ also is a null set. Since the set Z of points where F does not have a derivative equal to $f(x)$ is the union of these two sets, the set Z is a null set.

If F has a right hand derivative $F'_+(x) = f(x)$ at the point $x \in I$, then for any $\alpha > 0$ there exists an $s > 0$ such that if $v \in I$ is any number with $x < v < x + s$, then

$$\left|\frac{F(v) - F(x)}{v - x} - f(x)\right| \le \alpha.$$

Negating this assertion, if $x \in E$, then there exists $\alpha(x) > 0$ such that for every $s > 0$ there exists a point $v_{x,s} \in I$ with $x < v_{x,s} < x + s$ and such that

$$(5.\iota) \qquad \left|\frac{F(v_{x,s}) - F(x)}{v_{x,s} - x} - f(x)\right| > \alpha(x),$$

whence it follows that

$$(5.\kappa) \qquad \Big|[F(v_{x,s}) - F(x)] - f(x)(v_{x,s} - x)\Big| > \alpha(x)(v_{x,s} - x).$$

Fix $n \in \mathbb{N}$ and let $E_n := \{x \in E : \alpha(x) \ge 1/n\}$. Given $\varepsilon > 0$, since f is integrable, there exists a gauge δ_ε on I such that if $\dot{\mathcal{P}}$ is a δ_ε-fine partition of I, then

$$(5.\lambda) \qquad \left|S(f; \dot{\mathcal{P}}) - \int_I f\right| \le \varepsilon/n.$$

Now let $\mathcal{F}_n := \{[x, v_{x,s}] : x \in E_n,\ 0 < s \le \delta_\varepsilon(x)\}$; then $\mathcal{F}_n$ is a Vitali covering for E_n. By the Vitali Covering Theorem there exist intervals $I_1 := [x_1, v_1], \cdots, I_p := [x_p, v_p]$ in $\mathcal{F}_n$ and a sequence $(J_i)_{p+1}^\infty$ of closed intervals such that

$$(5.\mu) \qquad E_n \subseteq \bigcup_{i=1}^{p} I_i \cup \bigcup_{i=p+1}^{\infty} J_i \qquad \text{and} \qquad \sum_{i=p+1}^{\infty} l(J_i) \le \varepsilon.$$

We now consider the sum

$$(5.\nu) \quad \sum_{i=1}^{p} \left|f(x_i)(v_i - x_i) - \int_{x_i}^{v_i} f\right| = \sum_{i=1}^{p} \Big|f(x_i)(v_i - x_i) - [F(v_i) - F(x_i)]\Big|.$$

It follows from (5.κ) with $\alpha(x_i) \geq 1/n$ that the sum on the right in (5.ν) is greater than

$$(5.\xi) \qquad (1/n)\sum_{i=1}^{p}(v_i - x_i).$$

On the other hand, since $x_i \leq v_i \leq x_i + \delta_\varepsilon(x_i)$ for $i = 1, \cdots, p$, the ordered pairs $\{(I_i, x_i)\}_{i=1}^p$ form a subpartition of a δ_ε-fine partition of I for which (5.λ) holds. Therefore, by Corollary 5.4 of the Saks-Henstock Lemma, we conclude that the sum in (5.ν) is less than or equal to $2\varepsilon/n$. If we combine this estimate with (5.ξ), we find (after multiplying by n) that

$$(5.o) \qquad \sum_{i=1}^{p}(v_i - x_i) \leq n\sum_{i=1}^{p}\left|f(x_i)(v_i - x_i) - \int_{x_i}^{v_i} f\right| \leq 2\varepsilon.$$

But, in view of (5.μ), we conclude that E_n is contained in a countable union of intervals with total length $\leq 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that the set E_n is a null set. Therefore, since $E = \bigcup_{n=1}^{\infty} E_n$ and each E_n is a null set, we conclude that E is a null set. **Q.E.D.**

Theorem 4.11 has now been completely proved.

Characterization of Null Functions

We now establish some equivalent properties for a function to be a null function in the sense of Definition 2.4(b).

• **5.10 Characterization of null functions.** *Let $\psi : I \to \mathbb{R}$ where $I := [a, b]$. Then the following statements are equivalent:*

(a) *$|\psi|$ is a null function on I.*

(b) *ψ is a null function on I.*

(c) *$\psi \in \mathcal{R}^*(I)$ and $\int_a^r \psi = 0$ for every rational $r \in \mathbb{Q} \cap I$.*

(d) *An indefinite integral $\Psi_c(x) := \int_c^x \psi$ of ψ vanishes identically on I.*

(e) *$|\psi| \in \mathcal{R}^*(I)$ and $\int_a^b |\psi| = 0$.*

(f) *$|\psi| \in \mathcal{R}^*(I)$ and $\int_a^x |\psi| = 0$ for all $x \in I$.*

Proof. (a) $\Leftrightarrow$ (b) Since $\{x \in I : |\psi(x)| \neq 0\} = \{x \in I : \psi(x) \neq 0\}$, it follows that $|\psi|$ is a null function if and only if ψ is a null function.

(b) $\Rightarrow$ (c) If ψ is a null function, Example 2.6(b) implies that ψ is in $\mathcal{R}^*(I)$. By Corollary 3.8, the restriction of ψ to every subinterval $[a, r]$ is integrable. Since these restrictions are null functions, Example 2.6(b) implies that $\int_a^r \psi = 0$.

(c) ⇒ (d) If $\psi \in \mathcal{R}^*(I)$ and $\Psi_c(x) = \int_c^x \psi$, then since $\Psi_c(r) = 0$ for all $r \in \mathbb{Q} \cap I$, it follows from the continuity of Ψ_c that it vanishes identically on I.

(d) ⇒ (b) If the indefinite integral Ψ_c of ψ vanishes identically, then $\Psi_c'(x) = 0$ for all $x \in I$. By the Differentiation Theorem 5.9, we conclude that $\psi(x) = \Psi_c'(x) = 0$ except for x in some null set. Therefore, ψ is a null function.

(a) ⇒ (e) This assertion follows from Example 2.6(b) applied to $|\psi|$.

(e) ⇒ (f) If $|\psi| \in \mathcal{R}^*(I)$ and $\int_a^b |\psi| = 0$, then Theorem 3.7 implies that the restrictions of $|\psi|$ to $[a, x]$ and $[x, b]$ are integrable. From Theorem 3.2, we have

$$0 = \int_a^b |\psi| = \int_a^x |\psi| + \int_x^b |\psi| \geq \int_a^x |\psi| \geq 0.$$

Therefore $\int_a^x |\psi| = 0$ for all $x \in I$.

(f) ⇒ (a) The hypothesis is that the indefinite integral of $|\psi|$ with base point a vanishes identically, whence it follows from the Differentiation Theorem 5.9 that $|\psi|$ is a null function. Q.E.D.

A Characterization of Indefinite Integrals

We conclude this section by giving a necessary and sufficient condition that a function F be an indefinite integral of a function $f \in \mathcal{R}^*([a, b])$. The reader will see that the proof of the second part of this theorem is essentially the same as the proofs of the Fundamental Theorems 4.5 and 4.7.

The following notion was explicitly formulated by Výborný [V; p. 427]. It uses the notion of (δ, E)-fineness of a subpartition given in Definition 5.1(d).

• **5.11 Definition.** A function $F : I \to \mathbb{R}$ is said to have **negligible variation** on a set $E \subseteq I$ and we write $F \in NV_I(E)$ if, for every $\varepsilon > 0$ there exists a gauge δ_ε on E such that if $\dot{\mathcal{P}}_0 := \{([u_j, v_j], t_j)\}_{j=1}^s$ is any (δ_ε, E)-fine subpartition of I, then

$$\sum_{j=1}^{s} |F(v_j) - F(u_j)| \leq \varepsilon. \tag{5.π}$$

It will be seen in the exercises that if $F \in NV_I(E)$, then F is continuous at every point of E. Conversely, if C is a countable set in I and $F : I \to \mathbb{R}$ is continuous at every point of C, then $F \in NV_I(C)$. However, when $Z \subset I$ is a null set, not every continuous function on I belongs to $NV_I(Z)$. For

example, the Cantor-Lebesgue singular function $\Lambda : [0,1] \to \mathbb{R}$, introduced in 4.17, is monotone and continuous on $[0,1]$, but it is not in $NV_{[0,1]}(\Gamma)$.

• **5.12 Characterization Theorem.** *A function $G : I \to \mathbb{R}$ is an indefinite integral of a function $f \in \mathcal{R}^*(I)$ if any only if there exists a null set $Z \subset I := [a,b]$ such that $G'(x) = f(x)$ for all $x \in I - Z$ and $G \in NV_I(Z)$.*

In this case, we have

$$(5.\rho) \qquad \int_a^x f = G(x) - G(a) \qquad \textit{for all} \quad x \in I.$$

Proof. $(\Rightarrow)$ If $f \in \mathcal{R}^*(I)$ and $F(x) := \int_a^x f$, then it follows from the Differentiation Theorem 5.9 that there exists a null set $Z \subset I$ such that $F'(x) = f(x)$ for all $x \in I - Z$. We define f_1 on I by $f_1(x) := f(x)$ for $x \in I - Z$ and $f_1(x) := 0$ for $x \in Z$. It follows from Exercise 3.C that $f_1 \in \mathcal{R}^*(I)$ and that F is also the indefinite integral of f_1 with base point a. Therefore, given $\varepsilon > 0$, there exists a gauge η_ε on I such that if $\dot{\mathcal{P}} \ll \eta_\varepsilon$, then

$$\Big| \int_a^b f_1 - S(f_1; \dot{\mathcal{P}}) \Big| \le \tfrac{1}{2}\varepsilon.$$

Now let $\dot{\mathcal{P}}_0 := \{([u_j, v_j], t_j)\}_{j=1}^s$ be any (η_ε, Z)-fine subpartition of I. Then $\dot{\mathcal{P}}_0$ is a subset of some η_ε-fine partition $\dot{\mathcal{P}}$ of I. It follows from Corollary 5.4 that

$$\sum_{j=1}^s \Big| f_1(t_j)(v_j - u_j) - \int_{u_j}^{v_j} f_1 \Big| \le \varepsilon.$$

But since $f_1(t_j) = 0$ and $\int_{u_j}^{v_j} f_1 = F(v_j) - F(u_j)$ for $j = 1, \cdots, s$, we conclude that $\sum_{j=1}^s |F(v_j) - F(u_j)| \le \varepsilon$. Since $\dot{\mathcal{P}}_0$ is an arbitrary (η_ε, Z)-fine subpartition of I, we infer that F has negligible variation on Z.

If G is any indefinite integral of f, then $G = F + G(a)$ so that $G'(x) = f(x)$ for all $x \in I - Z$, and since $G(v_j) - G(u_j) = F(v_j) - F(u_j)$, it follows that $G \in NV_I(Z)$. Also, $F(x) = G(x) - G(a)$ so that equation $(5.\rho)$ results.

$(\Leftarrow)$ Suppose that Z is a null set and that $G \in NV_I(Z)$ is differentiable on $I - Z$. We define $f(x) := G'(x)$ for $x \in I - Z$ and $f(x) := 0$ for $x \in Z$. We will show that $f \in \mathcal{R}^*(I)$ and that G is an indefinite integral of f.

Given $\varepsilon > 0$, we will construct a gauge for f. If $t \in I - Z$, choose $\delta_\varepsilon(t) > 0$ as in the Straddle Lemma 4.4 such that if $t \in [u, v] \subseteq [t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)]$, then

$$\big| G(v) - G(u) - f(t)(v - u) \big| \le \varepsilon(v - u).$$

For $t \in Z$, choose $\delta_\varepsilon(t) > 0$ as required in Definition 5.11. Thus we have defined a gauge δ_ε on I.

We now let $\dot{\mathcal{P}} := \{([u_i, v_i], t_i)\}_{i=1}^n$ be a δ_ε-fine partition of I. Using the telescoping sum $G(b) - G(a) = \sum_{i=1}^n [G(v_i) - G(u_i)]$, it is readily seen that

$$\left|G(b) - G(a) - S(f;\dot{\mathcal{P}})\right| = \left|\sum_{i=1}^n [G(v_i) - G(u_i) - f(t_i)(v_i - u_i)]\right|.$$

We break this sum into sums over terms where $t_i \in Z$ (where $f(t_i) = 0$), and over terms where $t_i \in I - Z$. We conclude that this sum is

$$\leq \sum_{t_i \in Z} |G(v_i) - G(u_i)| + \sum_{t_i \in I-Z} |G(v_i) - G(u_i) - f(t_i)(v_i - u_i)|$$

$$\leq \varepsilon + \sum_{t_i \in I-Z} \varepsilon(v_i - u_i) \leq \varepsilon(1 + b - a).$$

Since $\varepsilon > 0$ is arbitrary, then $f \in \mathcal{R}^*(I)$ and $G(b) - G(a) = \int_a^b f$. Since the same argument can be applied to any interval $[a, x] \subseteq I$, the assertion is proved. **Q.E.D.**

This theorem can be readily used to show that every L-integrable function on I belongs to $\mathcal{R}^*(I)$, provided that we know that the indefinite integral of an L-integrable function is "absolutely continuous". (See Section 14.) Another characterization of the indefinite integral of functions in $\mathcal{R}^*(I)$ is given in the book of Gordon [G-3; p. 147], and will be mentioned in Section 14.

Exercises

5.A If $\Pi_1 := \{J_j\}_{j=1}^s$ is a subpartition of I, show that there exists a subpartition Π_2 of I such that $\Pi_1 \cup \Pi_2$ is a *partition* of I.

5.B If δ is a gauge on I and $\dot{\mathcal{P}}_1 := \{(J_j, t_j)\}_{j=1}^s$ is a δ-fine subpartition of I, show that there exists a δ-fine subpartition $\dot{\mathcal{P}}_2$ of I such that $\dot{\mathcal{P}}_1 \cup \dot{\mathcal{P}}_2$ is a δ-fine *partition* of I.

5.C Let $\alpha = 1$ and $\beta = -1$ in Example 2.2(a) and let δ_ε be as in that example.

(a) If $\dot{\mathcal{P}}_0 := \{(I_j, t_j)\}_{j=1}^k$, show that $|S(f;\dot{\mathcal{P}}_0) - \int_{U(\dot{\mathcal{P}}_0)} f| = 2(c - x_{k-1}) \leq \varepsilon$.

(b) If $\dot{\mathcal{P}}_1 := \{(I_j, t_j)\}_{j=k+1}^n$, show that $|S(f;\dot{\mathcal{P}}_1) - \int_{U(\dot{\mathcal{P}}_1)} f| = 0$.

5.D Let f be as in Exercise 5.C. If the partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ of I is δ_ε-fine, show that $\sum_{i=1}^n |f(t_i)l(I_i) - \int_{I_i} f| = 2(c - x_{k-1}) \leq \varepsilon$.

5.E Let $\alpha = 1$ and $\beta = -100$ in Example 2.2(a) and let $\dot{\mathcal{P}}_0$ and $\dot{\mathcal{P}}_1$ be as in Exercise 5.C. Evaluate $|S(f; \dot{\mathcal{P}}_0) - \int_{U(\dot{\mathcal{P}}_0)} f|$ and $|S(f; \dot{\mathcal{P}}_1) - \int_{U(\dot{\mathcal{P}}_1)} f|$.

5.F Write out the details of the proof in Theorem 5.6 that the indefinite integral F with base point a is continuous from the left at every point of $(a, b]$.

5.G If F is any indefinite integral of $f \in \mathcal{R}^*([a, b])$, show that F is continuous on I.

5.H If $I := [a, b]$, show that the collection $\{B[r; 1/n] : r \in I \cap \mathbb{Q},\ n \in \mathbb{N}\}$ of closed balls is a Vitali covering for I.

5.I Let $E \subseteq [a, b]$ and let $\mathcal{F}$ be a Vitali covering for E. If $J := (\alpha, \beta)$ is an open interval with $E \subset J$, show that $\mathcal{F}_J := \{I \in \mathcal{F} : I \subseteq J\}$ is a Vitali covering for E.

5.J If $(I_i)_{i=1}^\infty$ is a sequence of nonoverlapping closed intervals contained in $[a, b]$, show that $\sum_{i=1}^\infty l(I_i) \leq b - a$. [Hint: See Exercise 1.S.]

5.K Let $\mathcal{F}$ be a Vitali cover for a set $E \subseteq [a, b]$ and let $(I_i)_{i=1}^\infty$ be a sequence of pairwise disjoint closed intervals as constructed in the proof of the Vitali Covering Theorem 5.8. Show that $E - \bigcup_{i=1}^\infty I_i$ is a null set.

5.L Show that the Cantor-Lebesgue function Λ of 4.17 has a left hand derivative at 2/3, but that it does not have a right hand derivative there.

5.M Formulate a necessary and sufficient condition for the function F not to have a left hand derivative $F'_-(x)$ equal to $f(x)$ at $x \in (a, b]$ that is similar to the one in formula (5.ι).

5.N If $F : I \to \mathbb{R}$ belongs to $NV_I(E)$, show that F is continuous at every point $c \in E$.

5.O If $F : I \to \mathbb{R}$ is continuous at every point of a countable set $C \subset I$, show that $F \in NV_I(C)$.

5.P Show that the Cantor-Lebesgue function $\Lambda : [0, 1] \to \mathbb{R}$ of 4.17 does not belong to $NV_{[0,1]}(\Gamma)$, where Γ is the Cantor set.

5.Q (a) Show that the sum of two null functions is a null function.

(b) If $f, g : I \to \mathbb{R}$ and f is a null function and $|g(x)| \leq |f(x)|$ a.e., then g is a null function.

(c) If $f_n : I \to \mathbb{R}$ are null functions on I and $|f_n(x)| \leq 1/2^n$ a.e. on I, show that $f(x) := \sum_{n=1}^{\infty} f_n(x)$ is a null function.

(d) Let (f_n) be a sequence of null functions on I and let $g(x) := \sum_{n=1}^{\infty} f_n(x)$ when this series converges, and $g(x) := 0$ otherwise. Show that g is a null function.

5.R Show that a function $F : I \to \mathbb{R}$ has negligible variation on a set $E \subseteq I$ if and only if for every $\varepsilon > 0$ there exists a gauge η_ε on E such that if $\{([u_j, v_j], t_j)\}_{j=1}^{s}$ is any (η_ε, E)-fine subpartition of E, then $\sum_{j=1}^{s} \{F(v_j) - F(u_j)\} \leq \varepsilon$.

5.S Let $f : [a, b] \to \mathbb{C}$ be a complex-valued integrable function. Given $\varepsilon > 0$, let δ_ε be a gauge such that if $\dot{\mathcal{P}} \ll \delta_\varepsilon$, then $|S(f; \dot{\mathcal{P}}) - \int_I f| \leq \varepsilon$. Show that if $\dot{\mathcal{P}}_0$ is a δ_ε-fine subpartition of $[a, b]$, then (5.δ) holds with 2ε replaced by 4ε.

5.T Show that the Differentiation Theorem 5.9 holds for complex-valued integrable functions.

5.U Show that a complex-valued function is a null function if and only if its real and imaginary parts are null functions.

Oskar Perron
(May 7, 1880–February 22, 1975)

Courtesy of Frau Irmgard Hellerbrand,
Munich, Germany

Measurable Functions

It is now convenient to introduce a very important class of functions; namely, the collection of *measurable functions*. This collection contains almost all of the functions that we will consider in this book; indeed, it is difficult to construct any function that is not measurable. We will see that there is a very close connection between the collection $\mathcal{M}(I)$ of measurable functions on a compact interval $I := [a, b]$ and the collection $\mathcal{R}^*(I)$ of (generalized Riemann) integrable functions. In fact,

(i) every integrable function is measurable,

(ii) every measurable function that is bounded both below and above by integrable functions is integrable, and

(iii) the product of an integrable and a monotone function is integrable.

We will defer the proofs of (ii) and (iii) to later sections. However, the statements of these results are sufficiently simple that we announce them here.

Measurable Functions

Although measurable functions can be defined in several ways, in dealing with a compact interval I it is easiest to employ the following definition.

$\diamond$ **6.1 Definition.** A function $f : I \to \mathbb{R}$ is said to be **measurable** on $I := [a, b]$ if there exists a sequence $(s_k)_{k=1}^{\infty}$ of step functions on I such that

$$f(x) = \lim_{k \to \infty} s_k(x) \qquad \text{a.e. on } I. \tag{6.α}$$

The collection of all measurable functions on I will be denoted by $\mathcal{M}(I)$.

In other words, a function f is measurable if and only if there exist a null set $Z \subset I$ and a sequence (s_k) of step functions such that

$$(6.\beta) \qquad f(x) = \lim_{k\to\infty} s_k(x) \qquad \text{for all} \quad x \in I - Z.$$

Since a regulated function (see Definition 3.15) on I is the *uniform* limit of a sequence of step functions, it is trivial that every regulated function is measurable. However, it is easy to see that not every measurable function is a regulated function; for example, see Exercises 3.R and 6.B.

◇ **6.2 Theorem.** *Every step function, every continuous function, every monotone function, and every null function on I belongs to $\mathcal{M}(I)$.*

Proof. This result follows immediately from the definitions. Q.E.D.

◇ **6.3 Theorem.** *Let $f, g \in \mathcal{M}(I)$ for $I := [a, b]$ and let $c \in \mathbb{R}$.*

(a) *Then $\mathcal{M}(I)$ also contains the functions*

$$cf, \quad |f|, \quad f+g, \quad f-g, \quad f\cdot g.$$

(b) *If $h \in \mathcal{M}(I)$ and $h(x) \neq 0$ on I, then $1/h \in \mathcal{M}(I)$.*

(c) *If $k : I \to \mathbb{R}$ is such that $k(x) = f(x)$ a.e. on I, then $k \in \mathcal{M}(I)$.*

(d) *If $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, then the composition $\varphi \circ f \in \mathcal{M}(I)$.*

Proof. (a) Let $Z \subset I$ be a null set and let (s_k) be a sequence of step functions such that $(6.\beta)$ holds. Now $|s_k|$ is a step function on I for each $k \in \mathbb{N}$ and the Triangle Inequality implies that

$$0 \le \big||f(x)| - |s_k(x)|\big| \le \big|f(x) - s_k(x)\big| \to 0$$

for all $x \in I - Z$. Therefore $|f| \in \mathcal{M}(I)$.

The remaining assertions in (a) follow from the basic properties of limits of sequences, the fact that the sum and product of step functions are step functions, and the fact that the union of two null sets is a null set.

(b) Let $W \subset I$ be a null set and (u_k) be a sequence of step functions such that $h(x) = \lim u_k(x)$ for all $x \in I - W$. We let $\bar{u}_k(x) := 1/k$ when $u_k(x) = 0$ and $\bar{u}_k(x) := u_k(x)$ elsewhere on I. Thus $(\bar{u}_k)$ is a sequence of nonzero step functions that converges to h on $I - W$. Moreover, $1/\bar{u}_k$ is a step function, and the sequence $(1/\bar{u}_k)$ converges to $1/h$ on $I - W$. Therefore $1/h \in \mathcal{M}(I)$.

(c) If $(s_k) \to f$ on $I - Z$ and if there exists a null set $Y \subset I$ such that $h = f$ on $I - Y$, then $(s_k) \to h$ on $I - (Z \cup Y)$.

(d) Let $(s_k) \to f$ on $I - Z$ and note that the composition $\varphi \circ s_k$ is a step function on I. Since φ is continuous on $\mathbb{R}$ and $f(x) = \lim s_k(x)$ for $x \in I - Z$, it follows that $(\varphi \circ f)(x) = \varphi(f(x)) = \lim \varphi(s_k(x))$ for all $x \in I - Z$. Q.E.D.

We now introduce certain ways of constructing new functions from given ones. Note that these constructions make sense for any real-valued functions.

• **6.4 Definitions.** Let $f, g, h : I \to \mathbb{R}$.

(a) We define the **maximum** of f and g, denoted $f \vee g$ or $\max\{f, g\}$, by

$$(f \vee g)(x) := \max\{f(x), g(x)\} \qquad \text{for all} \quad x \in I.$$

We define the **minimum** of f and g, denoted $f \wedge g$ or $\min\{f, g\}$, by

$$(f \wedge g)(x) := \min\{f(x), g(x)\} \qquad \text{for all} \quad x \in I.$$

(b) We define the **positive and negative parts** of f, denoted f^+ and f^-, respectively, by

$$f^+ := f \vee 0 \qquad \text{and} \qquad f^- := (-f) \vee 0.$$

(c) We define the **middle function** of f, g, h, denoted $\text{mid}\{f, g, h\}$, by

$$\text{mid}\{f, g, h\}(x) := \text{mid}\{f(x), g(x), h(x)\} \qquad \text{for all} \quad x \in I.$$

Here, on the right, $\text{mid}\{a, b, c\}$ denotes the middle of the real numbers a, b, c.

• **6.5 Lemma.** *The following equalities hold:*

(a) $f \vee g = \frac{1}{2}\{f + g + |f - g|\}$ *and* $f \wedge g = \frac{1}{2}\{f + g - |f - g|\}$.

(b) $f^+ = \frac{1}{2}\{f + |f|\}$ *and* $f^- = \frac{1}{2}\{|f| - f\}$.

(c) $f = f^+ - f^-$ *and* $|f| = f^+ + f^-$.

(d) $\text{mid}\{f, g, h\} = \max\{\min\{f, g\}, \min\{g, h\}, \min\{h, f\}\}$
$= \min\{\max\{f, g\}, \max\{g, h\}, \max\{h, f\}\}$

(e) $\min\{a, b, c\} = \min\{\min\{a, b\}, c\}$.

Proof. We leave the proofs as exercises for the reader. Q.E.D.

We now show that these combinations of measurable functions give rise to measurable functions.

• **6.6 Theorem.** *Let $f, g, h : I \to \mathbb{R}$, where $I := [a, b]$.*

(a) *$f \in \mathcal{M}(I) \Leftrightarrow f^+$ and f^- belong to $\mathcal{M}(I)$.*

(b) *$f, g \in \mathcal{M}(I) \Leftrightarrow \max\{f, g\}$ and $\min\{f, g\}$ belong to $\mathcal{M}(I)$.*

(c) *$f, g, h \in \mathcal{M}(I) \Rightarrow \operatorname{mid}\{f, g, h\} \in \mathcal{M}(I)$.*

Proof. (a) If $f \in \mathcal{M}(I)$, then Theorem 6.3(a) implies that $|f| \in \mathcal{M}(I)$, whence (by 6.3(a) and 6.5(b)) f^+ and f^- belong to $\mathcal{M}(I)$. Conversely, if f^+ and f^- belong to $\mathcal{M}(I)$, then (by 6.3(a) and 6.5(c)) f and $|f|$ belong to $\mathcal{M}(I)$.

We leave the proofs of the remaining assertions as exercises. Q.E.D.

Measurable Functions as Limits of Continuous Functions

It is useful to know that a function on I is measurable if and only if it is the a.e. limit of a sequence of *continuous* functions on I. We will make important use of the first part of this assertion in Theorem 6.8.

◇ **6.7 Theorem.** *A function $f \in \mathcal{M}(I)$ if and only if there exists a sequence (h_k) of continuous functions on I such that*

$$(6.\gamma) \qquad f(x) = \lim_{k\to\infty} h_k(x) \qquad \text{for a.e.} \quad x \in I.$$

Proof. ($\Leftarrow$) Let $Z \subset I$ be a null set and (h_k) be a sequence of continuous functions such that $f(x) = \lim h_k(x)$ for all $x \in I - Z$. Since h_k is continuous, it is a regulated function, so there exists a step function s_k such that

$$|h_k(x) - s_k(x)| \le 1/k \qquad \text{for all} \quad x \in I.$$

Therefore we have

$$\begin{aligned} 0 \le |f(x) - s_k(x)| &\le |f(x) - h_k(x)| + |h_k(x) - s_k(x)| \\ &\le |f(x) - h_k(x)| + 1/k \end{aligned}$$

for all $x \in I$, whence it follows that $f(x) = \lim s_k(x)$ for all $x \in I - Z$.

($\Rightarrow$) Let $Z \subset I$ be a null set and let (s_k) be a sequence of step functions satisfying $(6.\beta)$. We will construct a sequence of continuous functions that converges a.e. to f. (The idea is simple, although the notation gets complicated.)

We may assume that each s_k is continuous at the endpoints a, b of I, since this can be obtained by redefining $s_k(a) := \lim_{x\to a+} s_k(x)$ and $s_k(b) := \lim_{x\to b-} s_k(x)$. For fixed $k \in \mathbb{N}$, the step function s_k is discontinuous at a

finite number of points in the open interval (a, b), which can be enclosed in disjoint closed intervals $J_1^k, \cdots, J_{m_k}^k$ with total length $\leq 1/2^k$.

For convenience, let $K_k := J_1^k \cup \cdots \cup J_{m_k}^k$ for $k \in \mathbb{N}$. We now define h_k to equal s_k on $I - K_k$ and, on each interval $J_j^k =: [a_j^k, b_j^k]$, we define h_k to be the linear function joining the points

$$\left(a_j^k, s_k(a_j^k)\right) \qquad \text{and} \qquad \left(b_j^k, s_k(b_j^k)\right).$$

Evidently h_k is continuous on I.

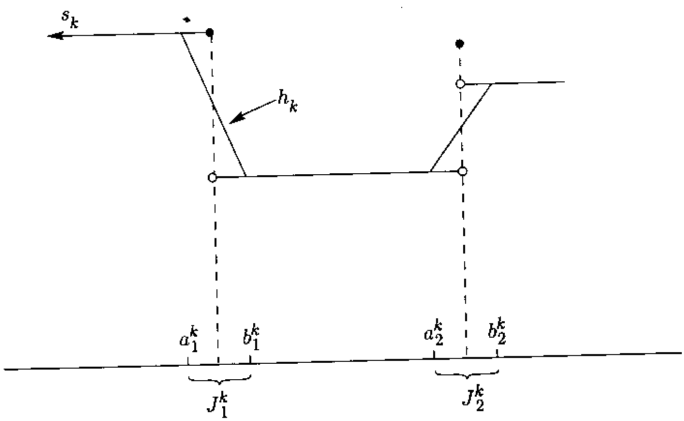

Figure 6.1

We claim that $(h_k) \to f$ a.e. on I. If $c \in I - Z$ belongs to only a finite number of the sets K_k, then $h_k(c) = s_k(c)$ for all sufficiently large k, in which case $\lim h_k(c) = \lim s_k(c) = f(c)$. Thus we need to be concerned only about the set W of all points that belong to infinitely many sets K_k, $k \in \mathbb{N}$. Let $\varepsilon > 0$ be given and let k_ε be such that $1/2^{k_\varepsilon} \leq \varepsilon$. Now K_k is contained in the union of a finite collection of closed intervals with total length $\leq 1/2^k$, so the set $\bigcup_{k>k_\varepsilon} K_k$ is contained in the union of a countable collection of closed intervals with total length $\leq \sum_{k>k_\varepsilon} 1/2^k = 1/2^{k_\varepsilon} \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that the set W of all points that belong to infinitely many sets K_k is a null set, and it follows that $h_k(x) \to f(x)$ for all $x \in I - (Z \cup W)$. Since $Z \cup W$ is a null set, the proof is complete. Q.E.D.

It will be proved in Theorem 9.2 that the a.e. limit of a sequence of measurable functions is a measurable function.

The Measurability of Integrable Functions

We now show that it follows from Theorem 4.11 (proved in Section 5) that an integrable function on a compact interval $I := [a, b]$ is measurable.

• **6.8 Measurability Theorem.** *If $f \in \mathcal{R}^*(I)$, then $f \in \mathcal{M}(I)$.*

Proof. Let F be defined on the interval $[a, b+1]$ by

$$F(x) := \begin{cases} \int_a^x f & \text{if} \quad x \in [a, b), \\ \int_a^b f & \text{if} \quad x \in [b, b+1]. \end{cases} \tag{6.δ}$$

The Differentiation Theorem 5.9 implies that there exists a null set $Z \subset I$ such that if $x \in I - Z$, then $F'(x) = f(x)$. Therefore, if we introduce the difference quotients $g_k(x) := k\{F(x + 1/k) - F(x)\}$ for $x \in I, k \in \mathbb{N}$, then $g_k(x) \to f(x)$ for all $x \in I - Z$.

By Theorem 5.6, the function F and hence the functions g_k are continuous on I. Since f is the a.e. limit of a sequence of continuous functions, it follows from (the easy part of) Theorem 6.7 that f is measurable. Q.E.D.

Integrating Measurable Functions

A measurable function is *not necessarily* integrable. For example, the function $h : [0, 1] \to \mathbb{R}$, defined by $h(0) := 0$ and $h(x) := 1/x$ for $x \in (0, 1]$, is measurable but not integrable. What goes wrong here is that the function h is "too large". This is the typical situation. The next result asserts that if a measurable function is bounded below and above by integrable functions, then it is integrable.

• **6.9 Integrability Theorem.** *Suppose $f \in \mathcal{M}(I)$; then $f \in \mathcal{R}^*(I)$ if and only if there exist $\alpha, \omega \in \mathcal{R}^*(I)$ such that*

$$\alpha(x) \le f(x) \le \omega(x) \qquad \textit{for a.e.} \quad x \in I. \tag{6.ε}$$

The implication $\Rightarrow$ is trivial, since we can take $\alpha = \omega = f$. The proof of the nontrivial portion ($\Leftarrow$) of this important theorem is given in Theorem 9.1, as a consequence of the Lebesgue Dominated Convergence Theorem. However, there are some useful corollaries that can be stated here.

6.10 Corollary. *If $f \in \mathcal{M}(I)$ is bounded and I is a compact interval, then both f and $|f|$ are in $\mathcal{R}^*(I)$.*

Proof. We have seen in Example 2.1(a) that constant functions are in $\mathcal{R}^*(I)$ when I is a compact interval. Since bounded functions are bounded

both below and above by constant functions, such functions are in $\mathcal{R}^*(I)$. Similarly for $|f|$. Q.E.D.

• **6.11 Corollary.** *If f and $|f|$ are in $\mathcal{R}^*(I)$ and $h \in \mathcal{M}(I)$ is bounded on I, then the product $f \cdot h$ belongs to $\mathcal{R}^*(I)$.*

Proof. If $|h(x)| \le M$ for all $x \in I$, then we can take $\alpha(x) := -M|f(x)|$ and $\omega(x) := M|f(x)|$ for $x \in I$. Then $\alpha(x) \le f(x) \cdot h(x) \le \omega(x)$ for $x \in I$ and the theorem applies. Q.E.D.

It will be seen in an exercise that the product of two integrable functions is not necessarily integrable, even when one of them is bounded or continuous. However, the product of a function $f \in \mathcal{R}^*(I)$ and a monotone function belongs to $\mathcal{R}^*(I)$. We now state this important result formally.

• **6.12 Multiplier Theorem.** *If $f \in \mathcal{R}^*(I)$ and if φ is monotone on $I := [a,b]$, then the product $f \cdot \varphi$ belongs to $\mathcal{R}^*(I)$.*

A detailed proof of this result will be given in Theorem 10.12.

Some Examples

We now show how some of these theorems can be applied.

6.13 Examples. (a) The **Beta function** (of Euler) is defined by

$$B(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1}\,dx. \tag{6.ζ}$$

If $p \ge 1$ and $q \ge 1$, then the integrand is continuous and the integral exists.

If $0 < p < 1$, then the integrand becomes infinite as $x \to 0+$. However, there exists a constant $K > 0$ such that

$$0 \le x^{p-1}(1-x)^{q-1} \le Kx^{p-1} \qquad \text{for} \quad 0 < x \le \tfrac{1}{2}.$$

Since the function $x \mapsto Kx^p/p$ is an f-primitive of $x \mapsto Kx^{p-1}$ on $[0, \frac{1}{2}]$, it follows from the Integrability Theorem 6.9 that the function $x \mapsto x^{p-1}(1-x)^{q-1}$ is integrable on $[0, \frac{1}{2}]$ when $p > 0$.

A similar argument, which we leave to the reader, shows that if $q > 0$, then $x \mapsto x^{p-1}(1-x)^{q-1}$ is integrable on $[\frac{1}{2}, 1]$. Therefore, we conclude that the integral (6.ζ) exists when $p > 0$ and $q > 0$.

(b) Consider the integral $\int_0^b x^r \sin(\pi/x)\,dx$ for $r \in \mathbb{R}$.

More precisely, let $f_r(x) := x^r \sin(\pi/x)$ for $x \in (0,b]$ and $f_r(0) := 0$. We will first show that $f_r \in \mathcal{R}^*([0,b])$ when $r > -2$.

Let $S(x) := x^{r+2}\cos(\pi/x)$ for $x \in (0, b]$ and let $S(0) := 0$. Since $r+2 > 0$, the function S is continuous on $I := [0, b]$; moreover, it is differentiable on $(0, b]$ and a calculation shows that

$$S'(x) = (r+2)x^{r+1}\cos(\pi/x) + \pi x^r \sin(\pi/x) \qquad \text{for} \quad x \in (0, b].$$

The Fundamental Theorem 4.7 implies that $S' \in \mathcal{R}^*(I)$. If $r > -1$, the first term in S' is continuous and belongs to $\mathcal{R}^*(I)$; hence (by Theorem 3.1) the second term also belongs to $\mathcal{R}^*(I)$.

If $-2 < r \leq -1$, then the first term in S' is not continuous at $x = 0$, but we have

$$-x^{r+1} \leq x^{r+1}\cos(\pi/x) \leq x^{r+1} \qquad \text{for} \quad x \in (0, b].$$

Since $-1 < r+1$, the maps $x \mapsto \pm x^{r+1}$ have f-primitives $x \mapsto \pm x^{r+2}/(r+2)$, and so they belong to $\mathcal{R}^*(I)$. The Integrability Theorem 6.9 then implies that the first term in S' belongs to $\mathcal{R}^*(I)$, whence the second term also belongs to $\mathcal{R}^*(I)$. Thus $f_r \in \mathcal{R}^*(I)$ when $r > -2$, as claimed.

(c) We show that $f_{-2} \notin \mathcal{R}^*(I)$.

Assume that $f_{-2} \in \mathcal{R}^*([0, b])$ and let $u \in (0, b]$ be a number of the form $2/(2k+1)$ for some $k \in \mathbb{N}$. If Φ_u is the indefinite integral

$$\Phi_u(x) := \int_u^x f_{-2} \qquad \text{for} \quad x \in [0, b],$$

then Theorem 4.11 implies that Φ_u is continuous on $[0, b]$.

If we let $F(x) := (1/\pi)\cos(\pi/x)$ for $x \in (0, b]$, then $F'(x) = f_{-2}(x)$ for $x \in (0, b]$, so that F is a primitive of f_{-2} on any compact interval in $(0, b]$. Since $F(u) = 0$, it follows that $\Phi_u(x) = \int_u^x f = F(x)$ for all $x \in (0, b]$. But this is a contradiction, since F does not have a limit as $x \to 0+$.

(d) We show that $f_r \notin \mathcal{R}^*(I)$ when $r < -2$.

Suppose that there exists $r < -2$ such that $f_r \in \mathcal{R}^*(I)$. If we let $c := -2 - r > 0$, then the function $\varphi(x) := x^c$ is increasing on $[0, b]$. The Multiplier Theorem 6.12 implies that the product $\varphi \cdot f_r$ belongs to $\mathcal{R}^*(I)$. But since this product equals f_{-2}, we have obtained a contradiction. Thus $f_r \notin \mathcal{R}^*(I)$ when $r < -2$, as claimed.

We can summarize the results of (b) – (d) in the assertion:

$$(6.\eta) \qquad \int_0^b x^r \sin(\pi/x)\,dx \quad \text{exists} \quad \Leftrightarrow \quad r > -2.$$

Measurable and Integrable Sets

We will now introduce the notions of a "measurable subset" and of an "integrable subset" of a compact interval $I := [a, b]$. It will be seen later that these two notions coincide in the context of a compact interval. However, they do not coincide for sets that are not bounded, so it is well to distinguish between them. These notions will only be introduced here; additional results will be given in Section 10.

◇ **6.14 Definition.** If $E \subseteq I := [a, b]$, then its **characteristic function** $\mathbf{1}_E$ is defined by

$$\mathbf{1}_E(x) := \begin{cases} 1 & \text{if} \quad x \in E, \\ 0 & \text{if} \quad x \in I - E. \end{cases} \tag{6.θ}$$

(a) A set $E \subseteq I$ is said to be **measurable** if its characteristic function $\mathbf{1}_E$ is a measurable function. The collection of all measurable subsets of I is sometimes denoted by $\mathbb{M}(I)$.

(b) A set $E \subseteq I$ is said to be **integrable** if its characteristic function $\mathbf{1}_E$ is an integrable function. The collection of all integrable subsets of I is sometimes denoted by $\mathbb{I}(I)$. If $E \in \mathbb{I}(I)$, we define the **measure** of E to be

$$|E| := \int_a^b \mathbf{1}_E. \tag{6.ι}$$

6.15 Remarks. (a) The Measurability Theorem 6.8 implies that $\mathbb{I}(I) \subseteq \mathbb{M}(I)$ when $I := [a, b]$. Conversely, the Integrability Theorem 6.9 (which will be proved in Section 9) implies that $\mathbb{M}(I) \subseteq \mathbb{I}(I)$ when $I := [a, b]$. However, in Part 2 of this book, we will consider the case of unbounded sets, and it will be seen that the corresponding notions do *not* coincide.

(b) It is an exercise to show that the notion of a measurable [respectively, integrable] set, or the value of its measure, does not depend on the compact interval I containing the set.

Integrals over Measurable Sets

It is sometimes convenient to use the notion of the integral of a function over a measurable (or an integrable) set.

◇ **6.16 Definition.** Let $f \in \mathcal{R}^*(I)$ and let $E \subseteq I := [a, b]$ be a measurable set. If the product $f \cdot \mathbf{1}_E$ is in $\mathcal{R}^*(I)$, then we say that f is **integrable on** E and write $f \in \mathcal{R}^*(E)$. The **integral of f over E** is defined to be

$$\int_E f := \int_I f \cdot \mathbf{1}_E. \tag{6.κ}$$

We will now show that if $f \in \mathcal{R}^*(I)$ and $E \subseteq I$ is an integrable set, then the product $f \cdot \mathbf{1}_E$ may *not* be in $\mathcal{R}^*(I)$, so that f may not belong to $\mathcal{R}^*(E)$.

6.17 Examples. (a) Let $c_n := 1 - 1/2^n$ for $n = 0, 1, 2, \cdots$ and let

$$E := \bigcup_{k=0}^{\infty} [c_{2k}, c_{2k+1}).$$

This set E is the union of a countable collection of pairwise disjoint intervals in $[0, 1]$. It is an exercise to show that E is a measurable set. Moreover, it was seen in Example 4.20(b) that the characteristic function $\mathbf{1}_E$ of E belongs to $\mathcal{R}^*([0,1])$, so that E is an integrable set.

(b) Let κ be the function in Example 2.8(a), which was seen to belong to $\mathcal{R}^*([0,1])$. Clearly $(\kappa \cdot \mathbf{1}_E)(x) = 2^{2k+1}/(2k+1)$ for $x \in [c_{2k}, c_{2k+1}), k = 0, 1, 2, \cdots$, and $(\kappa \cdot \mathbf{1}_E)(x) = 0$ otherwise. But since $l([c_{2k}, c_{2k+1})) = 1/2^{2k+1}$, then

$$\int_0^{c_{2k+1}} \kappa \cdot \mathbf{1}_E = \frac{1}{1} + \frac{1}{3} + \cdots + \frac{1}{2k+1}.$$

Since $\kappa \cdot \mathbf{1}_E \geq 0$, if this product is integrable on $[0,1]$, then it follows from Theorems 3.7 and 3.2 that

$$\int_0^1 \kappa \cdot \mathbf{1}_E \geq \frac{1}{1} + \frac{1}{3} + \cdots + \frac{1}{2k+1}$$

for all $k \in \mathbb{N}$. But since the series $\sum 1/(2k+1)$ is divergent, this is a contradiction. Therefore $\kappa \cdot \mathbf{1}_E$ does not belong to $\mathcal{R}^*([0,1])$.

We will now show that if *both* f and $|f|$ belong to $\mathcal{R}^*(I)$ and if E is measurable, then $f \in \mathcal{R}^*(E)$.

$\diamond$ **6.18 Theorem.** *If f and $|f|$ belong to $\mathcal{R}^*(I)$ and $E \subseteq I$ is a measurable set, then f and $|f|$ belong to $\mathcal{R}^*(E)$.*

Proof. From Definition 6.14, the function $\mathbf{1}_E$ belongs to $\mathcal{M}(I)$. By Theorem 6.3, the product $f \cdot \mathbf{1}_E \in \mathcal{M}(I)$ and since

$$-|f| \leq f \cdot \mathbf{1}_E \leq |f|,$$

it follows from the Integrability Theorem 6.9 that $f \cdot \mathbf{1}_E \in \mathcal{R}^*(I)$. Therefore, we conclude that $f \in \mathcal{R}^*(E)$. Q.E.D.

We note that this proof made use of the Integrability Theorem 6.9, which is proved in Section 9.

Exercises

6.A If s is a step function on $[a, b]$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is any function, show that the composition $\varphi \circ s$ is a step function.

6.B Show that $f(x) := 1/x$ for $x \in (0, 1]$ and $f(0) := 0$ is measurable on $[0, 1]$ but is not regulated.

6.C Let $f, g \in \mathcal{M}(I)$. (a) Write out a proof that $f + g \in \mathcal{M}(I)$.
(b) Write out a proof that $f \cdot g \in \mathcal{M}(I)$.

6.D Show directly that the (nonintegrable) function λ in Example 2.8(b) is measurable.

6.E Show directly that the function λ in Example 2.8(c) is the limit of a sequence of continuous functions.

6.F (a) Write out a proof of Lemma 6.5(a).
(b) Write out a proof of Lemma 6.5(d, e).
(c) Prove that $\max\{f, g, h\} = \max\{\max\{f, g\}, \max\{g, h\}\}$.

6.G Show that a measurable function on $[a, b]$ is the a.e. limit of a sequence of functions having continuous derivatives on $[a, b]$.

6.H If $f \in \mathcal{M}(I)$, show that the function $g(x) := \text{Arctan}\big(f(x)\big)$ is bounded and measurable.

6.I Give an example of a function $f \in \mathcal{R}^*(I)$ whose square f^2 is not integrable.

6.J If κ is the function in Example 2.8(a), find a bounded measurable function m such that $\kappa \cdot m$ is not integrable.

6.K Find a continuous function m in Exercise 6.J.

6.L If $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f, g \in \mathcal{M}(I)$, prove that the function defined by $h(x) := H(f(x), g(x))$ for $x \in I$ also belongs to $\mathcal{M}(I)$.

6.M Let $(E_n)_{n=1}^{\infty}$ be a sequence of subsets on $\mathbb{R}$.

(a) Show that the set U of points that belong to *infinitely many* of the E_n is given by $U = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$. (The set U is called the **limit superior** of the sequence (E_n).)

(b) Show that the set L of points that belong to *all but a finite number* of the E_n is given by $L = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k$. (The set L is called the **limit inferior** of the sequence (E_n).)

6.N Show that every interval $J \subseteq [a, b]$ is an integrable (and hence measurable) set and that $|J| = l(J)$.

6.O Show that the notion of a measurable subset $E \subseteq I := [a, b]$ does not depend on the choice of the interval I containing E.

6.P Show that the notion of an integrable set $E \subseteq I := [a, b]$ does not depend on the choice of the interval I containing E. Show that the value of its measure also does not depend on I.

6.Q Show that if E and F are measurable sets in I, then the sets $E \cap F$, $E - F$ and $E \cup F$ are also measurable. [Hint: $\mathbf{1}_{E\cap F} = \mathbf{1}_E \cdot \mathbf{1}_F$.]

6.R Assuming 6.10, show that if E, F are integrable sets in I, then we have $|E \cup F| + |E \cap F| = |E| + |F|$. In particular, if $E \cap F = \emptyset$, show that $|E \cup F| = |E| + |F|$.

6.S Show that $E \subseteq I$ is a null set if and only if E is integrable and $|E| = 0$.

6.T If $b > 0$, show that $\int_0^b x^r \cos(\pi/x)\, dx$ exists if and only if $r > -2$.

6.U Show that the function $g(x) := (1/x)\sin(\pi/x^2)$ for $x \in (0, 1]$ and $g(0) := 0$ belongs to $\mathcal{R}^*([0, 1])$.

6.V If E and F are disjoint measurable sets in I and if $f \in \mathcal{R}^*(E) \cap \mathcal{R}^*(F)$, show that $f \in \mathcal{R}^*(E \cup F)$ and that

$$\int_{E\cup F} f = \int_E f + \int_F f.$$

6.W Show that the union of an arbitrary countable collection of pairwise disjoint intervals is a measurable set. Show that it is an integrable set.

Absolute Integrability

We now introduce the collection $\mathcal{L}(I)$ of "absolutely integrable" functions on an interval $I := [a, b]$; that is, functions $f \in \mathcal{R}^*(I)$ such that $|f| \in \mathcal{R}^*(I)$. This is precisely the collection of Lebesgue integrable functions, although it is arrived at from a more general point of view. We will show that the functions in this class can be characterized by a condition on their indefinite integrals; another characterization will be given in Section 14.

It will be shown at the end of this section (in Theorem 7.13) that if f and g in $\mathcal{R}^*(I)$ are such that $f(x) \leq \omega(x)$ and $g(x) \leq \omega(x)$ for some function $\omega \in \mathcal{R}^*(I)$ and all $x \in I$, then the functions $f \vee g = \max\{f, g\}$ and $f \wedge g = \min\{f, g\}$ also belong to $\mathcal{R}^*(I)$. This result is an important tool in the proof of theorems that will be given in Section 8.

Absolute Integrability

If $f : I \to \mathbb{R}$ is R-integrable (= Riemann integrable) on $I := [a, b]$, then the function $|f|$ defined by $|f|(x) := |f(x)|$ for $x \in I$ is also R-integrable [B-S; p. 216] and

$$\left| \int_I f \right| \leq \int_I |f|.$$

The same conclusion follows if f is L-integrable (= Lebesgue integrable); see [B-1; p. 43]. However, this conclusion does *not* follow if f is R*-integrable.

• **7.1 Definition.** An integrable function $f : I \to \mathbb{R}$ is said to be **absolutely integrable** on I if $|f|$ is also integrable on I. The collection of all absolutely integrable functions on I is denoted by $\mathcal{L}(I)$, or by $\mathcal{L}([a, b])$. A function that is integrable on I but not absolutely integrable is said to be **conditionally integrable** on I.

Of course, if $f \in \mathcal{R}^*(I)$ and $f(x) \ge 0$ for all $x \in I$, then f is absolutely integrable on I.

7.2 Examples. **(a)** Let $\kappa : [0,1] \to \mathbb{R}$ be the function in Example 2.8(a), which was shown there to be integrable on $[0,1]$ to $\sum_{k=1}^{\infty}(-1)^{k+1}/k$. Evidently $\lambda := |\kappa|$ is given by $\lambda(1) := 0$ and $\lambda(x) := 2^k/k$ for $x \in [c_{k-1}, c_k)$, $k \in \mathbb{N}$, where $c_k := 1 - 1/2^k$. (See Figure 2.4 for a sketch of the graph of λ.)

If λ were integrable, Theorems 3.7 and 3.2 would imply that for $n \in \mathbb{N}$, then

$$\int_0^1 \lambda = \int_0^{c_n} \lambda + \int_{c_n}^1 \lambda \ge \int_0^{c_n} \lambda.$$

Now the restriction of λ to the interval $[0, c_n]$ is a step function, and

$$\int_0^{c_n} \lambda = \sum_{k=1}^{n} \frac{1}{k}.$$

Since the harmonic series diverges, the function λ does not belong to $\mathcal{R}^*(I)$.

(b) Let $F(x) := x\cos(\pi/x)$ for $x \in (0,1]$ and let $F(0) := 0$. Then

$$F'(x) = \cos(\pi/x) + (\pi/x)\sin(\pi/x) \qquad \text{for} \quad x \in (0,1]$$

and set $F'(0) := 0$, so F' is continuous at every point of $(0,1]$. Therefore both F' and $|F'|$ are integrable on every closed subinterval in $(0,1]$. If we let

$$a_k := 2/(2k+1) \qquad \text{and} \qquad b_k := 1/k \qquad \text{for} \quad k \in \mathbb{N},$$

then $F(a_k) = 0$ and $F(b_k) = (-1)^k/k$. Moreover, $0 < a_k < b_k < a_{k-1} < b_{k-1} < 1$ for $k > 1$. The Fundamental Theorem 4.5 and Corollary 3.5 imply that

$$\frac{1}{k} = |F(b_k) - F(a_k)| = \left| \int_{a_k}^{b_k} F' \right| \le \int_{a_k}^{b_k} |F'|.$$

If $|F'| \in \mathcal{R}^*([0,1])$, we would have (by 3.2 and 3.9) the inequality

$$\sum_{k=1}^{n} \frac{1}{k} \le \sum_{k=1}^{n} \int_{a_k}^{b_k} |F'| \le \int_0^1 |F'| \qquad \text{for all} \quad n \in \mathbb{N}.$$

However, from the divergence of the harmonic series we infer that $|F'|$ does not belong to $\mathcal{R}^*([0,1])$ so that F' is not absolutely integrable.

Characterization of Absolute Integrability

To establish the absolute integrability of a function, it is often useful to look at its indefinite integrals. It turns out that the absolutely integrable functions are precisely those integrable functions whose indefinite integrals do not "oscillate too much" in a sense we will now define.

• **7.3 Definition.** Let $\varphi : [a, b] \to \mathbb{R}$. We define the **variation** of φ over the compact interval $I := [a, b]$ to be

$$(7.\alpha) \qquad \mathrm{Var}(\varphi; I) := \sup\Big\{\sum_{i=1}^{n} |\varphi(x_i) - \varphi(x_{i-1})| : \mathcal{P} = \{x_i\}_{i=1}^{n}\Big\} \le \infty,$$

where the supremum is taken over all partitions $\mathcal{P}$ of I. We say that the function φ has (or is of) **bounded variation on** I if $\mathrm{Var}(\varphi; I) < \infty$. The collection of all functions on I that have bounded variation on I is denoted by $BV(I)$.

7.4 Examples. **(a)** If $\varphi : I \to \mathbb{R}$ is an increasing function, then

$$|\varphi(x_i) - \varphi(x_{i-1})| = \varphi(x_i) - \varphi(x_{i-1}),$$

so that we obtain a telescoping sum:

$$\sum_{i=1}^{n} |\varphi(x_i) - \varphi(x_{i-1})| = \sum_{i=1}^{n} \{\varphi(x_i) - \varphi(x_{i-1})\} = \varphi(b) - \varphi(a).$$

Therefore, it follows that $\mathrm{Var}(\varphi; I) = \varphi(b) - \varphi(a)$. Similarly, if $\psi : I \to \mathbb{R}$ is a decreasing function, then $\mathrm{Var}(\psi; I) = \psi(a) - \psi(b)$. Therefore, monotone functions on I belong to $BV(I)$.

(b) Define $f : [0, 1] \to \mathbb{R}$ by $f(1) := 0$ and $f(x) := (-1)^{k+1}$ for all x in $[c_{k-1}, c_k)$, where $c_k := 1 - 1/2^k$, $k \in \mathbb{N}$. (See Figure 7.1 on the next page.)

We claim that f does not belong to $BV([0, 1])$. To see this, let $m \in \mathbb{N}$ and choose the partition:

$$x_0 := 0, \quad x_1 := c_1, \quad x_2 := c_2, \quad \cdots, \quad x_{m-1} := c_{m-1}, \quad x_m := 1.$$

Then $|f(x_1) - f(x_0)| = 2, \quad |f(x_2) - f(x_1)| = 2, \ \cdots, \ |f(x_m) - f(x_{m-1})| = 1,$ so that we have

$$\mathrm{Var}(f; [0, 1]) \ge 1 + 2(m - 1) = 2m - 1.$$

Since $m \in \mathbb{N}$ is arbitrary, we see that $\mathrm{Var}(f; [0, 1]) = \infty$ and the bounded function f does *not* have bounded variation on $[0, 1]$.

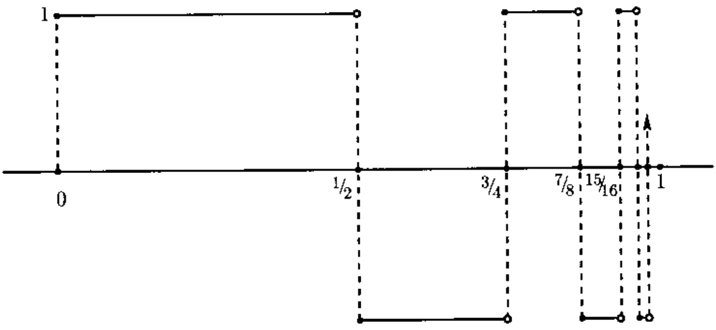

Figure 7.1 Graph of f.

Remark. Example 7.4(a) shows that a function in $BV(I)$ need not be continuous on I.

It is an exercise to show that if $\varphi : I \to \mathbb{R}$ has bounded variation on I, and if $k \in \mathbb{R}$, then the multiple $k\varphi$ also has bounded variation on I. Similarly, the sum or difference of two functions that have bounded variation on I also have bounded variation on I. Conversely, it can be shown that: *Any function that has bounded variation on a compact interval I is the difference of two increasing functions on I*. This is an important theorem, due to Camille Jordan (1838–1922), and facilitates working with functions of bounded variation. However, since we believe that its proof should be worked out by the reader, we leave it as Exercise 7.J.

We now show that the absolutely integrable functions are precisely those integrable functions whose indefinite integrals have bounded variation.

• **7.5 Characterization of Absolute Integrability.** *Let $f \in \mathcal{R}^*([a,b])$. Then $|f|$ is integrable if and only if the indefinite integral $F(x) := \int_a^x f$ has bounded variation on I. In this case,*

$$\int_I |f| = \mathrm{Var}(F; I). \tag{7.β}$$

Proof. ($\Rightarrow$) If $|f|$ is integrable and if $\mathcal{Q} := \{\xi_0, \xi_1, \cdots, \xi_m\}$ is any partition of I, then Corollaries 3.5 and 3.9 imply that

$$\sum_{i=1}^{m} |F(\xi_i) - F(\xi_{i-1})| = \sum_{i=1}^{m} \Big| \int_{\xi_{i-1}}^{\xi_i} f \Big| \le \sum_{i=1}^{m} \int_{\xi_{i-1}}^{\xi_i} |f| = \int_a^b |f|.$$

Therefore, the indefinite integral $F \in BV(I)$ and $\mathrm{Var}(F; I) \le \int_I |f|$.

($\Leftarrow$) Although the proof is a bit delicate, the strategy is easy to grasp: First find a partition $\mathcal{Q}$ whose sum approximates $\mathrm{Var}(F;I)$; then use the partition points in $\mathcal{Q}$ to generate finer partitions whose Riemann sums for $|f|$ are close to $\mathrm{Var}(F;I)$.

Suppose that the indefinite integral $F \in BV(I)$ so that $\mathrm{Var}(F;I) < \infty$. Given $\varepsilon > 0$, let $\mathcal{Q} := \{\xi_0, \xi_1, \cdots, \xi_m\}$ be a partition of I such that

$$\mathrm{Var}(F;I) - \varepsilon \le \sum_{i=1}^{m} |F(\xi_i) - F(\xi_{i-1})| \le \mathrm{Var}(F;I).$$

We note that if $\xi^* \in (\xi_{i-1}, \xi_i)$, then it follows from the additivity of the integral (Theorem 3.7) and the Triangle Inequality that

$$|F(\xi_i) - F(\xi_{i-1})| \le |F(\xi^*) - F(\xi_{i-1})| + |F(\xi_i) - F(\xi^*)|.$$

By induction, we conclude that if we adjoin a finite number of additional points to the partition $\mathcal{Q}$, then the sum corresponding to $\sum |F(\xi_i) - F(\xi_{i-1})|$ will be made larger, but still will not exceed the number $\mathrm{Var}(F;I)$, which is the supremum of all such sums.

Now let δ_ε be a gauge such that for any δ_ε-fine partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n}$ of I we have $|S(f;\dot{\mathcal{P}}) - \int_a^b f| \le \varepsilon$. If we apply Corollary 5.5 of the Saks-Henstock Lemma to the entire partition $\dot{\mathcal{P}}$ we infer that

$$(7.\gamma) \qquad \Big| \sum_{i=1}^{n} |f(t_i)| l(I_i) - \sum_{i=1}^{n} \Big| \int_{I_i} f \Big| \Big| \le 2\varepsilon.$$

Let $E := \{\xi_i : i = 0, 1, \cdots, m\}$ and define a gauge δ_ε^* on $[a,b]$ by

$$(7.\delta) \qquad \delta_\varepsilon^*(t) := \min\{\delta_\varepsilon(t), \tfrac{1}{2}\,\mathrm{dist}(t, E - \{t\})\}.$$

If a partition $\dot{\mathcal{P}}^*$ is δ_ε^*-fine, then it is also δ_ε-fine so that $(7.\gamma)$ also holds for $\dot{\mathcal{P}}^*$. Also every point $\xi_j \in E$ is a tag of at least one subinterval in $\dot{\mathcal{P}}^*$ and we use the right-left procedure to obtain the points $\xi_1, \cdots, \xi_{m-1}$ as tags for two subintervals by adding a finite number of points to the partition points in $\dot{\mathcal{P}}^*$. Now let u_i, τ_i $(i = 1, \cdots, p)$ denote the partition points and the tags in $\dot{\mathcal{P}}^*$, and let $J_i := [u_{i-1}, u_i]$. As observed above, we have

$$(7.\varepsilon) \quad \mathrm{Var}(F;I) - \varepsilon \le \sum_{i=1}^{p} |F(u_i) - F(u_{i-1})| = \sum_{i=1}^{p} \Big| \int_{J_i} f \Big| \le \mathrm{Var}(F;I).$$

If we combine the inequalities $(7.\gamma)$ and $(7.\varepsilon)$, we obtain

$$\begin{aligned} |S(|f|;\dot{\mathcal{P}}^*) - \mathrm{Var}(F;I)| &\le \Big| \sum_{i=1}^{p} |f(\tau_i)| l(J_i) - \sum_{i=1}^{p} \Big| \int_{J_i} f \Big| \Big| \\ &\quad + \Big| \sum_{i=1}^{p} \Big| \int_{J_i} f \Big| - \mathrm{Var}(F;I) \Big| \le 2\varepsilon + \varepsilon = 3\varepsilon, \end{aligned}$$

provided that $\dot{\mathcal{P}}^*$ is δ_ε^*-fine. Since $\varepsilon > 0$ is arbitrary, we conclude that $|f|$ is integrable with integral $\mathrm{Var}(F; I)$. Q.E.D.

7.6 Examples. **(a)** Let $\kappa : [0,1] \to \mathbb{R}$ denote the function considered in Example 2.8(a). We consider its indefinite integral $K(x) := \int_0^x \kappa$ with base point 0. (See Figure 7.2.)

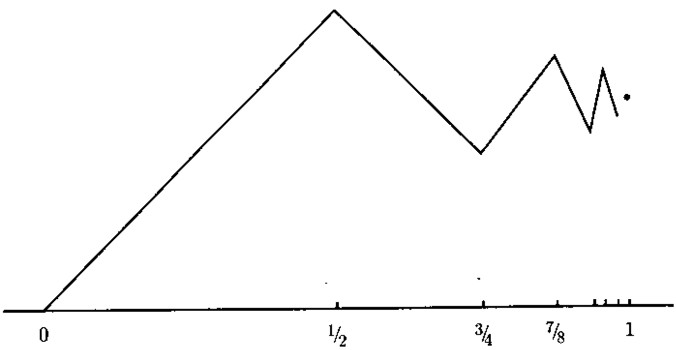

Figure 7.2 Graph of K.

It is easily seen that $K(0) = 0$ and $K(c_n) = \sum_{k=1}^n (-1)^{k+1}/k$ where $c_n := 1 - 1/2^n$, $n \in \mathbb{N}$, so that

$$|K(c_n) - K(c_{n-1})| = 1/n \tag{7.ζ}$$

when $n \geq 1$. Therefore, if $\mathcal{P}$ is the partition of $[0,1]$ with the partition points

$$y_0 := 0, \quad y_1 := c_1, \quad y_2 := c_2, \quad \cdots, \quad y_{n-1} := c_{n-1}, \quad y_n := 1,$$

then we have

$$\sum_{i=1}^n |K(y_i) - K(y_{i-1})| \geq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}.$$

Therefore $K \notin BV([0,1])$, so that κ is not absolutely integrable on $[0,1]$, as was seen in Example 2.8(a).

(b) Let $F(x) := x\cos(\pi/x)$ for $x \in (0,1]$ and $F(0) := 0$. We have seen in Example 7.2(b) that F' is not absolutely integrable on $[0,1]$. We will now

show that F does not have bounded variation, giving another proof of this assertion. In fact, for each $n > 2$, consider the partition

$$x_0 := 0, \quad x_1 := \frac{1}{n}, \quad x_2 := \frac{1}{n-1}, \quad \cdots, \quad x_{n-1} := \frac{1}{2}, \quad x_n := 1.$$

Since $F(x_0) = 0$, $F(x_1) = (-1)^n/n$, $F(x_2) = (-1)^{n-1}/(n-1), \cdots$, $F(x_{n-1}) = 1/2$, $F(x_n) = -1$, it follows that

$$\begin{aligned}
&|F(x_1) - F(x_0)| + |F(x_2) - F(x_1)| + \cdots + |F(x_n) - F(x_{n-1})| \\
&\quad = \frac{1}{n} + \Big(\frac{1}{n} + \frac{1}{n-1}\Big) + \Big(\frac{1}{n-1} + \frac{1}{n-2}\Big) + \cdots + \Big(1 + \frac{1}{2}\Big) \\
&\quad = 1 + 2\Big(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\Big).
\end{aligned}$$

Since $n \in \mathbb{N}$ is arbitrary, it follows from the divergence of the harmonic series that $F \notin BV([0,1])$. Hence F' is not absolutely integrable on $[0,1]$.

(c) Let $G(x) := x^2 \cos(\pi/x^2)$ for $x \in (0,1]$ and $G(0) := 0$. Then G has a derivative at every point of $[0,1]$; indeed,

$$G'(x) = 2x\cos(\pi/x^2) + (2\pi/x)\sin(\pi/x^2) \qquad \text{for} \quad x \in (0,1]$$

and $G'(0) = 0$. However G does not have bounded variation on $[0,1]$. For, if $n \in \mathbb{N}$, consider the partition

$$x_0 := 0, \quad x_1 := \frac{1}{\sqrt{n}}, \quad x_2 := \frac{1}{\sqrt{n-1}}, \quad \cdots, \quad x_{n-1} := \frac{1}{\sqrt{2}}, \quad x_n := 1.$$

Since $G(x_0) = 0$, $G(x_1) = (-1)^n/n$, $G(x_2) = (-1)^{n-1}/(n-1), \cdots$, $G(x_{n-1}) = 1/2$, $G(x_n) = -1$, it follows that

$$\begin{aligned}
&|G(x_1) - G(x_0)| + |G(x_2) - G(x_1)| + \cdots + |G(x_n) - G(x_{n-1})| \\
&\qquad = \frac{1}{n} + \Big(\frac{1}{n} + \frac{1}{n-1}\Big) + \Big(\frac{1}{n-1} + \frac{1}{n-2}\Big) + \cdots + \Big(1 + \frac{1}{2}\Big) \\
&\qquad = 1 + 2\Big(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\Big).
\end{aligned}$$

Since $n \in \mathbb{N}$ is arbitrary, then $G \notin BV([0,1])$.

The Comparison Test

The next result is a major tool for establishing absolute integrability.

• **7.7 Comparison Test for Absolute Integrability.** *If $f, g \in \mathcal{R}^*(I)$ and $|f(x)| \le g(x)$ for $x \in I := [a,b]$, then $f \in \mathcal{L}(I)$. Moreover, we have*

$$(7.\eta) \qquad \Big|\int_I f\Big| \le \int_I |f| \le \int_I g.$$

Proof. Let $F(x) := \int_a^x f$ so that if $\mathcal{P} := \{[x_{i-1}, x_i]\}_{i=1}^n$ is a partition of I, then (Corollary 3.3) we have

$$|F(x_i) - F(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} f \right| \leq \int_{x_{i-1}}^{x_i} g.$$

Therefore it follows that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \leq \int_a^b g,$$

whence $\operatorname{Var}(F; I) \leq \int_a^b g$. Theorem 7.5 implies that $|f| \in \mathcal{R}^*(I)$ and that $\int_I |f| \leq \int_I g$. The inequality (7.η) now follows from Corollary 3.5. Q.E.D.

Properties of Absolutely Integrable Functions

The hard work has been done for this section. We now obtain some useful corollaries. First we will show that the set $\mathcal{L}(I)$ of absolutely integrable functions on a compact interval forms a vector space (in the sense of Appendix F).

• **7.8 Theorem.** *If $f, g : I \to \mathbb{R}$ are in $\mathcal{L}(I)$ and if $c \in \mathbb{R}$, then cf and $f + g$ are in $\mathcal{L}(I)$.*

Proof. Since f and $|f|$ are integrable on I, the first assertion follows from the observation that $|cf|(x) = |c|\,|f(x)|$ for all $x \in I$. Since $|f|$ and $|g|$ are integrable on I, it follows from Theorem 3.1 that $|f| + |g|$ is also integrable on I. But by the Triangle Inequality, we have $|f + g| \leq |f| + |g|$, whence the conclusion follows from the Comparison Test 7.7. Q.E.D.

• **7.9 Theorem.** *If $f \in \mathcal{R}^*(I)$, the following assertions are equivalent:*

(a) $f \in \mathcal{L}(I)$.

(b) *There exists $\omega \in \mathcal{L}(I)$ such that $f(x) \leq \omega(x)$ for all $x \in I$.*

(c) *There exists $\alpha \in \mathcal{L}(I)$ such that $\alpha(x) \leq f(x)$ for all $x \in I$.*

Proof. (a) $\Rightarrow$ (b) Let $\omega := f$.

(b) $\Rightarrow$ (a) Note that $f = \omega - (\omega - f)$. Since $\omega - f$ is integrable and $\omega - f \geq 0$, it follows that $\omega - f \in \mathcal{L}(I)$. Now apply Theorem 7.8.

We leave it to the reader to show that (a) $\Leftrightarrow$ (c). Q.E.D.

We now give some elementary conditions that imply that $f \in \mathcal{L}(I)$.

7.10 Corollary. *If $f \in \mathcal{R}^*(I)$, the following assertions imply $f \in \mathcal{L}(I)$:*

(a) *f is bounded above on I.*

(b) *f is bounded below on I.*

(c) *f is bounded on I.*

Proof. A constant function is in $\mathcal{L}(I)$. Q.E.D.

We recall from Definition 6.4 the meaning of the functions denoted by:

$$f \vee g = \max\{f, g\}, \quad f \wedge g = \min\{f, g\}, \quad f^+ \quad \text{and} \quad f^-.$$

• **7.11 Theorem.** *If $f \in \mathcal{R}^*(I)$, then the following are equivalent:*

(a) *$f \in \mathcal{L}(I)$.*

(b) *f^+ and f^- are in $\mathcal{R}^*(I)$.*

(c) *f^+ and f^- are in $\mathcal{L}(I)$.*

Proof. (a) $\Rightarrow$ (b) Apply Lemma 6.5(b) and Theorem 3.1.

(b) $\Rightarrow$ (c) Both $f^+ \geq 0$ and $f^- \geq 0$, so they belong to $\mathcal{L}(I)$.

(c) $\Rightarrow$ (a) Apply Lemma 6.5(c) and Theorem 7.8. Q.E.D.

• **7.12 Theorem.** *If $f, g \in \mathcal{R}^*(I)$, then the following are equivalent:*

(a) *f and g are in $\mathcal{L}(I)$.*

(b) *$f \vee g$ is in $\mathcal{L}(I)$.*

(c) *$f \wedge g$ is in $\mathcal{L}(I)$.*

Proof. (a) $\Rightarrow$ (b) Apply Lemma 6.5(a) and Theorem 7.8.

(b) $\Rightarrow$ (a) Since $f, g \leq f \vee g$, we may apply Theorem 7.9.

We leave the proof that (a) $\Leftrightarrow$ (c) to the reader. Q.E.D.

The final result applies to functions that are merely integrable; it will be used in Section 8.

• **7.13 Theorem.** *Let $f, g, \alpha, \omega \in \mathcal{R}^*(I)$.*

(a) *If $f \leq \omega$ and $g \leq \omega$, then $f \vee g$ and $f \wedge g$ are in $\mathcal{R}^*(I)$.*

(b) *If $\alpha \leq f$ and $\alpha \leq g$, then $f \vee g$ and $f \wedge g$ are in $\mathcal{R}^*(I)$.*

Proof. (a) The hypothesis implies that $f \vee g \leq \omega$. From Lemma 6.5(a) we have that $0 \leq |f - g| = 2f \vee g - f - g \leq 2\omega - f - g$. From the Comparison Test 7.7 we have that $|f - g|$ is integrable. Now apply Lemma 6.5(a) and Theorem 3.1 to conclude that $f \vee g$ and $f \wedge g$ are integrable.

(b) We leave the proof to the reader. Q.E.D.

Remark. If f and g are integrable, then in general it is *not true* that $f \vee g$ and $f \wedge g$ are integrable. For example, if f is conditionally integrable, then *neither* $f^+ = f \vee 0$ *nor* $f^- = (-f) \vee 0$ is integrable.

Exercises

7.A Suppose that $\varphi \in BV([a,b])$.

(a) Show that $\mathrm{Var}(\varphi;[a,b]) = 0$ if and only if $\varphi(x) = \varphi(a)$ for all $x \in [a,b]$.

(b) Show that $|\varphi(b) - \varphi(a)| \le \mathrm{Var}(\varphi;[a,b])$.

7.B Suppose that $\varphi \in BV([a,b])$ and that $a \le x \le y \le b$.

(a) Show that $|\varphi(x) - \varphi(y)| \le \mathrm{Var}(\varphi;[x,y]) \le \mathrm{Var}(\varphi;[a,b])$.

(b) Show that $\mathrm{Var}(\varphi;[a,x]) + |\varphi(x) - \varphi(y)| \le \mathrm{Var}(\varphi;[a,y])$.

7.C If $\varphi \in BV([a,b])$, show that $|\varphi(y)| \le |\varphi(a)| + \mathrm{Var}(\varphi;[a,b])$ for all $y \in [a,b]$. Thus: *Every function in $BV([a,b])$ is bounded on the interval $[a,b]$.*

7.D (a) If $\varphi : [a,b] \to \mathbb{R}$ satisfies the Lipschitz condition $|\varphi(x) - \varphi(y)| \le K|x-y|$ for some $K > 0$ and all $x, y \in [a,b]$, show that $\varphi \in BV([a,b])$ and that $\mathrm{Var}(\varphi;[a,b]) \le K(b-a)$.

(b) Give an example of a continuous function in $BV([a,b])$ that does not satisfy a Lipschitz condition.

7.E Let $\varphi, \psi \in BV([a,b])$ and let $k \in \mathbb{R}$.

(a) Show that $k\varphi \in BV([a,b])$ and $\mathrm{Var}(k\varphi;[a,b]) = |k|\,\mathrm{Var}(\varphi;[a,b])$.

(b) Show that $\varphi \pm \psi \in BV([a,b])$ and $\mathrm{Var}(\varphi \pm \psi;[a,b]) \le \mathrm{Var}(\varphi;[a,b]) + \mathrm{Var}(\psi;[a,b])$.

(c) Show that the product $\varphi\psi \in BV([a,b])$ and that $\mathrm{Var}(\varphi\psi;[a,b]) \le M\{\mathrm{Var}(\varphi;[a,b]) + \mathrm{Var}(\psi;[a,b])\}$, where $|\varphi(x)| \le M$ and $|\psi(x)| \le M$ for all $x \in [a,b]$.

(d) If $\varphi \in BV([a,b])$ and $|\varphi(x)| \ge m$ for some $m > 0$ and all $x \in [a,b]$, show that $1/\varphi \in BV([a,b])$ and $\mathrm{Var}(1/\varphi;[a,b]) \le (1/m^2)\,\mathrm{Var}(\varphi;[a,b])$.

7.F (a) Show that if $\varphi \in BV([a,b])$, then $|\varphi| \in BV([a,b])$.

(b) Give an example of a function $\varphi : [a, b] \to \mathbb{R}$ such that $\varphi \notin BV([a, b])$, but such that $|\varphi| \in BV([a, b])$.

(c) Show that if $\varphi : [a, b] \to \mathbb{R}$ is continuous, then $|\varphi| \in BV([a, b])$ implies that φ belongs to $BV([a, b])$.

7.G Let $\varphi \in BV([a, b])$ and let $c \in (a, b)$. Show that $\varphi \in BV([a, b])$ if and only if $\varphi \in BV([a, c]) \cap BV([c, b])$.

7.H Let $\varphi : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$. If $\varphi \in BV([a, b])$, show that

$$\text{(7.}\theta\text{)} \qquad \operatorname{Var}(\varphi; [a, b]) = \operatorname{Var}(\varphi; [a, c]) + \operatorname{Var}(\varphi; [c, b]).$$

[Hint: To establish $\leq$, we let $\mathcal{P}$ be a partition of $[a, b]$ and consider $\mathcal{P} \cup \{c\}$. To show $\geq$, let $\mathcal{P}_1$ and $\mathcal{P}_2$ be suitable partitions of $[a, c]$ and $[c, b]$ and consider $\mathcal{P}_1 \cup \mathcal{P}_2$.]

7.I (a) Show that $\operatorname{Var}(\sin x; [0, \pi]) = 2$.

(b) Evaluate $\operatorname{Var}(x^2; [-1, 1])$.

(c) Let $\varphi(x) := x \sin(\pi/x)$ for $x \in (0, 1]$ and $\varphi(0) := 0$. Show that $\operatorname{Var}(\varphi; [0, 1]) = \infty$.

(d) Evaluate $\operatorname{Var}(\sin 2x; [0, 2\pi])$.

(e) Evaluate $\operatorname{Var}(|\cos x|; [0, 6\pi])$.

7.J A theorem of Camille Jordan asserts: *A function is in $BV([a, b])$ if and only if it is the difference of two increasing functions on $[a, b]$.* One direction follows immediately from Exercise 7.E(b). We are concerned here with the other direction.

(a) Let $\varphi \in BV([a, b])$, $V(x) := \operatorname{Var}(\varphi; [a, x])$ and $W(x) := V(x) - \varphi(x)$ for $x \in [a, b]$. Show that V and W are increasing functions on $[a, b]$ and that $\varphi = V - W$. [Hint: Use Exercises 7.B and 7.H.]

(b) Let V be as in (a) and let $Y(x) := V(x) + \varphi(x)$. Show that Y is an increasing function on $[a, b]$ and $\varphi = Y - V$.

7.K (a) A function in $BV([a, b])$ can be written as the difference of increasing functions in infinitely many ways. For, if $\varphi = v - w$, where v, w are increasing, and if h is any increasing function on $[a, b]$, then $\varphi = v_h - w_h$, where $v_h := v + h$ and $w_h := w + h$.

(b) If v, w are as in (a), show that $\operatorname{Var}(\varphi; [a, b]) \leq v\big|_a^b + w\big|_a^b$.

(c) Let $v_0(x) := \frac{1}{2}\{\operatorname{Var}(\varphi; [a, x]) + \varphi(x)\}$ and $w_0(x) := \frac{1}{2}\{\operatorname{Var}(\varphi; [a, x]) - \varphi(x)\}$ for $x \in [a, b]$. Show that v_0 and w_0 are increasing functions and that $\varphi = v_0 - w_0$ and $V = v_0 + w_0$.

(d) Show that the decomposition $\varphi = v_0 - w_0$, given in part (c), is *minimal* in the sense that $v_0\big|_a^b \le v\big|_a^b$ and $w_0\big|_a^b \le w\big|_a^b$ for any other decomposition $\varphi = v - w$ of φ into the difference of increasing functions.

7.L Let $\varphi \in BV([a,b])$ with $\varphi(0) = 0$ and suppose that the derivative φ' exists on $[a,b]$. Show that $\varphi' \in \mathcal{L}([a,b])$ and that the minimal increasing functions v_0, w_0 for φ (cf. Exercise 7.K) are given by $v_0(x) = \int_a^x (\varphi')^+$ and $w_0(x) = \int_a^x (\varphi')^-$ for $x \in [a,b]$.

7.M Let $\varphi \in BV([a,b])$ and let $c \in [a,b)$.

(a) If $V(x) := \mathrm{Var}(\varphi; [a,x])$ is right [resp., left] continuous at $c \in [a,b)$, show that φ is right [resp., left] continuous at c. [Hint: Use Exercise 7.B(b).]

(b) If φ is right continuous at $c \in [a,b)$, show that V is right continuous at c. [Hint: Given $\varepsilon > 0$ let $\delta > 0$ be such that if $y \in [c, c+\delta]$ then $|\varphi(y) - \varphi(c)| < \varepsilon$. Let $\mathcal{P} = \{x_i\}$ be a partition of $[c,b]$ with $|x_1 - c| \le \delta$ and $\mathrm{Var}(\varphi; [c,b]) \le \sum |\varphi(x_i) - \varphi(x_{i-1})| + \varepsilon$. Show that $0 \le V(y) - V(c) < 2\varepsilon$.]

(c) Show that if $\varphi \in BV([a,b])$ and $c \in (a,b)$, then φ is continuous at c if and only if V is continuous at c.

7.N (a) Let $(\varphi_k)_{k=1}^{\infty}$ be a sequence in $BV([a,b])$ with $\varphi(x) = \lim_k \varphi_k(x)$ for all $x \in [a,b]$. Give an example to show that φ is not necessarily in $BV([a,b])$.

(b) Show that the sequence in (a) can be uniformly convergent. (See Definition 8.2.)

(c) Suppose that (φ_k) is as in (a) and that $\mathrm{Var}(\varphi_k; [a,b]) \le K$ for some $K > 0$ and all $k \in \mathbb{N}$. Show that $\varphi \in BV([a,b])$.

7.O Express the following functions as differences of increasing functions:

(a) $\sin x$ on $[0, 2\pi]$.

(b) $\cos x$ on $[0, 2\pi]$.

(c) $\sin^2 x$ on $[0, 2\pi]$.

(d) $x - 2\lfloor x \rfloor$ on $[0, 4]$.

(e) $\sin x - x\cos x$ on $[0, 4\pi]$.

7.P Show that there exists a continuous function $\varphi \in BV([a,b])$ such that for every partition $\mathcal{P} = \{x_i\}$ of $[a,b]$, one has $\sum |\varphi(x_i) - \varphi(x_{i-1})| <$

$\mathrm{Var}(\varphi; [a, b])$. Thus, the variation is not always "attained" by any partition $\mathcal{P}$ even when φ is continuous.

7.Q If K is the indefinite integral in Example 7.6(a), determine the slope of the graph of K on the interval (c_{n-1}, c_n).

7.R Give a specific example of a function $f \in \mathcal{R}^*([a, b])$ such that neither f^+ nor f^- is in $\mathcal{L}([a, b])$. Can you give an example of an integrable function g such that g^+ is integrable, but g^- is not?

7.S If any two of the functions $f, |f|, f^+, f^-$ are in $\mathcal{R}^*([a, b])$, show that all four of them are in $\mathcal{L}([a, b])$.

7.T Give an example of functions $f, g \in \mathcal{R}^*([a, b])$ such that neither $f \wedge g$ nor $f \vee g$ is in $\mathcal{R}^*([a, b])$.

7.U Give an example of $f, g \in \mathcal{R}^*([a, b])$ such that f and g are not absolutely integrable, but $f + g$ is absolutely integrable.

7.V If $f \in \mathcal{L}([a, b])$ and $n \in \mathbb{N}$, show (without using Theorem 6.9 or Corollary 6.10) that the n-truncate $f^{[n]} := \mathrm{mid}\{-n, f, n\}$ belongs to $\mathcal{L}([a, b])$.

7.W Show that a complex-valued function is in $BV([a, b])$ if and only if its real and imaginary parts are in $BV([a, b])$.

Convergence Theorems

Most functions that are encountered in elementary calculus are either (piecewise) monotone or (piecewise) continuous; in any case they are generally R-integrable. However, even in relatively elementary mathematical applications it is necessary to consider the limits of sequences or series of functions, and these limiting relations often give rise to more complicated functions. Indeed, the limit f of a sequence (f_k) of R-integrable functions on an interval I *may not* be R-integrable. Further, even when this limit is R-integrable, its R-integral *may not* be the limit of the sequence $(\int_I f_k)$ of integrals. These assertions will now be shown by some simple examples.

8.1 Examples. **(a)** Let $\mathbb{Q}_1 := \{r_1, r_2, \cdots\}$ be an enumeration of the rational numbers in $[0,1]$. We let $f_k(x) := 1$ if $x = r_1, \cdots, r_k$, and $f_k(x) := 0$ otherwise. Then the sequence (f_k) converges at every point of $[0,1]$ to the Dirichlet discontinuous function defined by $f(x) := 1$ if $x \in \mathbb{Q}_1$ and $f(x) := 0$ otherwise.

Although all of the functions f_k are R-integrable with R-integral equal to 0, the limit function is not R-integrable (although it was seen in Example 2.3(a) to be in $\mathcal{R}^*([0,1])$ with integral equal to 0).

(b) Let $g_k : [0,1] \to \mathbb{R}$ $(k > 2)$ be defined by

$$g_k(x) := \begin{cases} k^2 x & \text{for} \quad 0 \le x \le 1/k, \\ -k^2(x - 2/k) & \text{for} \quad 1/k < x \le 2/k, \\ 0 & \text{for} \quad 2/k < x \le 1. \end{cases}$$

[Note that g_k is a piecewise linear continuous function, whose graph passes through the points $(0,0)$, $(1/k,k)$, $(2/k,0)$, and $(1,0)$.]

We claim that the sequence (g_k) converges at every point in $[0,1]$ to the zero function $g(x) := 0$. Indeed, $g_k(0) = 0$ for all $k \in \mathbb{N}$; also, if $x > 0$, then there exists $k_x \in \mathbb{N}$ such that $k_x \geq 2/x$, whence $g_k(x) = 0$ for all $k \geq k_x$. Therefore $\lim g_k(x) = 0$ for all $x \in [0,1]$. Since all of the g_k are continuous, they are R-integrable, and so is the limit function. However, $\int_0^1 g_k = 1$ for all $k \in \mathbb{N}$, while $\int_0^1 g = 0$.

(c) Lest the reader get the impression that the source of the difficulties exhibited in the preceding examples springs from the "artificiality" in the definition of the functions, we offer an additional example. Let $h_k(x) := 2kxe^{-kx^2}$ for $k \in \mathbb{N}$, and let $h(x) := 0$ for $x \in [0,1]$. Since $H_k(x) := -e^{-kx^2}$ is a primitive of h_k on $[0,1]$, then

$$\int_0^1 h_k = H_k\Big|_0^1 = 1 - e^{-k} \to 1.$$

However, $\int_0^1 h = 0$, so that $\int_0^1 h = 0 \neq 1 = \lim \int_0^1 h_k$.

Example 8.1(a) gives a *bounded* sequence of R-integrable functions whose limit is not R-integrable. Both Examples 8.1(b, c) are sequences of *continuous* functions that converge at every point of the interval to a continuous function; however, the R-integral of the limit is *not* the limit of the sequence of R-integrals. A key to the source of difficulty in these examples is the fact that both functions become very large at certain points. For example, in Example 8.1(b) we have $g_k(1/k) = k$, and in Example 8.1(c) we have $h_k(1/\sqrt{2k}\,) = \sqrt{2k/e}$.

The first difficulty in 8.1(a) is one that can be cured by enlarging the class of integrable functions. However, Examples 8.1(b, c) cannot be cured by such an enlargement, since the functions involved are continuous. Instead, we will need to control the mode of convergence of the sequence of functions in order to be able to take limits with confidence.

In 1885, the Italian mathematician Cesare Arzelà (1847–1912) proved a useful theorem for the R-integral: *Suppose that the functions $f_k : I \to \mathbb{R}$ are R-integrable, that they converge at every point of I to an R-integrable function f, and that the sequence is bounded on $I := [a,b]$ (that is, there exists $B > 0$ such that $|f_k(x)| \leq B$ for all $x \in I$). Then $\int_I f = \lim \int_I f_k$.*

Note that Arzelà's theorem does not apply to Example 8.1(a), since the limit function is not R-integrable. It also does not apply to Examples 8.1(b, c), since these sequences are not bounded.

The Limit Theorems

In the remainder of this section we will present a number of theorems that give conditions under which the limit of a sequence of R*-integrable functions is integrable and the integral of the limit function equals the limit of the sequence of integrals.

In order to facilitate reading, we will present these theorems roughly in the order of *increasing* generality. Our first result is the most familiar one and has the strongest hypothesis (uniform convergence); this result will be familiar to the reader from the Riemann integral. We then establish versions of the Monotone Convergence Theorem, Fatou's Lemma and the Dominated and Mean Convergence Theorems, that are extensions to our integral of results from the Lebesgue theory of integration. These results are very important results for later work. After that we will give a sufficient condition (called equi-integrability) that is intrinsic to the generalized Riemann integral. Finally, we present two results that are of interest chiefly because they give necessary and sufficient conditions; however, they are not easy to apply.

Uniform Convergence

We have seen in Example 8.1(a) that the limit of a sequence of integrable functions is not always integrable. We now show that there is a familiar condition that will guarantee the integrability of the limit function.

• **8.2 Definition.** A sequence (f_k) of functions on an interval I to $\mathbb{R}$ is said to **converge uniformly** (or to be **uniformly convergent**) on I to a function f if, for every $\varepsilon > 0$ there exists $K_\varepsilon \in \mathbb{N}$ such that if $k \geq K_\varepsilon$ and $x \in I$, then $|f_k(x) - f(x)| \leq \varepsilon$.

We will assume that the reader has some familiarity with this notion of convergence. While uniform convergence is a very severe restriction, it remains an important mode of convergence. We recall that if a sequence of R-integrable functions on a compact interval I converges uniformly on I to a function f on I, then f is R-integrable on I and $\int_I f = \lim \int_I f_k$ (see [B-S; p.237]). We now prove a generalization of that result to functions in $\mathcal{R}^*(I)$.

8.3 Uniform Convergence Theorem. *If the sequence $(f_k) \in \mathcal{R}^*(I)$ converges to f uniformly on $I := [a, b]$, then $f \in \mathcal{R}^*(I)$ and*

$$(8.*) \qquad \int_I f = \lim_{k\to\infty} \int_I f_k.$$

Proof. Given $\varepsilon > 0$, there exists K_ε such that if $k \geq K_\varepsilon$ and $x \in I$, then $|f_k(x) - f(x)| \leq \varepsilon$. Consequently, if $h, k \geq K_\varepsilon$, then

$$-2\varepsilon \leq f_h(x) - f_k(x) \leq 2\varepsilon \qquad \text{for} \quad x \in [a, b].$$

Corollary 3.3 and Theorem 3.1 imply that

$$-2\varepsilon(b-a) \leq \int_I f_h - \int_I f_k \leq 2\varepsilon(b-a),$$

whence $|\int_I f_h - \int_I f_k| \leq 2\varepsilon(b-a)$. Since $\varepsilon > 0$ is arbitrary, the sequence $(\int_I f_k)$ is a Cauchy sequence in $\mathbb{R}$ and so converges to some number, say $A \in \mathbb{R}$.

We now show that $f \in \mathcal{R}^*(I)$ with integral A. Let $\varepsilon > 0$ be given and let K_ε be as above. If $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is any tagged partition of I and $k \geq K_\varepsilon$, then

$$\begin{aligned}\left|S(f_k;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})\right| &= \left|\sum_{i=1}^n \{f_k(t_i) - f(t_i)\} l(\dot{I}_i)\right| \\ &\leq \sum_{i=1}^n \left|f_k(t_i) - f(t_i)\right| l(I_i) \leq \sum_{i=1}^n \varepsilon l(I_i) = \varepsilon(b-a).\end{aligned}$$

Now choose a fixed number $r \geq K_\varepsilon$ such that $|\int_I f_r - A| < \varepsilon$. Let $\delta_{r,\varepsilon}$ be a gauge on I such that $|\int_I f_r - S(f_r;\dot{\mathcal{P}})| \leq \varepsilon$ whenever $\dot{\mathcal{P}}$ is $\delta_{r,\varepsilon}$-fine. Then

$$\begin{aligned}\left|S(f;\dot{\mathcal{P}}) - A\right| &\leq \left|S(f;\dot{\mathcal{P}}) - S(f_r;\dot{\mathcal{P}})\right| + \left|S(f_r;\dot{\mathcal{P}}) - \int_I f_r\right| + \left|\int_I f_r - A\right| \\ &\leq \varepsilon(b-a) + \varepsilon + \varepsilon = \varepsilon(b-a+2).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then $f \in \mathcal{R}^*(I)$ and $\int_I f = A$. Q.E.D.

Although the hypotheses of the Uniform Convergence Theorem are highly restrictive, it is a very useful result.

Monotone Convergence

The next result is a generalization of an important theorem proved in 1906 for the Lebesgue integral by the Italian mathematician Beppo Levi (1875–1961). The theorem presented here applies to generalized Riemann integrable functions; it requires pointwise (rather than uniform) convergence of the sequence, but assumes that the convergence is monotone in the following sense.

• **8.4 Definition.** A sequence $(f_k) : I \to \mathbb{R}$ is said to be **increasing** on I if

$$f_k(x) \le f_{k+1}(x) \qquad \text{for all} \quad x \in I,\ k \in \mathbb{N}.$$

It is said to be **decreasing** on I if

$$f_k(x) \ge f_{k+1}(x) \qquad \text{for all} \quad x \in I,\ k \in \mathbb{N}.$$

A sequence is said to be **monotone** on I if it is either increasing on I or decreasing on I.

We note that the sequence in Example 8.1(a) is increasing on $[0,1]$; however, the sequences in Examples 8.1(b, c) are not monotone.

The proof of the next theorem is rather delicate, and makes use of the Saks-Henstock Lemma.

• **8.5 Monotone Convergence Theorem.** *Let (f_k) be a monotone sequence in $\mathcal{R}^*(I)$, and let $f(x) := \lim f_k(x)$ for all $x \in I := [a,b]$. Then $f \in \mathcal{R}^*(I)$ if and only if the sequence $(\int_I f_k)$ is bounded in $\mathbb{R}$. In this case:*

$$(8.*) \qquad \int_I f = \lim_{k\to\infty} \int_I f_k.$$

Moreover, if $f_1 \in \mathcal{L}(I)$, then $f \in \mathcal{L}(I)$.

Proof. We will discuss the case of an increasing sequence of functions.

($\Rightarrow$) If $f \in \mathcal{R}^*(I)$, then since $f_1(x) \le f_k(x) \le f_{k+1}(x) \le f(x)$ for all $x \in I$, Corollary 3.3 implies that the real sequence $(\int_I f_k)$ is increasing and bounded.

($\Leftarrow$) Let $A := \sup\{\int_I f_k : k \in \mathbb{N}\}$, so that the increasing sequence $(\int_I f_k)$ converges to A. If $\varepsilon > 0$, let $r \in \mathbb{N}$ be such that $1/2^{r-2} < \varepsilon$ and

$$(8.\alpha) \qquad 0 \le A - \int_I f_r < \varepsilon.$$

Since $f_k \in \mathcal{R}^*(I)$, for each $k \in \mathbb{N}$ there exists a gauge δ_k on I such that if $\dot{\mathcal{P}}$ is a δ_k-fine partition of I, then $|S(f_k;\dot{\mathcal{P}}) - \int_I f_k| \le 1/2^k$. Also, since $f(x) = \lim f_k(x)$, then for each $x \in I$ there exists an integer $k(x) \ge r$ with

$$(8.\beta) \qquad 0 \le f(x) - f_{k(x)}(x) < \varepsilon.$$

Now define $\delta_\varepsilon(t) := \delta_{k(t)}(t)$ for $t \in I$, so that δ_ε is a gauge on I. We will use this gauge to show that f is integrable with integral A. Thus, if

$\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is a δ_ε-fine partition of I, we want to show that $|S(f;\dot{\mathcal{P}})-A|$ is suitably small. By the Triangle Inequality we have

$$(8.\gamma)\qquad \begin{aligned} \left|S(f;\dot{\mathcal{P}}) - A\right| \le & \Big|\sum_{i=1}^n f(t_i)l(I_i) - \sum_{i=1}^n f_{k(t_i)}(t_i)l(I_i)\Big| \\ & + \Big|\sum_{i=1}^n f_{k(t_i)}(t_i)l(I_i) - \sum_{i=1}^n \int_{I_i} f_{k(t_i)}\Big| \\ & + \Big|\sum_{i=1}^n \int_{I_i} f_{k(t_i)} - A\Big|. \end{aligned}$$

(i) From $(8.\beta)$, the first term on the right in $(8.\gamma)$ is dominated by

$$\sum_{i=1}^n \big|f(t_i) - f_{k(t_i)}(t_i)\big| l(I_i) \le \sum_{i=1}^n \varepsilon l(I_i) = (b-a)\varepsilon.$$

(ii) The second term on the right in $(8.\gamma)$ is dominated by

$$(8.\delta)\qquad \sum_{i=1}^n \big|f_{k(t_i)}(t_i)l(I_i) - \int_{I_i} f_{k(t_i)}\big|.$$

To estimate this sum, let $s := \max\{k(t_1), \cdots, k(t_n)\} \ge r$. We note that the finite sum $(8.\delta)$ can be written as an iterated sum: first over all values of i such that $k(t_i) = p$ for some natural number $p \ge r$, and then over $p = r, \cdots, s$. Consider all those tags t_i with $k(t_i) = p$ for fixed p. Each corresponding subinterval I_i is contained in the closed ball with center t_i and radius $\delta_\varepsilon(t_i) = \delta_{k(t_i)}(t_i) = \delta_p(t_i)$. Therefore the collection of pairs $\{(I_i, t_i) : k(t_i) = p\}$ forms a δ_p-fine subpartition. Consequently, Corollary 5.4 of the Saks-Henstock Lemma implies that

$$(8.\varepsilon)\qquad \sum_{k(t_i)=p} \Big|f_{k(t_i)}(t_i)l(I_i) - \int_{I_i} f_{k(t_i)}\Big| \le \frac{1}{2^{p-1}}.$$

If we sum $(8.\varepsilon)$ over $p = r, \cdots, s$, we find that the second term in $(8.\gamma)$ is dominated by

$$\sum_{p=r}^s \frac{1}{2^{p-1}} < \sum_{p=r}^\infty \frac{1}{2^{p-1}} = \frac{1}{2^{r-2}} < \varepsilon.$$

(iii) We now estimate the third term in $(8.\gamma)$. Since the sequence (f_k) is increasing and $r \le k(t_i) \le s$, then $f_r \le f_{k(t_i)} \le f_s$ and Corollary 3.3 implies that

$$\int_{I_i} f_r \le \int_{I_i} f_{k(t_i)} \le \int_{I_i} f_s.$$

Summing these inequalities for $i = 1, \cdots, n$ and using Corollary 3.9, we obtain

$$\int_I f_r \leq \sum_{i=1}^{n} \int_{I_i} f_{k(t_i)} \leq \int_I f_s.$$

Hence it follows from (8.α) that

$$A - \varepsilon \leq \sum_{i=1}^{n} \int_{I_i} f_{k(t_i)} \leq A,$$

so that the third term in (8.γ) is also dominated by ε.

Combining these three estimates, we conclude that if $\dot{\mathcal{P}}$ is δ_ε-fine, then

$$|S(f;\dot{\mathcal{P}}) - A| \leq (b-a)\varepsilon + \varepsilon + \varepsilon = (b-a+2)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then f is integrable on I with integral $A = \lim \int_I f_k$.

The final assertion follows from Theorem 7.9(c). Q.E.D.

The Monotone Convergence Theorem is a *very important result* and will be frequently used in the following.

Fatou's Lemma

We now obtain a result that is often useful when the sequence is not monotone. It was proved in 1906 by Pierre Fatou (1878–1929) for the Lebesgue integral. First we need a lemma.

• **8.6 Lemma.** *Let $f_k, \alpha \in \mathcal{R}^*(I)$ be such that*

$$(8.\zeta) \qquad \alpha(x) \leq f_k(x) \quad \textit{for} \quad x \in I,\ k \in \mathbb{N}.$$

Then $\inf\{f_k\}$ *belongs to* $\mathcal{R}^*(I)$.

Proof. The inequality (8.ζ) shows that $\inf\{f_k\}$ exists and is $\geq \alpha$. If $k \in \mathbb{N}$, we let $\psi_k := f_1 \wedge \cdots \wedge f_k$. Theorem 7.13(b) and mathematical induction imply that $\psi_k \in \mathcal{R}^*(I)$. Moreover, the sequence (ψ_k) is decreasing on I to $\inf\{f_k\}$. But, since $\int_I \psi_k \geq \int_I \alpha$, the Monotone Convergence Theorem implies that $\lim \psi_k \in \mathcal{R}^*(I)$. Since it is clear that $\inf\{f_k\} = \lim \psi_k$, the assertion is proved. Q.E.D.

Fatou's Lemma and the proof of the Dominated Convergence Theorem make use of the notions of the limit inferior and the limit superior. These concepts and their main properties are introduced in Appendix A for those readers who are not familiar with them.

• **8.7 Fatou's Lemma.** *Let $f_k, \alpha \in \mathcal{R}^*(I)$ be such that*

$$\alpha(x) \le f_k(x) \quad \text{for} \quad x \in I,\ k \in \mathbb{N} \tag{8.ζ}$$

and that

$$\liminf_{k\to\infty} \int_I f_k < \infty. \tag{8.η}$$

Then $\liminf_{k\to\infty} f_k$ *belongs to* $\mathcal{R}^*(I)$ *and*

$$-\infty < \int_I \liminf_{k\to\infty} f_k \le \liminf_{k\to\infty} \int_I f_k < \infty. \tag{8.θ}$$

Proof. If $\varphi_k := \inf\{f_m : m \ge k, m \in \mathbb{N}\}$ for $k \in \mathbb{N}$, then Lemma 8.6 implies that $\varphi_k \in \mathcal{R}^*(I)$. Since $\alpha(x) \le \varphi_k(x) \le f_k(x)$ for $x \in I,\ k \in \mathbb{N}$, we have

$$\int_I \alpha \le \int_I \varphi_k \le \int_I f_k.$$

By a well-known property of the limit inferior (see Theorem A.3(h)), we have

$$\int_I \alpha \le \liminf_{k\to\infty} \int_I \varphi_k \le \liminf_{k\to\infty} \int_I f_k. \tag{8.ι}$$

Now (φ_k) is an increasing sequence that converges on I to $\varphi := \liminf f_k$. Therefore, (8.ι) implies that the increasing sequence $(\int_I \varphi_k)$ is convergent and hence is bounded. The Monotone Convergence Theorem 8.5 then implies that $\varphi = \lim \varphi_k = \liminf f_k$ belongs to $\mathcal{R}^*(I)$ and that $\int_I \varphi = \lim \int_I \varphi_k \in \mathbb{R}$. If we use (8.$\eta$), we obtain the inequality (8.θ). Q.E.D.

[Students at the University of Illinois become very proficient at remembering Fatou's Lemma by recalling the acronym ILINILLINI = the Integral of the Limit INferior $\cdots$.]

Remark. We have stated Fatou's Lemma in a form that is somewhat more general than usual; for a still more general form, see Exercise 8.L. We emphasize the importance of the hypothesis (8.ζ), without which the conclusion may fail (see Exercise 8.J). Sometimes, only the case $\alpha = 0$ is considered and it is asserted that the dual version of Fatou's Lemma (involving limits superior) is false. In fact, there is a completely dual version of Fatou's Lemma (see Exercise 8.K).

Dominated and Mean Convergence Theorems

Our next result is an extension to the generalized Riemann integral of a theorem proved in 1908 by Lebesgue. It is a key result in integration theory, and its importance cannot be overestimated. A slight extension of this result is given in Exercise 8.M.

• **8.8 Dominated Convergence Theorem.** *Let (f_k) be a sequence in $\mathcal{R}^*(I)$ with $f(x) = \lim f_k(x)$ for all $x \in I := [a, b]$. Suppose that there exist functions $\alpha, \omega \in \mathcal{R}^*(I)$ such that*

$$(8.\kappa) \qquad \alpha(x) \le f_k(x) \le \omega(x) \quad \text{for} \quad x \in I,\ k \in \mathbb{N}.$$

Then $f \in \mathcal{R}^*(I)$ and

$$(8.*) \qquad \int_I f = \lim_{k\to\infty} \int_I f_k.$$

Proof. The hypothesis implies that $f(x) = \lim f_k(x) = \liminf f_k(x) \in \mathbb{R}$ for all $x \in I$. It follows from $(8.\kappa)$ and Corollary 3.3 that

$$\int_I \alpha \le \int_I f_k \le \int_I \omega \quad \text{for} \quad k \in \mathbb{N},$$

whence both $\liminf \int_I f_k$ and $\limsup \int_I f_k$ are in $\mathbb{R}$. Fatou's Lemma 8.7 then implies that $f \in \mathcal{R}^*(I)$ and that

$$(8.\lambda) \qquad \int_I f \le \liminf_{k\to\infty} \int_I f_k.$$

If we apply Fatou's Lemma to the sequence $(-f_k)$, together with the fact that $\liminf(-\xi_k) = -\limsup \xi_k$, then we conclude that

$$-\int_I f = \int_I (-f) \le \liminf_{k\to\infty} \int_I (-f_k) = -\limsup_{k\to\infty} \int_I f_k,$$

whence we infer that

$$(8.\mu) \qquad \limsup_{k\to\infty} \int_I f_k \le \int_I f.$$

If we combine $(8.\lambda)$ and $(8.\mu)$, we obtain $(8.*)$. Q.E.D.

Remark. It is worth noting that if at least *one* of the functions α, ω, f_n $(n \in \mathbb{N})$ in Theorem 8.8 belongs to $\mathcal{L}(I)$, then the limit function f also

belongs to $\mathcal{L}(I)$. That is also true for the next result, as is seen from Theorem 7.9.

We now show that the very same hypotheses employed in the Dominated Convergence Theorem 8.8 can be used to obtain a rather different conclusion. This conclusion, given in (8.ν), is usually referred to as the **mean convergence** of the sequence (f_n) to f. We will encounter this mode of convergence frequently in later sections.

• **8.9 Mean Convergence Theorem.** *Let (f_k) be a sequence in $\mathcal{R}^*(I)$ with $f(x) = \lim f_k(x)$ for all $x \in I$. Suppose there exist functions $\alpha, \omega \in \mathcal{R}^*(I)$ such that*

$$(8.\kappa) \qquad \alpha(x) \le f_k(x) \le \omega(x) \qquad \text{for} \quad x \in I,\ k \in \mathbb{N}.$$

Then $f - f_k \in \mathcal{L}(I)$ for each $k \in \mathbb{N}$ and

$$(8.\nu) \qquad \lim_{k\to\infty} \int_I |f - f_k| = 0.$$

Proof. It follows from the Dominated Convergence Theorem 8.8 that $f \in \mathcal{R}^*(I)$ whence $f - f_k \in \mathcal{R}^*(I)$. If we take the limit in (8.κ) as $k \to \infty$, we conclude that $\alpha(x) \le f(x) \le \omega(x)$ for all $x \in I$. Therefore we have

$$-(\omega - \alpha) \le f - f_k \le \omega - \alpha,$$

whence $0 \le |f - f_k| \le \omega - \alpha \in \mathcal{L}(I)$. The Comparison Test 7.7 then implies that $f - f_k \in \mathcal{L}(I)$. By Theorem 8.8 applied to $\tilde{f}_k := |f - f_k| \to 0$, $\tilde{\alpha} := 0$, and $\tilde{\omega} := \omega - \alpha$, we conclude that (8.$\nu$) holds. Q.E.D.

Equi-integrability

It is well known that the interchange of the order of iterated limits is possible when one of the limits is uniform in an appropriate sense. We have already encountered this phenomenon in the Uniform Convergence Theorem 8.3. We will now give another instance of it, when the integrability of the sequence is uniform in the sense that, given $\varepsilon > 0$, then *the same* gauge function is valid simultaneously for the integrability of all of the functions in the sequence. (This concept is due to Jaroslav Kurzweil, who established Theorem 8.11.)

• **8.10 Definition.** A collection $\mathcal{F} \subset \mathcal{R}^*(I)$ is said to be **equi-integrable** on $I := [a, b]$ if, for each $\varepsilon > 0$ there exists a gauge δ_ε on I such that if $\dot{\mathcal{P}}$ is any δ_ε-fine partition of I and $f \in \mathcal{F}$, then $|S(f; \dot{\mathcal{P}}) - \int_I f| \le \varepsilon$.

We will show that equi-integrability and pointwise convergence of a sequence in $\mathcal{R}^*(I)$ imply that the limit function is integrable and that equation (8.$*$) holds.

• **8.11 Equi-integrability Theorem.** *If $(f_k) \in \mathcal{R}^*(I)$ is equi-integrable on $I := [a, b]$, and if $f(x) = \lim f_k(x)$ for all $x \in I$, then $f \in \mathcal{R}^*(I)$ and*

$$(8.*) \qquad \int_I f = \lim_{k\to\infty} \int_I f_k.$$

A proof of this result is given in [B-S; p. 303]. Instead of repeating that proof here, we will prove a more general theorem due to R. A. Gordon [G-4] from which it clearly follows. Gordon's theorem has the advantage that it gives a necessary and sufficient condition for (8.$*$).

• **8.12 Theorem (Gordon).** *Suppose that $(f_k) \in \mathcal{R}^*(I)$ and that $f(x) = \lim f_k(x)$ for all $x \in I$. Then $f \in \mathcal{R}^*(I)$ and*

$$(8.*) \qquad \int_I f = \lim_{k\to\infty} \int_I f_k$$

if and only if for every $\varepsilon > 0$ there exists a gauge δ_ε on I such that if $\dot{\mathcal{P}}$ is any δ_ε-fine partition, there exists $K_{\dot{\mathcal{P}}} \in \mathbb{N}$ such that if $k \geq K_{\dot{\mathcal{P}}}$ then

$$(8.\xi) \qquad \left| S(f_k; \dot{\mathcal{P}}) - \int_I f_k \right| \leq \varepsilon.$$

Proof. ($\Leftarrow$) We will first show that $(\int_I f_k)$ is a Cauchy sequence.

Given $\varepsilon > 0$, the condition implies that there exists a gauge δ_ε such that if $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is δ_ε-fine, then there exists $K_{\dot{\mathcal{P}}} \in \mathbb{N}$ such that if $k \geq K_{\dot{\mathcal{P}}}$ then (8.ξ) holds. Since $\dot{\mathcal{P}}$ contains only a finite number of tags $t_1, \cdots, t_n$ and since $f(t) = \lim f_k(t)$ for all $t \in I$, there exists $K_\varepsilon \geq K_{\dot{\mathcal{P}}}$ such that if $h, k \geq K_\varepsilon$, then

$$(8.o) \qquad \left| S(f_k; \dot{\mathcal{P}}) - S(f_h; \dot{\mathcal{P}}) \right| \leq \sum_{i=1}^n \left| f_k(t_i) - f_h(t_i) \right| l(I_i) \leq \varepsilon(b-a).$$

If we let $h \to \infty$ in (8.o), we obtain

$$(8.\pi) \qquad \left| S(f_k; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{P}}) \right| \leq \varepsilon(b-a) \qquad \text{for} \quad k \geq K_\varepsilon.$$

Moreover, if $h, k \geq K_\varepsilon$, we can use (8.ξ) and (8.o) to obtain

$$\begin{aligned}\left| \int_I f_k - \int_I f_h \right| &\leq \left| \int_I f_k - S(f_k; \dot{\mathcal{P}}) \right| + \left| S(f_k; \dot{\mathcal{P}}) - S(f_h; \dot{\mathcal{P}}) \right| \\ &\quad + \left| S(f_h; \dot{\mathcal{P}}) - \int_I f_h \right| \leq \varepsilon + \varepsilon(b-a) + \varepsilon = \varepsilon(2 + b - a).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $(\int_I f_k)$ is a Cauchy sequence and so converges, say to $A \in \mathbb{R}$. If we let $h \to \infty$ in this last inequality, we obtain

$$(8.\rho) \qquad \Big| \int_I f_k - A \Big| \le \varepsilon(2 + b - a) \qquad \text{for} \quad k \ge K_\varepsilon.$$

We now show that $f \in \mathcal{R}^*(I)$ with integral A. For, if $\dot{\mathcal{P}}$ is a δ_ε-fine partition of I and $k \ge K_\varepsilon \ge K_{\dot{\mathcal{P}}}$, then

$$\begin{aligned} \big|S(f;\dot{\mathcal{P}}) - A\big| &\le \big|S(f;\dot{\mathcal{P}}) - S(f_k;\dot{\mathcal{P}})\big| + \big|S(f_k;\dot{\mathcal{P}}) - \int_I f_k\big| + \big| \int_I f_k - A\big| \\ &\le \varepsilon(b-a) + \varepsilon + \varepsilon(2+b-a) = \varepsilon(3 + 2b - 2a)\,, \end{aligned}$$

where we have used $(8.\pi), (8.\xi)$ and $(8.\rho)$. Since $\varepsilon > 0$ is arbitrary, then f is integrable with integral A and $(8.*)$ holds.

$(\Rightarrow)$ If $(8.*)$ holds and $\varepsilon > 0$, there exists $M_\varepsilon \in \mathbb{N}$ such that if $k \ge M_\varepsilon$ then

$$(8.\sigma) \qquad \Big| \int_I f - \int_I f_k \Big| \le \varepsilon/3.$$

Since $f \in \mathcal{R}^*(I)$, there exists a gauge δ_ε such that if $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is a partition with $\dot{\mathcal{P}} \ll \delta_\varepsilon$ then

$$(8.\tau) \qquad \Big| \int_I f - S(f;\dot{\mathcal{P}}) \Big| \le \varepsilon/3.$$

Now choose $K_{\dot{\mathcal{P}}} \ge M_\varepsilon$ such that if $k \ge K_{\dot{\mathcal{P}}}$ then $|f_k(t_i) - f(t_i)| \le \varepsilon/3(b-a)$ for $i = 1, \cdots, n$, whence

$$(8.\upsilon) \qquad |S(f_k;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})| \le \sum_{i=1}^n |f_k(t_i) - f(t_i)| l(I_i) \le \varepsilon/3.$$

Consequently, if $k \ge K_{\dot{\mathcal{P}}} \ge M_\varepsilon$, then

$$\begin{aligned} \Big|S(f_k;\dot{\mathcal{P}}) - \int_I f_k\Big| \le \Big|S(f_k;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})\Big| &+ \Big|S(f;\dot{\mathcal{P}}) - \int_I f\Big| \\ &+ \Big| \int_I f - \int_I f_k \Big| \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

where we have used $(8.\upsilon), (8.\tau)$ and $(8.\sigma)$. Since $\varepsilon > 0$ is arbitrary, it follows that $(8.\xi)$ holds for $k \ge K_{\dot{\mathcal{P}}}$. Q.E.D.

Note. McLeod [ML; pp. 96–101] has shown that both the Monotone Convergence Theorem and the Dominated Convergence Theorem are consequences of the Equi-integrability Theorem. However, we have not used this procedure, since the details seem to be more complicated than the direct proofs that we have given. Gordon [G-1] has given another proof of these facts, using results from measure theory.

Another Necessary and Sufficient Condition

We now present another necessary and sufficient condition for the limit of a sequence of integrable functions to satisfy (8.$*$). The exact interpretation of the condition we will impose is certainly not transparent; however, if the integrals of the functions are to be close to the integral of f, it seems reasonable that the Riemann sums of the f_k should also approximate those of f. It will be observed that the gauges may vary with the index k in this theorem.

The reader will note that the condition in the theorem is obviously satisfied when the convergence of the (f_n) on the compact interval $I = [a, b]$ is uniform.

• **8.13 Theorem.** *Suppose that $(f_k) \in \mathcal{R}^*(I)$ and that $f(x) = \lim f_k(x)$ for all $x \in I$. Then $f \in \mathcal{R}^*(I)$ and*

$$\int_I f = \lim_{k\to\infty} \int_I f_k \tag{8.$*$}$$

if and only if for every $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that if $k \geq m_\varepsilon$ there exists a gauge γ_k on I such that if $\dot{\mathcal{P}}$ is any γ_k-fine partition of I then

$$\left|S(f_k; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{P}})\right| \leq \varepsilon. \tag{8.φ}$$

Proof. ($\Leftarrow$) We will first show that $(\int_I f_k)$ is a Cauchy sequence.

Given $\varepsilon > 0$ let m_ε be as in the condition; thus if $h, k \geq m_\varepsilon$, there exist gauges γ_h, γ_k such that if $\dot{\mathcal{P}} \ll \gamma_h$ then $|S(f_h; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{P}})| \leq \varepsilon$, and if $\dot{\mathcal{P}} \ll \gamma_k$ then $|S(f_k; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{P}})| \leq \varepsilon$. Further, since f_h and f_k are integrable, there exist gauges δ_h and δ_k such that if $\dot{\mathcal{P}} \ll \delta_h$ then $|S(f_h; \dot{\mathcal{P}}) - \int_I f_h| \leq \varepsilon$, and if $\dot{\mathcal{P}} \ll \delta_k$ then $|S(f_k; \dot{\mathcal{P}}) - \int_I f_h| \leq \varepsilon$. Now, let $\eta_\varepsilon := \min\{\gamma_h, \gamma_k, \delta_h, \delta_k\}$. Therefore, if $\dot{\mathcal{P}} \ll \eta_\varepsilon$ then we have

$$\begin{aligned}\left|\int_I f_h - \int_I f_k\right| &\leq \left|\int_I f_h - S(f_h; \dot{\mathcal{P}})\right| + \left|S(f_h; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{P}})\right| \\ &\quad + \left|S(f; \dot{\mathcal{P}}) - S(f_k; \dot{\mathcal{P}})\right| + \left|S(f_k; \dot{\mathcal{P}}) - \int_I f_k\right| \\ &\leq \varepsilon + \varepsilon + \varepsilon + \varepsilon = 4\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the sequence $(\int_I f_k)$ is a Cauchy sequence in $\mathbb{R}$, and therefore converges to some number $A \in \mathbb{R}$.

We now show that $f \in \mathcal{R}^*(I)$ and that $\int_I f = A$. For, if $\varepsilon > 0$, let m_ε be as in the condition, and let $k \geq m_\varepsilon$ be such that $|\int_I f_k - A| \leq \varepsilon$. Then there exists a gauge γ_k on I such that if $\dot{\mathcal{P}} \ll \gamma_k$ then $|S(f_k;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})| \leq \varepsilon$. Also, since $f_k \in \mathcal{R}^*(I)$, there exists a gauge δ_k such that if $\dot{\mathcal{P}} \ll \delta_k$ then $|S(f_k;\dot{\mathcal{P}}) - \int_I f_k| \leq \varepsilon$. Now let $\zeta_\varepsilon := \min\{\gamma_k, \delta_k\}$. It follows that if $\dot{\mathcal{P}} \ll \zeta_\varepsilon$ then we have

$$\begin{aligned} \left|S(f;\dot{\mathcal{P}}) - A\right| &\leq \left|S(f;\dot{\mathcal{P}}) - S(f_k;\dot{\mathcal{P}})\right| + \left|S(f_k;\dot{\mathcal{P}}) - \int_I f_k\right| \\ &\quad + \left|\int_I f_k - A\right| \leq 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then $f \in \mathcal{R}^*(I)$ and $\int_I f = A$, so that (8.$*$) holds.

($\Rightarrow$) Let $\varepsilon > 0$. Since $(\int_I f_k) \to \int_I f$, there exists $m_\varepsilon \in \mathbb{N}$ such that if $k \geq m_\varepsilon$ then $|\int_I f_k - \int_I f| \leq \varepsilon$. Now let $k \geq m_\varepsilon$ be fixed. Since $f_k \in \mathcal{R}^*(I)$, there exists a gauge δ_k such that if $\dot{\mathcal{P}} \ll \delta_k$ then $|\int_I f_k - S(f_k;\dot{\mathcal{P}})| \leq \varepsilon$. Since $f \in \mathcal{R}^*(I)$, there exists a gauge δ_0 such that if $\dot{\mathcal{P}} \ll \delta_0$ then $|\int_I f - S(f;\dot{\mathcal{P}})| \leq \varepsilon$. Now let $\gamma_k := \min\{\delta_0, \delta_k\}$, so that if $\dot{\mathcal{P}} \ll \gamma_k$ then

$$\begin{aligned} \left|S(f_k;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})\right| &\leq \left|S(f_k;\dot{\mathcal{P}}) - \int_I f_k\right| + \left|\int_I f_k - \int_I f\right| \\ &\quad + \left|\int_I f - S(f;\dot{\mathcal{P}})\right| \leq 3\varepsilon. \end{aligned}$$

Therefore the sequence (f_k) satisfies the condition in the theorem. Q.E.D.

8.14 Example. Let $I := [0,2]$ and let $f_k(x) := k$ for $x \in (1/k, 1/k + 1/k^2]$ and $f_k(x) := 0$ elsewhere in I. Then f_k is a step function with $\int_I f_k = 1/k$. Moreover $f_k(x) \to f(x) := 0$ for $x \in I$.

We claim that the sequence (f_k) satisfies the condition in Theorem 8.13. Indeed, let δ_k be the gauge defined by $\delta_k(t) := \frac{1}{2}\,\mathrm{dist}(t, \{1/k, 1/k + 1/k^2\})$ if $t \in I - \{1/k, 1/k + 1/k^2\}$ and $\delta_k(t) := 1/k^2$ otherwise. It is an exercise to show that if $\dot{\mathcal{P}}$ is a δ_k-fine partition of I, then $|S(f_k;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})| \leq 3/k$.

Therefore Theorem 8.13 applies to this sequence; however, neither the Monotone nor Dominated Convergence Theorems apply to it. It is also an exercise to show that the sequence (f_k) is *not* equi-integrable, so that Theorem 8.11 does not apply either. (See Exercise 8.X.)

In [B-3] it is shown that the version of Theorem 8.13 for $[a, \infty)$ can be used to prove Hake's Theorem 16.5. (However, the details of this proof

involve considerations that are very similar to the proof we will give in Section 16.)

Almost Everywhere Convergence

In the preceding we have assumed that $f(x) = \lim f_k(x)$ for *all* $x \in I$. However, in many cases it is more convenient to assume only that we have a sequence (g_k) such that $g(x) = \lim g_k(x)$ a.e. on I; that is, that there exists a null set $Z \subset I$ such that $g(x) = \lim g_k(x)$ for all $x \in I - Z$. Most of the preceding theorems extend almost immediately to this slightly more general mode of convergence. Indeed, we merely define $f_k(x) := g_k(x)$ for $x \in I - Z$ and $f_k(x) := 0$ for $x \in Z$, and apply the theorem. To be explicit, we have written out the statements of these results in Exercise 8.A.

A notable exception to this situation is the case of equi-integrability. Indeed, it is possible for two sequences $(f_k), (g_k)$ on I to be such that $f_k(x) = g_k(x)$ a.e. on I for each $k \in \mathbb{N}$ and for (f_k) to be equi-integrable, but (g_k) not be equi-integrable. For example, let $f_k(x) := 0$ for all $x \in [0,1]$, and let $g_k(0) := k$ and $g_k(x) := 0$ for $x \in (0,1]$. It is trivial that (f_k) is equi-integrable and is an exercise to show that (g_k) is not equi-integrable on $[0,1]$.

Controlled Convergence

There is a considerable literature dealing with the notion of "controlled convergence" of a sequence of integrable functions. However, this mode involves severe conditions on the sequence of indefinite integrals that makes its application difficult. Since it implies the equi-integrability of the sequence, we will not discuss this mode here, but refer the interested reader to the books of Lee [Le-1] and Lee and Výborný [L-V], as well as many notes in the *Real Analysis Exchange.*

However, we will state one result of this type, where restrictions are imposed on the indefinite integrals $F_k(x) = \int_a^x f_k$ of the sequence (f_k). This result is in the spirit of the Characterization Theorem 5.12.

Let $I := [a,b]$ and let $A, Z \subseteq I$. We say that the sequence $F_k : I \to \mathbb{R}$ of continuous functions is:

(i) **uniformly differentiable** to f_k on A if for every $\varepsilon > 0$ there exists a gauge δ_ε on A such that if $0 < |z - t| \le \delta_\varepsilon(t),\ z \in I,\ t \in A$, then

$$\left| \frac{F_k(z) - F_k(t)}{z - t} - f_k(t) \right| \le \varepsilon \qquad \text{for all} \quad k \in \mathbb{N}.$$

(ii) **uniformly** $NV_I(Z)$ if for every $\varepsilon > 0$ there exists a gauge δ_ε on Z such that if $\dot{\mathcal{P}} := \{([u_i, v_i], t_i)\}_{i=1}^n$ is a (δ_ε, Z)-fine subpartition of I, then

$$\sum_{i=1}^{n} |F_k(v_j) - F_k(u_j)| \le \varepsilon \qquad \text{for all} \quad k \in \mathbb{N}.$$

• **8.15 Theorem**. *Let (f_k) be a sequence of functions in $\mathcal{R}^*(I)$ such that $f(x) = \lim f_k(x)$ for all $x \in I$, and let $F_k(x) := \int_a^x f_k$. Suppose that there exists a null set $Z \subset I$ such that:*

(1) the (F_k) are uniformly differentiable to (f_k) on $I - Z$, and

(2) the (F_k) are uniformly $NV_I(Z)$.

Then $f \in \mathcal{R}^*(I)$ and $\int_I f_k \to \int_I f$.

Proof. The δ_ε given on $I - Z$ from the uniform differentiability and the δ_ε given on Z from the uniform $NV_I(Z)$ property yield a gauge on all of I, with respect to which the sequence (f_k) is equi-integrable. Now apply the Equi-integrability Theorem 8.11. Q.E.D.

Exercises

8.A (a) Let $(g_k) \in \mathcal{R}^*(I)$ be a sequence that converges uniformly a.e. to g (in the sense that there exists a null set $Z \subset I := [a, b]$ such that (g_k) converges uniformly to g on $I - Z$). Then $g \in \mathcal{R}^*(I)$ and $(*)$ $\int_I g = \lim \int_I g_k$.

(b) Let $(g_k) \in \mathcal{R}^*(I)$ be a monotone sequence and let $g : I \to \mathbb{R}$ be such that $g(x) = \lim g_k(x)$ a.e. on $I := [a, b]$. Show that $g \in \mathcal{R}^*(I)$ and that $(*)$ holds if and only if the sequence $(\int_I g_k)$ is bounded in $\mathbb{R}$.

(c) Suppose that $\alpha, g_k \in \mathcal{R}^*(I)$ are such that $\alpha(x) \le g_k(x)$ a.e. on $I := [a, b]$ for each $k \in \mathbb{N}$. Also suppose that $\liminf g_k(x) \in \mathbb{R}$ a.e. on I and that $\liminf \int_I g_k \in \mathbb{R}$. Show that $\int_I \liminf g_k \le \liminf \int_I g_k$.

(d) Let $(g_k) \in \mathcal{R}^*(I)$ converge a.e. to g on $I := [a, b]$ and suppose that there exist $\alpha, \omega \in \mathcal{R}^*(I)$ such that $\alpha(x) \le g_k(x) \le \omega(x)$ a.e. on I. Show that $g \in \mathcal{R}^*(I)$ and that $(*)$ holds.

8.B Let $I := [0, 1]$ and, if $k \ge 2$, let $f_k(x) := 1$ for $x \in [1/k, 2/k]$ and $f_k(x) := 0$ elsewhere in I. Show that $f_k(x) \to f(x) := 0$ on $[0, 1]$ as $k \to \infty$, and that $f_k(x) \le 1$. Show that (f_k) does not converge uniformly on I, but that $\int_I f = \lim \int_I f_k$, so that equality holds in Fatou's Lemma.

8.C Using the notation in 8.B, let $g_k(x) := kf_k(x)$ for $x \in I,\ k \in \mathbb{N}$.

(a) Show that $g_k(x) \to g(x) := 0$ as $k \to \infty$, but that (g_k) does not converge uniformly and (g_k) is not dominated by a function in $\mathcal{L}([0,1])$.

(b) Show that $0 = \int_I g < \lim \int_I g_k = 1$, so that a strict inequality holds in (8.κ).

(c) Show that $\int_I |g_k - g| \not\to 0$.

(d) Now let $h_k := -g_k$ and show that $\int_I \lim h_k > \lim \int_I h_k$. Hence the inequality (8.θ) is false for the sequence (h_k). Are the hypotheses of Fatou's Lemma satisfied by the (h_k) ?

8.D Using the notation in 8.B, let $p_k(x) := k^2 f_k(x)$ for $x \in I$, so that $p_k(x) \to p(x) := 0$ as $k \to \infty$. Show that $0 = \int_I p < \lim \int_I p_k = \infty$.

8.E (a) Let $f_k \in \mathcal{R}^*(I)$ be such that $0 \le f_k(x) \to f(x)$ for all $x \in I := [a,b]$ and $\int_I f = \lim \int_I f_k < \infty$. If $J \subset I$ is a subinterval of I, show that $\int_J f = \lim \int_J f_k$.

(b) Show that the concusion in (a) may fail if the hypothesis that $f_k(x) \ge 0$ is dropped.

8.F Let $I := [0,1]$; for $k \in \mathbb{N}$, let f_k be defined to equal 1 on the kth interval in the list:

$$[0,1],\quad [0,\tfrac{1}{2}], (\tfrac{1}{2},1],\quad [0,\tfrac{1}{4}], (\tfrac{1}{4},\tfrac{1}{2}], (\tfrac{1}{2},\tfrac{3}{4}], (\tfrac{3}{4},1],\quad [0,\tfrac{1}{8}], \cdots, (\tfrac{7}{8},1],\quad \cdots,$$

and to equal 0 elsewhere on I.

(a) Show that if $k \ge 2^n$ then $\int_I f_k \le 1/2^n$.

(b) Show that $(f_k(x))$ does not converge to 0 *for any* $x \in I$.

(c) Show that $\int_I f_k \to 0$ and that $\int_I |f_k - 0| \to 0$.

(d) Show that the subsequence (f_{2^k}) converges at every point of I.

8.G (a) If (f_k) is a monotone sequence in $\mathcal{R}^*(I)$, show that $f_k = f_1 + \varphi_k$ with $\varphi_k \in \mathcal{L}(I)$.

(b) If (g_k) is a sequence in $\mathcal{R}^*(I)$ and $\alpha \le g_k \le \omega$ for $\alpha, \omega \in \mathcal{R}^*(I)$, show that $g_k = g_1 + \psi_k$ where $|\psi_k| \le \lambda$ with $\lambda \in \mathcal{L}(I)$.

8.H Let $\kappa : [0,1] \to \mathbb{R}$ be the function in Example 2.8(a) and let $\kappa_k(x) := \kappa(x) + 1/k$ for $x \in I$.

(a) Show that $\kappa_k \in \mathcal{R}^*(I)$ but $\kappa_k \notin \mathcal{L}(I)$ for $k \in \mathbb{N}$.

(b) Show that the Uniform Convergence Theorem 8.3 applies even though none of the functions κ_k is bounded on I.

(c) Show that the Monotone Convergence Theorem 8.5 applies even though none of the κ_k belongs to $\mathcal{L}(I)$.

(d) Show that the Dominated Convergence Theorem 8.8 applies.

(e) Show that the Mean Convergence Theorem 8.9 holds, even though none of the κ_k belongs to $\mathcal{L}(I)$.

(f) Show that the Equi-integrability Theorem 8.11 applies.

8.I Let $(f_k) \in \mathcal{L}(I)$ be a decreasing sequence on $I := [a, b]$ with $0 \le f_k(x)$ for all $x \in I$. Show that $f_k \to 0$ a.e. if and only if $\int_I f_k \to 0$.

8.J Let $I := [0, 1]$ and for $k \in \mathbb{N}$, let $f_k(x) := -k$ for $x \in (0, 1/k]$, and let $f_k(x) := 0$ elsewhere in I. Show that the hypotheses of Fatou's Lemma are not satisfied, and that

$$\int_I \liminf_{k\to\infty} f_k = 0 > -1 = \liminf_{k\to\infty} \int_I f_k.$$

8.K Let $f_k, \omega \in \mathcal{R}^*(I)$ be such that $f_k(x) \le \omega(x)$ for $x \in I := [a, b]$, $k \in \mathbb{N}$.

(a) Show that $\sup\{f_k\}$ belongs to $\mathcal{R}^*(I)$.

(b) If $\limsup \int_I f_k > -\infty$, show that

$$-\infty < \limsup_{k\to\infty} \int_I f_k \le \int_I \limsup_{k\to\infty} f_k < \infty.$$

(This result is Fatou's Lemma for the limit superior.)

8.L (a) Let $f_k, \alpha_k, \alpha \in \mathcal{R}^*(I)$ be such that $\alpha_k \le f_k$ a.e. on I, $\alpha_k \to \alpha$ a.e. on I and $\int_I \alpha_k \to \int_I \alpha$. Suppose also that $\liminf \int_I f_k < \infty$. Then prove that $\liminf f_k \in \mathcal{R}^*(I)$ and that

$$-\infty < \int_I \liminf_{k\to\infty} f_k \le \liminf_{k\to\infty} \int_I f_k < \infty.$$

[Hint: Let $h_k := f_k - \alpha_k \ge 0$, and use Fatou's Lemma 8.7 and Theorem A.3(e).]

(b) Let $f_k, \omega_k, \omega \in \mathcal{R}^*(I)$ be such that $f_k \le \omega_k$ a.e. on I, $\omega_k \to \omega$ a.e. on I, and $\int_I \omega_k \to \int_I \omega$. Suppose also that $-\infty < \limsup \int_I f_k$. Then prove that $\limsup f_k \in \mathcal{R}^*(I)$ and that

$$-\infty < \limsup_{k\to\infty} \int_I f_k \le \int_I \limsup_{k\to\infty} f_k < \infty.$$

8.M Let $\alpha_k, \alpha, f_k, \omega_k, \omega$ belong to $\mathcal{R}^*(I)$ and suppose that $\alpha_k \le f_k \le \omega_k$ a.e. on I, that $\alpha_k \to \alpha$, $f_k \to f$ and $\omega_k \to \omega$ a.e. on I, and that $\int_I \alpha_k \to \int_I \alpha$ and $\int_I \omega_k \to \int_I \omega$. Prove that $f \in \mathcal{R}^*(I)$ and $\int_I f_k \to \int_I f$. (This exercise gives an extension of the Dominated Convergence Theorem 8.8 that is sometimes useful.)

8.N If $f \in \mathcal{L}(I)$ and $f^{[k]}$ is the k-truncate of f (see Exercise 7.V), show that $\int_I f = \lim \int_I f^{[k]}$ and that $\int_I |f^{[k]} - f| \to 0$.

8.O Let $0 \le f \in \mathcal{L}(I)$ and $E \subset I$ be an integrable set (see Definition 6.14(b)). Prove that $f \in \mathcal{R}^*(E)$ without using Theorem 6.18, which depends on the as yet unproved Theorem 6.9. [Hint: Consider $f_k := \min\{f, k\mathbf{1}_E\}$.]

8.P If $0 \le f \in \mathcal{L}(I)$, let $r > 0$ and consider the function e_r defined by $e_r(x) := 1$ whenever $f(x) > r$ and $e_r(x) := 0$ elsewhere in I. Show that e_r and $e_r \cdot f$ belong to $\mathcal{L}(I)$ and that

$$r\int_I e_r \le \int_I e_r \cdot f \le \int_I f.$$

[Hint: Consider the function $g_k(x) := \min\{1, k \cdot \max\{f(x) - r, 0\}\}$.]

8.Q If $f : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$, then $\lim \int_0^1 f(x^k)\,dx = f(0)$.

8.R Let $(f_k) \in \mathcal{R}^*(I)$ converge a.e. to f on I. Suppose there exists $\omega \in \mathcal{L}(I)$ such that $|f| \le \omega$ a.e. Show that $f \in \mathcal{L}(I)$. [Hint: Let $g_k := \operatorname{mid}\{-\omega, f_k, \omega\}$.]

8.S Suppose that $f \in \mathcal{R}^*(I)$, where $I := [a,b]$.
(a) If $c \in (a,b)$, let $f_c(x) := f(x)$ for $x \in [a,c]$ and $f_c(x) := 0$ for $x \in (c,b]$. Show that $f_c \in \mathcal{R}^*(I)$ and that $\lim_{c\to b-} \int_I f_c = \int_I f$.
(b) Show that the set $\{f \cdot \mathbf{1}_{[a,c]} : c \in (a,b)\}$ is equi-integrable on I.

8.T Let $\mathcal{F}$ be a collection of functions on $I := [a,b]$ that is equi-integrable and let $\mathcal{F}^*$ be the collection of pointwise limits of functions in $\mathcal{F}$. Show that $\mathcal{F}^*$ is equi-integrable on I.

8.U Show that the sequence $g_k(0) := k$ and $g_k(x) := 0$ for $x \in (0,1]$, $k \in \mathbb{N}$, is *not* equi-integrable on I.

8.V A collection $\mathcal{F}$ of functions on $I := [a,b]$ is said to be **equicontinuous** at $c \in I$ if for every $\varepsilon > 0$ there exists $\eta_\varepsilon(c) > 0$ such that if $|x - c| \le \eta_\varepsilon(c)$, $x \in I$, $F \in \mathcal{F}$, then $|F(x) - F(c)| \le \varepsilon$.

(a) Show that if $\mathcal{F}$ is equi-integrable on I and if $\{f(c) : f \in \mathcal{F}\}$ is a bounded set, then the collection $\mathcal{F}^i$ of indefinite integrals $F(x) := \int_a^x f$, $f \in \mathcal{F}$, is equicontinuous at c.

(b) If $\mathcal{F}$ is a collection of functions on $I := [a,b]$ that is equicontinuous at every point of I, show that it is **uniformly equicontinuous** on I in the sense that for any $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $|x - y| \le \eta_\varepsilon$, $x, y \in I$, $F \in \mathcal{F}$, then $|F(x) - F(y)| \le \varepsilon$. [Hint: Use the compactness of I.]

8.W (a) If (F_k) is a sequence of continuous functions on $I := [a,b] \to \mathbb{R}$ that is uniformly convergent to F on I, show that the sequence (F_k) is uniformly equicontinuous in the sense of the preceding exercise.

(b) If (F_k) is a sequence of functions that is uniformly equicontinuous on $I := [a,b]$ and if $F(x) = \lim F_k(x)$ for all $x \in I$, show that F is continuous and that the convergence is uniform on I. [Hint: If $\zeta > 0$ is given, then there exists a finite set $\{y_1, \cdots, y_m\}$ in I such that every point in I is within ζ of some one of the y_i.]

8.X Let (f_k) be the sequence in Example 8.14.

(a) Verify that if δ_k is the gauge in 8.14 and if $\dot{\mathcal{P}} \ll \delta_k$, then $|S(f_k;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})| \le 3/k$.

(b) Show that the Dominated Convergence Theorem does not apply.

(c) Let $F_k(x) := \int_0^x f_k$ for $x \in [0,1], k \in \mathbb{N}$. Show that the sequence (F_k) is not uniformly equicontinuous on $[0,1]$.

8.Y Let $g_k(x) := k$ for $x \in (1/k, 2/k]$, $g_k(x) := -k$ for $x \in (2/k, 3/k]$ and $g_k(x) := 0$ elsewhere in $I := [0,3]$. Show that $g_k(x) \to 0$ for all $x \in I$ and that the sequence (g_k) satisfies the condition in Theorem 8.13 on I. However, show that the sequence of indefinite integrals $G_k(x) := \int_0^x g_k$, for $k \in \mathbb{N}$, is not uniformly equicontinuous on I.

8.Z Suppose that (f_k) is a sequence of complex-valued functions in $\mathcal{L}(I)$ that converges to f a.e. on $I := [a,b]$ and that there exists $\omega \in \mathcal{L}(I)$ such that $|f_k| \le \omega$ a.e. on I. Show that $f \in \mathcal{L}(I)$ and that $\int_I f = \lim \int_I f_k$.

Integrability and Mean Convergence

As noted at the end of Section 8 (and in Exercise 8.A), most of the results given there extend without difficulty to the case where the sequences converge only almost everywhere. We will use that fact freely in this section and in the sequel.

The first result in this section is the basic "Integrability Theorem" that we announced in 6.9, but were unable to prove at that time. However, we are now in a position to give a proof of this important result, using the Dominated Convergence Theorem 8.8. An immediate consequence is the Measurable Limit Theorem asserting that the a.e. limit of a sequence of measurable functions is measurable, a result that was mentioned after Theorem 6.7. Other results that are closely related to the Monotone Convergence Theorem are then established.

We also introduce the notion of mean convergence, and establish the main results concerning this important mode of convergence in the space $\mathcal{L}(I)$ of absolutely integrable functions, such as the Completeness Theorem of Riesz and Fischer. It will also be seen that both the collection of step functions and the collection of continuous functions are dense in $\mathcal{L}(I)$. These facts are quite useful, since we often establish an assertion for the simpler functions, and then extend to arbitrary functions in $\mathcal{L}(I)$ by taking limits.

The Integrability Theorem

We now give the proof of this basic result.

• **9.1 Integrability Theorem.** *Suppose that $f \in \mathcal{M}(I)$, where $I := [a, b]$.*

(a) *Then $f \in \mathcal{R}^*(I)$ if and only if there exist $\alpha, \omega \in \mathcal{R}^*(I)$ such that*

$$\alpha(x) \le f(x) \le \omega(x) \qquad \text{for a.e.} \quad x \in I. \tag{9.α}$$

(b) *Moreover, $f \in \mathcal{L}(I)$ if and only if at least one of α, ω is in $\mathcal{L}(I)$.*

Proof. (a) ($\Rightarrow$) If $f \in \mathcal{R}^*(I)$, we can take $\alpha = \omega = f$.

($\Leftarrow$) Let (s_n) be a sequence of step functions such that $s_n(x) \to f(x)$ for a.e. $x \in I$ and let $\bar{s}_n := \operatorname{mid}\{\alpha, s_n, \omega\}$; therefore $\alpha(x) \le \bar{s}_n(x) \le \omega(x)$ for $x \in I$. Since it is clear that $\bar{s}_n(x) \to f(x)$ a.e. on I, the Dominated Convergence Theorem 8.8 implies that $f \in \mathcal{R}^*(I)$.

(b) Apply Theorem 7.9. Q.E.D.

Remarks. Two immediate consequences of the Integrability Theorem 9.1 are:

(a) *A bounded measurable function on a compact interval I is in $\mathcal{L}(I)$.*

(b) *Every measurable subset of a compact interval I is an integrable set; therefore, $\mathbb{M}(I) = \mathbb{I}(I)$ when $I := [a, b]$.*

• **9.2 Measurable Limit Theorem.** *If (f_n) is a sequence in $\mathcal{M}(I)$ and if $f_n \to f$ a.e. on $I := [a, b]$, then $f \in \mathcal{M}(I)$.*

Proof. Let $g_n(x) := \operatorname{Arctan}(f_n(x))$ and $g(x) := \operatorname{Arctan}(f(x))$ for $x \in I$. Theorem 6.3(d) implies that the g_n belong to $\mathcal{M}(I)$ and that

$$-\tfrac{1}{2}\pi < g_n(x) < \tfrac{1}{2}\pi \qquad \text{for all} \quad x \in I.$$

From the Integrability Theorem 9.1(b) we infer that $g_n \in \mathcal{L}(I)$. Since $f_n \to f$ a.e. on I, the continuity of Arctan implies that $g_n \to g$ a.e. on I, so the Dominated Convergence Theorem 8.8 implies that $g \in \mathcal{L}(I)$. Therefore, the Measurability Theorem 6.8 implies that $g \in \mathcal{M}(I)$. Since $f(x) = \tan(g(x))$ for $x \in I$, another application of Theorem 6.3(d) yields $f \in \mathcal{M}(I)$. Q.E.D.

Increasing Sequence Theorem

It is an elementary result that an increasing sequence of real numbers either converges to an element of $\mathbb{R}$, or "converges" to ∞.

We also recall that the a.e. version of the Monotone Convergence Theorem 8.5 asserts that if (φ_n) is an increasing sequence in $\mathcal{R}^*(I)$ and if $\varphi(x) = \lim \varphi_n(x)$ for a.e. $x \in I := [a, b]$, then $\varphi \in \mathcal{R}^*(I)$ if and only if the sequence of integrals $(\int_I \varphi_n)$ is bounded above in $\mathbb{R}$, in which case equation (8.$*$) holds.

We now show that if an increasing sequence (φ_n) in $\mathcal{R}^*(I)$ is such that $(\int_I \varphi_n)$ is bounded above, then this sequence of functions converges a.e. to a

finite-valued φ that belongs to $\mathcal{R}^*(I)$. This result will be used several times in the sequel.

• **9.3 Increasing Sequence Theorem.** *Let $(\varphi_n)_{n=1}^{\infty}$ be an increasing sequence in $\mathcal{R}^*(I)$ such that $\sup\{\int_I \varphi_n : n \in \mathbb{N}\} \leq K$.*

(a) *Then the set $Z := \{x \in I : \lim \varphi_n(x) = \infty\}$ is a null set.*

(b) *If we define $\varphi(x) := \lim \varphi_n(x) \in \mathbb{R}$ when $x \in I - Z$ and $\varphi(x) := 0$ when $x \in Z$, then $\varphi \in \mathcal{R}^*(I)$,*

$$(9.\beta) \qquad \int_I \varphi = \lim_{n\to\infty} \int_I \varphi_n \qquad \text{and} \qquad \lim_{n\to\infty} \int_I |\varphi - \varphi_n| = 0.$$

Proof. (a) Our strategy is to show that the characteristic function $\mathbf{1}_Z$ is a null function; therefore, Z is a null set. We will obtain $\mathbf{1}_Z$ as a limit.

We may assume that $0 \leq \varphi_n \in \mathcal{L}(I)$ for $n \in \mathbb{N}$; otherwise, we consider the sequence $(\tilde{\varphi}_n)$ defined by $\tilde{\varphi}_n := \varphi_n - \varphi_1$. For $n, m \in \mathbb{N}$, we define $\psi_{n,m}$ on I by

$$\psi_{n,m} := \min\{1, \tfrac{1}{m}\varphi_n\} \leq \tfrac{1}{m}\varphi_n.$$

Since $\psi_{n,m} \geq 0$, it follows from Theorems 6.6(b) and 9.1(b) that $\psi_{n,m} \in \mathcal{L}(I)$, and it is evident that

$$(9.\gamma) \qquad 0 \leq \int_I \psi_{n,m} \leq K/m.$$

Since $(\varphi_n)_n$ is increasing, so is the sequence $(\psi_{n,m})_n$ for each fixed $m \in \mathbb{N}$. Further, for $m \in \mathbb{N}$, we have

$$(9.\delta) \qquad \Psi_m(x) := \lim_{n\to\infty} \psi_{n,m}(x) = \begin{cases} 1 & \text{if} \quad x \in Z, \\ \min\{1, \frac{1}{m}\varphi\} & \text{if} \quad x \in I - Z. \end{cases}$$

If we use $(9.\gamma)$ and the Monotone Convergence Theorem 8.5, we conclude that $0 \leq \Psi_m \in \mathcal{L}(I)$ and that

$$0 \leq \int_I \Psi_m \leq K/m.$$

But $(\Psi_m)_m$ is a decreasing sequence of nonnegative functions on I. Hence, by the Monotone Convergence Theorem 8.5, we infer that $\Psi := \lim \Psi_m$ belongs to $\mathcal{L}(I)$ and that $\int_I \Psi = 0$. But $(9.\delta)$ implies that $\Psi(x) = 1$ when $x \in Z$ and $\Psi(x) = 0$ when $x \in I - Z$. Thus $\Psi = \mathbf{1}_Z$ is a null function and Z is a null set.

(b) Defining φ as indicated, the Monotone Convergence Theorem 8.5 implies that $\varphi \in \mathcal{L}(I)$ and the first limit in $(9.\beta)$ holds. Since we have

$|\varphi(x)-\varphi_n(x)| \le \varphi(x)$ for a.e. $x \in I$, and $|\varphi(x)-\varphi_n(x)| \to 0$ a.e., the second limit in (9.β) follows from the Mean Convergence Theorem 8.8. Q.E.D.

The next result is a reformulation of the Increasing Sequence Theorem 9.3 for a series of functions.

• **9.4 Beppo Levi's Theorem.** *Let $(\psi_k)_k$ be a sequence in $\mathcal{L}(I)$ with $K := \sum_{k=1}^{\infty} \int_I |\psi_k| < \infty$. Then there exists a null set $Z \subset I$ such that the series*

$$\sum_{k=1}^{\infty} \psi_k(x) \tag{9.ε}$$

converges absolutely to a function $\tau \in \mathcal{L}(I)$ for $x \in I - Z$. In addition, we have

$$\int_I \tau = \sum_{k=1}^{\infty} \int_I \psi_k, \tag{9.ζ}$$

and if (τ_n) denotes the sequence of partial sums of (9.ε), then $\int_I |\tau_n - \tau| \to 0$.

Proof. Let $\tau_n(x) := \sum_{k=1}^{n} \psi_k(x)$ and $\varphi_n(x) := \sum_{k=1}^{n} |\psi_k(x)|$ for $x \in I$ and $n \in \mathbb{N}$, so that τ_n and φ_n belong to $\mathcal{L}(I)$ and

$$\int_I |\tau_n| \le \sum_{k=1}^{n} \int_I |\psi_k| = \int_I \varphi_n \le K.$$

By the Increasing Sequence Theorem 9.3 there exists a null set $Z \subset I$ such that the limit $\varphi(x) := \lim_n \varphi_n(x)$ exists in $\mathbb{R}$ for all $x \in I - Z$. If we define $\varphi(x) := 0$ for $x \in Z$, then $\varphi \in \mathcal{L}(I)$.

The convergence of $(\varphi_n(x))$ on $I - Z$ implies the absolute convergence (and hence the convergence) of the series (9.ε) to $\tau(x)$ for $x \in I - Z$. Moreover, since $|\tau_n| \le \varphi$ a.e., it follows from the Dominated Convergence Theorem 8.8 that

$$\int_I \tau = \lim_{n\to\infty} \int_I \tau_n = \lim_{n\to\infty} \int_I \sum_{k=1}^{n} \psi_k = \sum_{k=1}^{\infty} \int_I \psi_k,$$

and from the Mean Convergence Theorem 8.9 that $\int_I |\tau_n - \tau| \to 0$. Q.E.D.

The Seminorm on $\mathcal{L}(I)$

We have noted in Theorem 7.8 that the space $\mathcal{L}(I)$ of absolutely integrable functions on a bounded closed interval $I \subset \mathbb{R}$ forms a *vector space* in the

sense of Appendix F. We will show that $\mathcal{L}(I)$ is a *seminormed* space in the sense of the following definition.

• **9.5 Definition.** If V is a real (or complex) vector space, then a function $N : V \to \mathbb{R}$ is said to be a **seminorm** on V in case it satisfies:

(i) $N(v) \geq 0$ for all $v \in V$;

(ii) $N(v) = 0$ when $v = 0$;

(iii) $N(cv) = |c|N(v)$ for all $v \in V$ and scalars c;

(iv) $N(u+v) \leq N(u) + N(v)$ for all $u, v \in V$.

A seminorm N is said to be a **norm** if, instead of (ii), it satisfies:

(ii*) $N(v) = 0$ if and only if $v = 0$.

The reader is certainly already familiar with norms and seminorms on certain vector spaces. We will recall a few examples in an exercise; others will be given in Appendix I. It will be seen in some exercises that seminorms give rise to semimetrics (in the sense of Appendix G).

• **9.6 Definition.** **(a)** If $f \in \mathcal{L}(I)$, we define $\|f\|$ or $\|f\|_1$ by

$$\|f\| := \int_I |f|.$$

(b) If $f, f_n \in \mathcal{R}^*(I)$ and $f - f_n \in \mathcal{L}(I)$ for all $n \in \mathbb{N}$, we say that the sequence (f_n) **converges in mean** (or **converges in** $\mathcal{L}(I)$) to f in case

$$\lim_{n\to\infty} \|f_n - f\| = 0. \tag{9.η}$$

In this case we sometimes write

$$f_n \to f \quad \text{[mean] on} \quad I.$$

Remarks. **(a)** We sometimes write $\mathcal{L}^1(I)$ instead of $\mathcal{L}(I)$.

(b) Mean convergence is usually considered when the functions f, f_n belong to $\mathcal{L}(I)$, but it also makes sense when the differences $f - f_n$ are in $\mathcal{L}(I)$.

• **9.7 Lemma.** *The mapping* $f \mapsto \|f\|$ *in Definition* 9.6 *is a seminorm on* $\mathcal{L}(I)$. *Moreover,*

$$\|f\| = 0 \quad \Leftrightarrow \quad f = 0 \text{ a.e.} \quad \Leftrightarrow \quad f \text{ is a null function on } I.$$

Proof. Property (i) of Definition 9.5 is obvious, as is (ii) (where 0 also denotes the function identically equal to the number $0 \in \mathbb{R}$). Property (iii)

follows from Theorem 3.1, since $|cf(x)| = |c| \cdot |f(x)|$ for all $x \in I$, so that

$$\|cf\| = \int_I |cf| = \int_I |c| \cdot |f| = |c| \int_I |f| = |c| \cdot \|f\|.$$

The Triangle Inequality implies that

$$|(f+g)(x)| = |f(x) + g(x)| \le |f(x)| + |g(x)| \qquad \text{for} \quad x \in I.$$

Therefore, it follows from Corollary 3.3 and Theorem 3.1 that

$$\|f + g\| = \int_I |f + g| \le \int_I |f| + \int_I |g| = \|f\| + \|g\|,$$

which is (iv). If $f(x) = 0$ a.e., then f is a null function; so, by Example 2.6(b), $\|f\| = \int_I |f| = 0$. Conversely, if $\int_I |f| = 0$, it was seen in Theorem 5.10 that $|f|$ (and therefore f) is a null function on I. Q.E.D.

Remarks. If we are given a seminorm N on a vector space V, then it is always possible to "identify" various elements of V and obtain a related norm $\tilde{N}$ on a vector space $\tilde{V}$ of equivalence classes in V. In fact, what one is doing in this process is replacing the vector space V by the quotient space of V modulo the subspace $V_N := \{v \in V : N(v) = 0\}$. Thus, an element of $\tilde{V} := V/V_N$ is an *equivalence class*:

$$[v]_N := \{u \in V : N(u - v) = 0\}.$$

It is easy to verify that, on the space $\tilde{V}$, the function $\tilde{N}$ defined by

$$\tilde{N}([v]_N) := N(u) \qquad \text{where} \quad u \in [v]_N,$$

is well defined and gives a norm.

One can readily apply the foregoing procedure to the seminorm $f \mapsto \|f\|$ on $\mathcal{L}(I)$ and obtain a proper norm on a space of equivalence classes of functions in $\mathcal{L}(I)$ modulo null functions. After years of saying this, yet continuing to think of functions (rather than equivalence classes of functions), this author has decided that this cure is worse than the disease: the mere fact that we have a seminorm on $\mathcal{L}(I)$, rather than a proper norm. *Therefore we will continue to deal with functions (and not equivalence classes) and work with a seminorm rather than a norm.* To remind the reader that we are dealing with functions, we will retain the notation $\mathcal{L}(I)$ or $\mathcal{L}^1(I)$, rather than $L(I)$, or $L^1(I)$, which is customary when dealing with the space of equivalence classes. And we must remember that $\|f\| = 0$ does not imply that $f = 0$, but only that $f = 0$ a.e.

• **9.8 Lemma.** *If $f, g \in \mathcal{L}(I)$, then*

$$(9.\theta) \qquad \left|\int_I f - \int_I g\right| \leq \|f - g\|,$$

$$(9.\iota) \qquad \Big|\|f\| - \|g\|\Big| \leq \|f - g\|.$$

Proof. Since $\pm(f(x) - g(x)) \leq |f(x) - g(x)|$, Corollary 3.3 implies that

$$\pm\left(\int_I f - \int_I g\right) \leq \int_I |f - g| = \|f - g\|,$$

proving $(9.\theta)$. To prove $(9.\iota)$, we use 9.5(iv) to obtain:

$$\|f\| \leq \|(f-g)+g\| \leq \|f-g\|+\|g\|, \quad \|g\| \leq \|(g-f)+f\| = \|f-g\|+\|f\|,$$

which combine to yield $(9.\iota)$. Q.E.D.

We first note that the mean convergence of a sequence in $\mathcal{L}(I)$ implies the convergence of the sequence of integrals and the sequence of seminorms.

• **9.9 Theorem.** *If $f_n, f \in \mathcal{L}(I)$ for $n \in \mathbb{N}$ and if $f_n \to f$ in mean, then*

$$\int_I f_n \to \int_I f \quad \text{and} \quad \|f_n\| \to \|f\|.$$

Proof. Apply $(9.\theta)$ and $(9.\iota)$. Q.E.D.

The following examples may be useful in understanding the nature of convergence in mean.

9.10 Examples. (a) Let (φ_n) be a sequence in $\mathcal{L}([a,b])$ that converges uniformly on $[a,b]$ to φ. It is an exercise to show that $\varphi \in \mathcal{L}([a,b])$ and that $\varphi_n \to \varphi$ in mean. We leave the details to the reader.

(b) Let $g_n(x) := n$ for $x \in (0, 1/n)$ and $g_n(x) := 0$ elsewhere in $[0,1]$, and let $g(x) := 0$ for all $x \in [0,1]$. It is clear that $g_n, g \in \mathcal{L}([0,1])$ and that $g_n(x) \to g(x)$ for all $x \in [0,1]$. However, $\|g_n - g\| = 1$ so that (g_n) does not converge in mean to g. Note also that $0 = \|g\| \neq \lim \|g_n\| = 1$.

(c) Let (f_k) be the sequence in Exercise 8.F. Recall that $(f_k(x))$ does not converge to $f(x) := 0$ at *any* point of $[0,1]$. Moreover, $\|f\| = 0 = \lim \|f_k\|$ and $\|f_k - f\| \to 0$ so that (f_k) converges in mean to f. Note that a subsequence of (f_k) converges a.e. to f.

(d) Let h_n be the piecewise linear function on $I := [-1, 1]$ passing through the points $(-1, 0), (0, 0), (1/n, 1), (1, 1)$, and let $h(x) := 0$ for $x \in [-1, 0]$ and $h(x) := 1$ for $x \in (0, 1]$. Then h_n is continuous on I for each $n \in \mathbb{N}$, but h is not continuous. Show that $h_n(x) \to h(x)$ for all $x \in I$, that the convergence is not uniform on I and that $h_n \to h$ in mean.

The Completeness Theorem

We next establish the important "completeness property" of the space $\mathcal{L}(I)$.

• **9.11 Definition.** A sequence (f_n) in $\mathcal{R}^*(I)$ is said to be a **mean Cauchy sequence** if for every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that if $m, n \geq N(\varepsilon)$ then $f_m - f_n \in \mathcal{L}(I)$ and $\|f_m - f_n\| \leq \varepsilon$.

It is an exercise to show that if a sequence (f_n) in $\mathcal{L}(I)$ converges in mean to a function f, then the sequence is a mean Cauchy sequence; the next theorem is the converse implication. This important result was proved independently in 1906 by Frigyes (= Frédéric) Riesz (1880–1956) and Ernst Fischer (1875–1954) and plays an important role in functional analysis.

• **9.12 Completeness Theorem.** *The space $\mathcal{L}(I)$ is complete in the sense that every mean Cauchy sequence converges in mean to a function in $\mathcal{L}(I)$.*

Proof. Let (f_n) be a mean Cauchy sequence in $\mathcal{L}(I)$. We will first show that there exists a subsequence (g_k) of (f_n) that converges a.e. and in mean to a function $g \in \mathcal{L}(I)$. We will then show that the full sequence (f_n) converges in mean to g.

Since (f_n) is a mean Cauchy sequence, by induction we find a strictly increasing sequence $n_1 < n_2 < \cdots < n_k < \cdots$ such that if $m, n \geq n_k$, then

$$\|f_m - f_n\| \leq 1/2^k.$$

Now let $g_k := f_{n_k}$ so that (g_k) is a subsequence of (f_n) and $\|g_{k+1} - g_k\| \leq 1/2^k$ for $k \in \mathbb{N}$. If we put $\psi_1 := g_1$ and $\psi_k := g_k - g_{k-1}$ for $2 \leq k \in \mathbb{N}$, then

$$\sum_{k=1}^{\infty} \|\psi_k\| = \|g_1\| + \sum_{k=2}^{\infty} \|g_k - g_{k-1}\| \leq \|g_1\| + 1.$$

By Beppo Levi's Theorem 9.4, the series $\sum_{k=1}^{\infty} \psi_k = g_1 + \sum_{k=2}^{\infty} (g_k - g_{k-1})$ converges a.e. and in mean to a function $g \in \mathcal{L}(I)$. But, since the nth partial sum of this telescoping series is g_n, we conclude that (g_n) converges a.e. and in mean to g. Moreover, $\|g - g_n\| \leq \sum_{k=n+1}^{\infty} \|g_k - g_{k-1}\| \leq 1/2^{n-1}$.

To show that the entire sequence (f_n) converges in mean to g, note that if $m \geq n_k$, then since $n_m \geq m \geq n_k$, we have $\|f_m - g_m\| \leq 1/2^k$. Therefore

$$\|f_m - g\| \leq \|f_m - g_m\| + \|g_m - g\| \leq 1/2^k + 1/2^{k-1} \leq 1/2^{k-2}.$$

Consequently, the sequence (f_m) converges in mean to g. Q.E.D.

Example 9.10(c) shows that a mean convergent sequence may not converge at any point. However, we now prove that a mean convergent sequence always has a *subsequence* that converges a.e. to its limit.

• **9.13 Corollary.** *If a sequence (f_n) converges in mean to $f \in \mathcal{L}(I)$, then there exists a subsequence of (f_n) that converges a.e. to f.*

Proof. Since $\|f_n - f\| \to 0$, it follows that the sequence (f_n) is a mean Cauchy sequence. By the proof of the Completeness Theorem 9.12, there exist a subsequence (g_k) of (f_n) and a function $g \in \mathcal{L}(I)$ such that $g_k \to g$ a.e. and in mean. Therefore,

$$0 \leq \|f - g\| \leq \|f - g_k\| + \|g_k - g\| \to 0,$$

so that $\|f - g\| = 0$, whence $f = g$ a.e. on I. Since $g_k \to g$ a.e. on I, we also have $g_k \to f$ a.e. Q.E.D.

Some Density Results

If $f : I \to \mathbb{R}$ and $n \in \mathbb{N}$, then the **n-truncate** of f is the function $f^{[n]}$ defined on $I := [a, b]$ by

$$f^{[n]} := \operatorname{mid}\{-n, f, n\}.$$

It is a consequence of Theorem 6.6(c) that if $f \in \mathcal{M}(I)$, then each truncate $f^{[n]}$ also belongs to $\mathcal{M}(I)$.

• **9.14 Truncation Theorem.** *If $f \in \mathcal{L}(I)$, then its sequence $(f^{[n]})$ of truncates converges everywhere and in mean to f.*

Proof. Since $|f^{[n]}(x)| \leq |f(x)|$ for all $x \in I$, it follows from the Integrability Theorem 9.1(b) that $f^{[n]} \in \mathcal{L}(I)$. Since $f^{[n]}(x) \to f(x)$ for all $x \in I$, the Mean Convergence Theorem 8.9 implies that $\|f^{[n]} - f\| \to 0$. Q.E.D.

We now show that a function in $\mathcal{L}(I)$ can be approximated arbitrarily closely by step functions and by continuous functions.

• **9.15 Density Theorem.** **(a)** *If $f \in \mathcal{L}(I)$ and $\varepsilon > 0$, then there exists a step function s_ε such that $\|f - s_\varepsilon\| \leq \varepsilon$.*

(b) *If $f \in \mathcal{L}(I)$ and $\varepsilon > 0$, then there exists a continuous function h_ε such that $\|f - h_\varepsilon\| \leq \varepsilon$.*

Proof. (a) It follows from the Truncation Theorem 9.14 that there exists $n \in \mathbb{N}$ such that $\|f - f^{[n]}\| \leq \frac{1}{2}\varepsilon$. Now fix n. Since $f^{[n]}$ is measurable, there exists a sequence of step functions (s_k) such that $s_k \to f^{[n]}$ a.e. on I. With no loss of generality, we may assume that $|s_k(x)| \leq n$ for all $x \in I$; otherwise, we replace s_k by its n-truncate. Now we apply the Mean Convergence Theorem 8.9 to conclude that $\|f^{[n]} - s_k\| \to 0$ as $k \to \infty$. Finally, choose k so large that $\|f^{[n]} - s_k\| \leq \frac{1}{2}\varepsilon$, whence we conclude from the Triangle Inequality that $\|f - s_k\| \leq \varepsilon$.

(b) Argue as in (a), using Theorem 6.7. Q.E.D.

• **9.16 Corollary.** **(a)** *If $f \in \mathcal{L}(I)$, then there exists a sequence (s_n) of step functions that converges a.e. and in mean to f.*

(b) *If $f \in \mathcal{L}(I)$, then there exists a sequence (h_n) of continuous functions that converges a.e. and in mean to f.*

We leave the details to the reader.

The Riemann-Lebesgue Theorem

We close this section by establishing a result that is of great importance in harmonic analysis. In addition this proof illustrates how the Density Theorem 9.15 is often applied by first establishing the assertion for a step (or continuous) function, and then taking limits.

• **9.17 Riemann-Lebesgue Lemma.** *If $f \in \mathcal{L}([a,b])$ and $\beta \in \mathbb{R}$, then*

$$\lim_{\alpha \to \infty} \int_a^b f(x) \sin(\alpha x + \beta)\, dx = 0.$$

Proof. If f is a constant function $f(x) = c$ for all $x \in [a_1, b_1] \subseteq I$, then the result is an immediate consequence of the fact that if $\alpha > 0$, then

$$\left| \int_{a_1}^{b_1} c \sin(\alpha x + \beta)\, dx \right| = \frac{|c|}{\alpha} \cdot \left| \cos(\alpha a_1 + \beta) - \cos(\alpha b_1 + \beta) \right| \leq \frac{2|c|}{\alpha} \to 0.$$

From the linearity of the integral, it follows that the statement is true if f is a step function on I.

Now, if $f \in \mathcal{L}(I)$ is arbitrary and $\varepsilon > 0$, the Density Theorem 9.15 implies that there exists a step function s_ε on I such that $\|f - s_\varepsilon\| \leq \varepsilon$.

Therefore,

$$\left|\int_a^b f(x)\sin(\alpha x+\beta)\,dx - \int_a^b s_\varepsilon(x)\sin(\alpha x+\beta)\,dx\right|$$
$$\le \int_a^b \left|\{f(x)-s_\varepsilon(x)\}\sin(\alpha x+\beta)\right|dx$$
$$\le \|f-s_\varepsilon\| \le \varepsilon(b-a).$$

Therefore we have that

$$\left|\int_a^b f(x)\sin(\alpha x+\beta)\,dx\right| \le \left|\int_a^b s_\varepsilon(x)(\sin\alpha x+\beta)\,dx\right| + \varepsilon(b-a),$$

whence the statement follows. Q.E.D.

Note. The Riemann-Lebesgue Theorem does *not* hold for $f \in \mathcal{R}^*(I)$. Indeed, Riemann [R; p. 260 ff.] showed that the Fourier coefficients of the function $f(x) := [x^\nu \cos(1/x)]'$ on $[0, 2\pi]$ do not converge to 0 when $0 < \nu < \frac{1}{2}$. See also Zygmund [Z; p. 19]. However, we will show in Example 12.3(c) that these coefficients are $o(n)$. For an interesting discussion of these matters, see Talvila [T-2].

This failure of the Riemann-Lebesgue Theorem for functions in $\mathcal{R}^*(I)$ implies that the theory of Fourier series for these functions is somewhat more complicated than for functions in $\mathcal{L}(I)$. However, many of the principal features of this theory still remain; see the book of Čelidze and Džvaršeĭšvili [C-D; Chapter 3].

Exercises

9.A Show that a set $Z \subseteq I$ is a null set if and only if there exists an increasing sequence (f_n) in $\mathcal{L}(I)$ such that $f_n(x) \to \infty$ for all $x \in Z$, and the sequence $(\int_I f_n)$ converges in $\mathbb{R}$.

9.B Show that a set $Z \subseteq I$ is a null set if and only if there exists a sequence (g_n) in $\mathcal{L}(I)$ such that $\sum_{n=1}^\infty |g_n(x)|$ diverges for all $x \in Z$ and $\sum_{n=1}^\infty \int_I |g_n|$ converges in $\mathbb{R}$.

9.C Let $\sum_{n=1}^\infty h_n$ be a series in $\mathcal{L}(I)$ such that the series $\sum_{n=1}^\infty \int_I |h_n|$ is convergent. Show that $\lim h_n = 0$ a.e.

9.D Let (f_n) be a sequence in $\mathcal{L}(I)$ such that $\sum_{n=1}^{\infty} f_n = 0$ a.e. and $\sum_{n=1}^{\infty} \int_I |f_n|$ is convergent in $\mathbb{R}$. Prove that $\sum_{n=1}^{\infty} \int_I f_n = 0$.

9.E Let (f_n) be a sequence of null functions. Show that $\sum_{n=1}^{\infty} f_n$ is absolutely convergent a.e. to 0.

9.F Let (g_n) be a sequence of nonnegative functions in $\mathcal{L}(I)$ that converges a.e. on I and with $\lim \int_I g_n = 0$. Prove that $\lim g_n = 0$ a.e.

9.G Let (f_n) be a sequence of nonnegative functions in $\mathcal{L}(I)$ such that $f_n(x) \to 0$ a.e., and there exists $M > 0$ such that

$$\int_I (f_1 \vee \cdots \vee f_n) \le M \qquad \text{for all} \quad n \in \mathbb{N}.$$

Prove that $\lim \int_I f_n = 0 = \lim \|f_n\|$.

9.H Let (g_n) be the sequence of nonnegative functions in $\mathcal{L}([0,1])$ given in Example 9.10(b), so that $g_n \to 0$ a.e., but $\lim \int_0^1 g_n \ne 0$. Show directly that $\lim \int_0^1 (g_1 \vee \cdots \vee g_n) = \infty$.

9.I Suppose that $f, g \in \mathcal{L}([0,1])$. For each of the following functions, either show that it belongs to $\mathcal{L}([0,1])$, or give a counterexample.

(a) f^2,	(b) $\sqrt{\lvert f\rvert}$,	(c) $\lvert\operatorname{mid}\{-1, f, 1\}\rvert^{1/2}$,
(d) $\operatorname{Arctan}(f)$,	(e) $\max\{\ln\lvert f\rvert, 0\}$,	(f) fg,
(g) $\sqrt{\lvert fg\rvert}$,	(h) $\lvert f\rvert/(1+\lvert g\rvert)$,	(i) $\sqrt{1+f^2}$.

9.J If N is a seminorm on a real vector space V and if $\sigma(u,v) := N(u-v)$ for $u, v \in V$, show that σ is a semimetric function in the sense of Appendix G. Show also that σ is a metric function if and only if N is a norm.

9.K (a) If l^∞ denotes the collection of all **bounded sequences** $u = (u_n)$ of elements in $\mathbb{R}$ or $\mathbb{C}$, show that l^∞ is a vector space (in the sense of Appendix F) under the operations:

$$c\,(u_n) := (cu_n) \qquad \text{and} \qquad (u_n) + (v_n) := (u_n + v_n).$$

Here the zero element is the zero sequence (0), and $-(u_n) = (-u_n)$.

(b) If we define $N_1(u) := \sup\{|u_n| : n \in \mathbb{N}\}$, show that N_1 is a norm on l^∞.

(c) If we define $N_2(u) := \sup\{|u_n| : n \geq 2\}$, show that N_2 is a seminorm on l^∞, but is not a norm on l^∞.

(d) Show that l^∞ is complete under both N_1 and N_2.

9.L (a) If l^1 denotes the collection of all sequences $u = (u_n)$ of elements in $\mathbb{R}$ or $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |u_n|$ is convergent in $\mathbb{R}$, show that l^1 is a vector space under the same operations as in Exercise 9.K. The collection l^1 is called the space of **absolutely convergent** series.

(b) If we define $N_1(u) := \sum_{n=1}^{\infty} |u_n|$, show that N_1 is a norm on l^1.

(c) If we define $N_2(u) := \sum_{n=2}^{\infty} |u_n|$, show that N_2 is a seminorm on l^1, but is not a norm on l^1.

(d) Show that l^1 is complete under both N_1 and N_2.

9.M Let $f, f_n \in \mathcal{L}(I)$ and let $f_n \to f$ a.e. Obtain E. Lieb's extension of Fatou's Lemma:

$$\lim_{n\to\infty} \big| \|f_n - f\| - \|f_n\| + \|f\| \big| = 0,$$

whence it follows that $\|f\| = \lim_n \big(\|f_n\| - \|f_n - f\|\big)$. [Hint: Use the Triangle Inequality twice to show that $\big||f_n - f| - |f_n| + |f|\big| \leq 2|f|$ and apply the Dominated Convergence Theorem.]

9.N Let $f, f_n \in \mathcal{L}(I)$ and let $f_n \to f$ a.e. Prove that $\|f_n - f\| \to 0$ if and only if $\|f_n\| \to \|f\|$.

9.O Write out the details of the proof of 9.15(b). In fact, show that if $f \in \mathcal{L}(I)$ and $\varepsilon > 0$, then there is a piecewise linear continuous function p_ε on I such that $\|f - p_\varepsilon\| < \varepsilon$.

9.P Suppose (φ_n) in $\mathcal{R}^*(I)$ is a mean Cauchy sequence in the sense that $\varphi_m - \varphi_n \in \mathcal{L}(I)$ for $m, n \in \mathbb{N}$ and $\lim_{m,n} \|\varphi_m - \varphi_n\| = 0$.

(a) Show that there exists a subsequence (φ_{n_k}) of (φ_n) and a function $\varphi \in \mathcal{R}^*(I)$ such that $\varphi_{n_k} \to \varphi$ a.e. and $\|\varphi_{n_k} - \varphi\| \to 0$.

(b) Show that $\|\varphi_n - \varphi\| \to 0$ and that $\int_I \varphi_n \to \int_I \varphi$.

9.Q (a) Suppose that f_n, α_n, α belong to $\mathcal{R}^*(I)$ and that $\alpha_n \leq f_n$ a.e. on I. Suppose also that $\|\alpha_n - \alpha\| \to 0$, that $f_n \to f$ a.e. and that $\liminf \int_I f_n < \infty$. Show that $f \in \mathcal{R}^*(I)$ and that $-\infty < \int_I f \leq \liminf \int f_n < \infty$. (Compare this extension of Fatou's Lemma with Exercise 8.L.)

(b) Suppose that f_n, ω_n, ω belong to $\mathcal{R}^*(I)$ and that $f_n \le \omega_n$ a.e. on I. Suppose also that $\|\omega_n - \omega\| \to 0$, that $f_n \to f$ a.e. and that $-\infty < \limsup \int_I f_n$. Show that $f \in \mathcal{R}^*(I)$ and $-\infty < \limsup \int_I f_n \le \int_I f < \infty$.

9.R Suppose that $\alpha_n, \alpha, f_n, \omega_n, \omega$ belong to $\mathcal{R}^*(I)$ and that $\alpha_n \le f_n \le \omega_n$ a.e. on I. Suppose also that $\|\alpha_n - \alpha\| \to 0$, $\|\omega_n - \omega\| \to 0$ and $f = \lim f_n$ a.e. on I. Prove that $f \in \mathcal{R}^*(I)$ and that $\int_I f_n \to \int_I f$. In addition, prove that $f_n - f \in \mathcal{L}(I)$ and that $\|f_n - f\| \to 0$. (Compare this extension of the Dominated Convergence Theorem 8.8 with Exercise 8.M.)

9.S Let $I := [a, b]$ and let $\mathcal{L}^2(I)$ denote the collection of all measurable functions $f : I \to \mathbb{R}$ such that $x \mapsto |f(x)|^2$ is integrable on I. The collection $\mathcal{L}^2(I)$ is called the space of **square integrable** functions on I. We define

$$\|f\|_2 := \left(\int_I |f|^2 \right)^{1/2}.$$

Sometimes we will denote $\mathcal{L}(I)$ by $\mathcal{L}^1(I)$ and $\|f\|$ by $\|f\|_1$. We will give certain results here that will be sharpened in the next exercise.

(a) Let $I := [0, 4]$ and $f(x) := 1$ for $x \in I$ and $g(x) := 1$ on $[0, \frac{1}{4}]$ and $g(x) := 0$ elsewhere on I. Show that $\|f\|_2 < \|f\|_1$ while $\|g\|_1 < \|g\|_2$.

(b) If $f \in \mathcal{L}^2(I)$, show that $f \in \mathcal{L}^1(I)$ and $\|f\|_1 \le (b - a) + \|f\|_2^2$. [Hint: $|t| \le 1 + t^2$ for $t \in \mathbb{R}$.]

(c) If $f, g \in \mathcal{L}^2(I)$, the product $fg \in \mathcal{L}^1(I)$ and $\|fg\|_1 \le \frac{1}{2}(\|f\|_2^2 + \|g\|_2^2)$.

(d) If $f, g \in \mathcal{L}^2(I)$, the sum $f + g \in \mathcal{L}^2(I)$ and $\|f + g\|_2^2 \le 2(\|f\|_2^2 + \|g\|_2^2)$.

9.T We consider $\mathcal{L}^2(I)$, where $I := [a, b]$, and will sharpen some inequalities in Exercise 9.S.

(a) Show that $2\|fg\|_1 \le t\|f\|_2^2 + (1/t)\|g\|_2^2$ for all $t > 0$. Show that, if $\|f\|_2 \ne 0$, then the right side of this inequality is minimized for $t = \|g\|_2 / \|f\|_2$. Now obtain the important **Schwarz (or Cauchy-Bunyakovskiĭ-Schwarz) Inequality**:

$$\|fg\|_1 \le \|f\|_2 \cdot \|g\|_2.$$

(b) If $f \in \mathcal{L}^2(I)$, then $f \in \mathcal{L}^1(I)$ and $\|f\|_1 \le \sqrt{b - a} \cdot \|f\|_2$.

(c) If $f, g \in \mathcal{L}^2(I)$, establish the Triangle Inequality: $\|f + g\|_2 \le \|f\|_2 + \|g\|_2$.

(d) Complete the proof that the map $f \mapsto \|f\|_2$ gives a seminorm on $\mathcal{L}^2(I)$ and that the null functions on I are precisely the functions with $\|f\|_2 = 0$.

(e) If (f_n) is a Cauchy sequence relative to the seminorm $\|\cdot\|_2$, show that there exists a subsequence (f_{n_k}) of (f_n) converging a.e. to some function f.

(f) Complete the proof that $\mathcal{L}^2(I)$ is complete under the seminorm $\|\cdot\|_2$.

9.U Show that if (f_n) is a Cauchy sequence relative to $\|\cdot\|_2$, then it is also a Cauchy sequence relative to $\|\cdot\|_1$, but give an example of a sequence in $\mathcal{L}^2(I)$ that is a Cauchy sequence relative to $\|\cdot\|_1$ but is not a Cauchy sequence relative to $\|\cdot\|_2$.

9.V Prove that $f \in \mathcal{L}(I)$ if and only if there exists a sequence (s_n) of step functions [or continuous functions] such that (i) $s_n \to f$ a.e., and that (ii) $\|s_m - s_n\| \to 0$ as $m, n \to \infty$.

Measure, Measurability, and Multipliers

We recall from Definition 6.14 that a subset E of a compact interval $I := [a, b]$ is measurable [respectively, integrable] if its characteristic function $\mathbf{1}_E$ belongs to $\mathcal{M}(I)$ [respectively, $\mathcal{L}(I)$]. From the Integrability Theorem 9.1 and the boundedness of $\mathbf{1}_E$, it follows that the set E is measurable if and only if it is integrable. In this case, we defined the **measure** of E to be

$$(10.\alpha) \qquad |E| := \int_I \mathbf{1}_E = \|\mathbf{1}_E\|.$$

In Exercises 6.O and 6.P it was seen that the notion of a measurable set, and the value of its measure, do not depend on the interval I that contains it. We will use the symbol $\mathbb{M}(I)$ to denote the collection of all measurable subsets of the interval I.

In this section we first obtain some important properties of the measure function, after which we give another characterization of measurable functions on I. We then consider integrals of $f \in \mathcal{L}(I)$ over measurable sets. Finally we obtain conditions under which the product $f \cdot m$ of functions in $\mathcal{R}^*(I)$ belongs to $\mathcal{R}^*(I)$; in particular, we prove the Multiplier Theorem stated in 6.12.

In Sections 18 and 19 we will give a detailed discussion of measurable sets in, and measurable functions on $\mathbb{R}$. Many (*but not all*) of the results here will be extended to certain unbounded sets and functions defined on

them. We will continue to mark our results with the flags • and ⋄, and will comment on these results in Section 17.

Sequences of Measurable Sets

We first consider properties of the function $E \mapsto |E|$ defined on $\mathbb{M}(I) \to \mathbb{R}$. Properties of this mapping will be of great importance in the following.

⋄ **10.1 Lemma.** *Let $I := [a, b]$ be a compact interval.*

(a) *If $E, F \in \mathbb{M}(I)$, then $E \cup F$, $E \cap F$, and $E - F$ belong to $\mathbb{M}(I)$ and*

$$|E \cup F| + |E \cap F| = |E| + |F|. \tag{10.β}$$

(b) *If $E, F \in \mathbb{M}(I)$ and $E \cap F = \emptyset$, then $|E \cup F| = |E| + |F|$.*

(c) *If $E, F \in \mathbb{M}(I)$ and $E \subseteq F$, then $|E| \le |F|$.*

Proof. (a) It is seen that $\mathbf{1}_{E \cup F} = \max\{\mathbf{1}_E, \mathbf{1}_F\}$ and $\mathbf{1}_{E \cap F} = \min\{\mathbf{1}_E, \mathbf{1}_F\}$. Thus, Theorem 7.12 implies that $E \cup F$ and $E \cap F$ belong to $\mathbf{M}(I)$ when E and F do. Since it is clear that $\mathbf{1}_{E-F} = \mathbf{1}_E - \mathbf{1}_{E \cap F}$, it follows from Theorem 3.1 that $E - F \in \mathbf{M}(I)$. Since $\mathbf{1}_{E \cup F} + \mathbf{1}_{E \cap F} = \mathbf{1}_E + \mathbf{1}_F$, Theorem 3.1 implies that

$$\begin{aligned} |E \cup F| + |E \cap F| &= \int_I \mathbf{1}_{E \cup F} + \int_I \mathbf{1}_{E \cap F} \\ &= \int_I \mathbf{1}_E + \int_I \mathbf{1}_F = |E| + |F|. \end{aligned}$$

(b) If $E \cap F = \emptyset$, then $\mathbf{1}_{E \cap F} = 0$ so that $|E \cap F| = 0$. Hence the conclusion follows from (10.β).

(c) If $E \subseteq F$, then $\mathbf{1}_E \le \mathbf{1}_F$ and Corollary 3.3 applies. Q.E.D.

It follows from Lemma 10.1 and mathematical induction that if $E_1, \cdots, E_m$ belong to $\mathbf{M}(I)$, then their union $E_1 \cup \cdots \cup E_m$ and their intersection $E_1 \cap \cdots \cap E_m$ belong to $\mathbb{M}(I)$. Further, the "finite subadditivity property":

$$|E_1 \cup \cdots \cup E_m| \le |E_1| + \cdots + |E_m|,$$

is always true, while if the sets E_i are pairwise disjoint (so that $E_i \cap E_j = \emptyset$ for $i \ne j$) then one has the "finite additivity relation":

$$|E_1 \cup \cdots \cup E_m| = |E_1| + \cdots + |E_m|.$$

⋄ **10.2 Theorem.** **(a)** *If $(E_k)_{k=1}^{\infty}$ is any sequence in $\mathbb{M}(I)$, then the union $\bigcup_{k=1}^{\infty} E_k$ and the intersection $\bigcap_{k=1}^{\infty} E_k$ also belong to $\mathbb{M}(I)$.*

(b) *If $E_1 \subseteq \cdots \subseteq E_k \subseteq E_{k+1} \subseteq \cdots$ is an increasing sequence in $\mathbb{M}(I)$, then*

$$(10.\gamma) \qquad \Big|\bigcup_{k=1}^{\infty} E_k\Big| = \sup\big\{|E_k| : k \in \mathbb{N}\big\} = \lim_{k\to\infty} |E_k|.$$

(c) *If $E_1 \supseteq \cdots \supseteq E_k \supseteq E_{k+1} \supseteq \cdots$ is a decreasing sequence in $\mathbb{M}(I)$, then*

$$(10.\delta) \qquad \Big|\bigcap_{k=1}^{\infty} E_k\Big| = \inf\big\{|E_k| : k \in \mathbb{N}\big\} = \lim_{k\to\infty} |E_k|.$$

(d) *If $(E_k)_{k=1}^{\infty}$ is a pairwise disjoint sequence in $\mathbb{M}(I)$ (so that $E_k \cap E_j = \emptyset$ for all $k \neq j$), then*

$$(10.\varepsilon) \qquad \Big|\bigcup_{k=1}^{\infty} E_k\Big| = \sum_{k=1}^{\infty} |E_k|.$$

(e) *If $(E_k)_{k=1}^{\infty}$ is an arbitrary sequence in $\mathbb{M}(I)$, then*

$$(10.\zeta) \qquad \Big|\bigcup_{k=1}^{\infty} E_k\Big| \leq \sum_{k=1}^{\infty} |E_k| \leq \infty.$$

Proof. (a) By hypothesis, $\mathbf{1}_{E_k} \in \mathcal{R}^*(I)$ for each $k \in \mathbb{N}$. Therefore, it follows from Lemma 8.6 (with $\alpha = 0$) that $\inf\{\mathbf{1}_{E_k} : k \in \mathbb{N}\}$ belongs to $\mathcal{R}^*(I)$. But since (see Exercise 10.B) this infimum is the characteristic function of the set $E := \bigcap_{k=1}^{\infty} E_k$, it follows that $E \in \mathbb{M}(I)$.

The statement about the union follows from the result dual to Lemma 8.6 (see Exercise 8.K with $\omega = 1$), or by using the fact that

$$\bigcup_{k=1}^{\infty} E_k = I - \bigcap_{k=1}^{\infty} (I - E_k).$$

(b) Here $(\mathbf{1}_{E_k})$ is an increasing sequence in $\mathcal{M}(I)$ with $\int_I \mathbf{1}_{E_k} \leq b - a$, so the Monotone Convergence Theorem 8.5 applies.

(c) Here the sequence $(\mathbf{1}_{E_k})$ is decreasing with $\int_I \mathbf{1}_{E_k} \geq 0$, so the Monotone Convergence Theorem 8.5 applies.

(d) If $F_n := E_1 \cup \cdots \cup E_n$ for $n \in \mathbb{N}$, then the sequence $(F_n)_{n=1}^{\infty}$ is increasing and $|F_n| = \sum_{k=1}^{n} |E_k|$ for each $n \in \mathbb{N}$. Since $\bigcup_{n=1}^{\infty} F_n = \bigcup_{k=1}^{\infty} E_k$, it

follows from (b) that

$$\Big|\bigcup_{k=1}^{\infty} E_k\Big| = \Big|\bigcup_{n=1}^{\infty} F_n\Big| = \lim_{n\to\infty} |F_n|$$
$$= \lim_{k\to\infty} \sum_{k=1}^{n} |E_k| = \sum_{k=1}^{\infty} |E_k|.$$

(e) Let $C_1 := E_1$ and $C_n := E_n - \bigcup_{k=1}^{n-1} E_k$ for $n \geq 2$. Then $(C_n)_{n=1}^{\infty}$ is a pairwise disjoint sequence in $\mathbb{M}(I)$ such that $\bigcup_{k=1}^{\infty} E_k = \bigcup_{n=1}^{\infty} C_n$, whence

$$\Big|\bigcup_{k=1}^{\infty} E_k\Big| = \Big|\bigcup_{n=1}^{\infty} C_n\Big| = \sum_{n=1}^{\infty} |C_n|.$$

But since $|C_n| \leq |E_n|$, the inequality (10.ζ) is immediate. Q.E.D.

The relations (10.γ) and (10.δ) exhibit the **monotone properties** of the measure function, the relation (10.ε) is called the **countable additivity property**, and relation (10.ζ) is called the **countable subadditivity property** of the measure.

The Limits Inferior and Superior

In Appendices A and B we define the limits inferior and superior of sequences of real numbers, and hence of sequences of real-valued functions. We now define closely related constructions of sequences of subsets of a set X.

If $(E_n)_{n=1}^{\infty}$ is a sequence of subsets of an arbitrary set X, then we define the **limit inferior** of the sequence to be the collection of all points in X that belong to *all but a finite number* of the sets E_n, and we define the **limit superior** to be the collection of all points in X that belong to *infinitely many* of the sets E_n. These sets are denoted by $\liminf E_n$ and $\limsup E_n$, respectively, and it is seen (as in Exercise 6.M) that

$$(10.\eta) \qquad \liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k \qquad \text{and} \qquad \limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k,$$

respectively. It is an interesting exercise (see 10.D) to show that if $\mathbf{1}_{E_n}$ denotes the characteristic function of the set E_n, then the characteristic functions of the sets $E_* := \liminf E_n$ and $E^* := \limsup E_n$ are given by

$$\mathbf{1}_{E_*} = \liminf_{n\to\infty} \mathbf{1}_{E_n} \qquad \text{and} \qquad \mathbf{1}_{E^*} = \limsup_{n\to\infty} \mathbf{1}_{E_n}.$$

From the definitions of these sets, it is evident that we always have the inclusion

$$(10.\theta) \qquad \liminf_{n\to\infty} E_n \subseteq \limsup_{n\to\infty} E_n.$$

If the equality holds in $(10.\theta)$, then we call this common set the **limit** of the sequence (E_n) and denote it by $\lim E_n$.

It is an easy exercise to show that if (A_n) is a decreasing sequence of sets (so that $A_1 \supseteq A_2 \supseteq \cdots$), or if (B_n) is an increasing sequence of sets (so that $B_1 \subseteq B_2 \subseteq \cdots$), then

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n \qquad \text{and} \qquad \lim_{n\to\infty} B_n = \bigcup_{n=1}^{\infty} B_n.$$

The following result, applied to sequences of measurable sets, is quite useful.

◇ **10.3 Lemma.** *Let $(E_n)_{n=1}^{\infty} \in \mathbf{M}(I)$, where I is a compact interval.*

(a) *Then $E_* := \liminf E_n$ and $E^* := \limsup E_n$ belong to $\mathbf{M}(I)$. Moreover, we have*

$$(10.\iota) \qquad |E_*| \le \liminf_{n\to\infty} |E_n| \le \limsup_{n\to\infty} |E_n| \le |E^*|.$$

(b) *If the series $\sum_{n=1}^{\infty} |E_n|$ is convergent in $\mathbb{R}$, then $|E_*| = |E^*| = 0$.*

Proof. (a) We make use of the observation that $\mathbf{1}_{E_*} = \liminf \mathbf{1}_{E_n}$ and Fatou's Lemma 8.7 (with $\alpha = 0$) to obtain

$$|E_*| = \int_I \mathbf{1}_{E_*} = \int_I \liminf_{n\to\infty} \mathbf{1}_{E_n} \le \liminf_{n\to\infty} \int_I \mathbf{1}_{E_n} = \liminf_{n\to\infty} |E_n|.$$

The second inequality in $(10.\iota)$ follows from the fact that the limit inferior of any sequence in $\mathbb{R}$ is less than or equal to the limit superior of this sequence (see Appendix A.3(a)), and the third inequality follows from the dual version of Fatou's Lemma (see Exercise 8.K) with $\omega = \mathbf{1}_I$.

(b) In view of (a), we always have $0 \le |E_*| \le |E^*|$, so it suffices to show that $|E^*| = 0$. Since $\sum_1^{\infty} |E_k|$ is convergent it follows that, given $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that $\sum_{k=K(\varepsilon)}^{\infty} |E_k| \le \varepsilon$. But since $E^* \subseteq \bigcup_{k=K(\varepsilon)}^{\infty} E_k$, then Lemma 10.1(c) and Theorem 10.2(e) imply that

$$0 \le |E^*| \le \Big| \bigcup_{k=K(\varepsilon)}^{\infty} E_k \Big| \le \sum_{k=K(\varepsilon)}^{\infty} |E_k| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $|E^*| = 0$. Q.E.D.

Remarks. **(a)** Inequality $(10.\iota)$ can be extended to infinite measures when the set $\bigcup_{n=1}^{\infty} E_n$ has finite measure.

(b) The conclusion in 10.3(b) is (part of) the **Borel-Cantelli Lemma** in probability theory. We will use it several times in later sections.

Measurable Functions

Measurable functions on the compact interval $I := [a, b]$ were introduced in Definition 6.1 as a.e. limits of step functions. We also established the Measurable Limit Theorem 9.2 that if (f_n) is a sequence in $\mathcal{M}(I)$ and if $f_n \to f$ a.e. on I, then $f \in \mathcal{M}(I)$.

We will now give another characterization of measurable functions that is often very useful. (In fact, this characterization is often taken as the *definition* of a measurable function.)

• **10.4 Characterization of Measurability.** *Let $I := [a, b]$ be a compact interval. Then the following statements about $f : I \to \mathbb{R}$ are equivalent:*

(a) *The function f belongs to $\mathcal{M}(I)$.*

(b) *For every $r \in \mathbb{R}$ the set $\{f < r\} := \{x \in I : f(x) < r\}$ is in $\mathbb{M}(I)$.*

(c) *For every $r \in \mathbb{R}$ the set $\{f \geq r\} := \{x \in I : f(x) \geq r\}$ is in $\mathbb{M}(I)$.*

(d) *For every $r \in \mathbb{R}$ the set $\{f \leq r\} := \{x \in I : f(x) \leq r\}$ is in $\mathbb{M}(I)$.*

(e) *For every $r \in \mathbb{R}$ the set $\{f > r\} := \{x \in I : f(x) > r\}$ is in $\mathbb{M}(I)$.*

Proof. We note from Theorem 6.6(a) that it is sufficient to prove that f^+ and f^- are in $\mathcal{M}(I)$, so we will suppose that $f(x) \geq 0$ for all $x \in I$.

(a) $\Rightarrow$ (b) Let $r \in \mathbb{R}$ be given. For $n \in \mathbb{N}$ define $\varphi_n : I \to \mathbb{R}$ by

$$\varphi_n(x) := \min\big\{1, n \cdot \max\{r - f(x), 0\}\big\}.$$

Theorems 6.3 and 6.6 imply that $\varphi_n \in \mathcal{M}(I)$, and the definition of φ_n implies that $0 \leq \varphi_n \leq 1$. By the Integrability Theorem 9.1(b), we have $\varphi_n \in \mathcal{L}(I)$ for all $n \in \mathbb{N}$. It is evident that $\varphi_n(x) \leq \varphi_{n+1}(x)$ for all $x \in I$, and is readily seen that $\lim \varphi_n(x) = 0$ when $r - f(x) \leq 0$, and $\lim \varphi_n(x) = 1$ when $r - f(x) > 0$. In other words, $\lim \varphi_n$ is the characteristic function $\mathbf{1}_{\{f<r\}}$, so it follows from the Monotone Convergence Theorem 8.5 that $\mathbf{1}_{\{f<r\}}$ belongs to $\mathcal{L}(I)$, whence $\{f < r\}$ is in $\mathbb{M}(I)$.

(b) $\Rightarrow$ (a) We suppose that f satisfies (b). Our strategy is to construct an increasing sequence of measurable (but *not* step) functions that converges to f at every point of I. To do this, we dissect the *range* of f into successively

finer portions and use this dissection to define an approximating measurable function. For fixed $n \in \mathbb{N}$, we divide $[0, \infty)$ into $4^n + 1$ pairwise disjoint subintervals by dividing $[0, 2^n)$ into the 4^n subintervals

$$[r/2^n, (r+1)/2^n) \qquad \text{for} \quad r = 0, 1, \cdots, 4^n - 1,$$

and we adjoin the unbounded interval $[2^n, \infty)$. If $r = 0, 1, \cdots, 4^n - 1$, we let

$$\begin{aligned} E_{r,n} &:= \{x \in I : r/2^n \le f(x) < (r+1)/2^n\} \\ &= \{f < (r+1)/2^n\} - \{f < r/2^n\}, \end{aligned}$$

and if $r = 4^n$, we let

$$E_{4^n,n} := \{x \in I : 2^n \le f(x)\} = I - \{f < 2^n\}.$$

From hypothesis (b), each set $E_{r,n}$ $(r = 0, 1, \cdots, 4^n)$ is measurable in I. Moreover, these sets are pairwise disjoint and their union is I.

We now define $\psi_n(x) := r/2^n$ for $x \in E_{r,n}$, so that

$$\psi_n = \sum_{r=0}^{4^n} (r/2^n) \cdot \mathbf{1}_{E_{r,n}}.$$

Since ψ_n is a finite linear combination of functions in $\mathcal{M}(I)$, it also belongs to $\mathcal{M}(I)$. We claim that the sequence (ψ_n) is a monotone increasing sequence. That is because, at the $(n+1)$st step, the dissections of the preceding intervals in the range of f are bisected, and subintervals of the interval $[2^n, 2^{n+1})$ are adjoined. It is an exercise to verify that $f(x) = \lim \psi_n(x)$ for all $x \in I$, and so the measurability of f follows from the Measurable Limit Theorem 9.2.

(b) $\Leftrightarrow$ (c) This equivalence follows from the relations

$$\{f \ge r\} = I - \{f < r\} \qquad \text{and} \qquad \{f < r\} = I - \{f \ge r\}.$$

(b) $\Leftrightarrow$ (d) This equivalence follows from the relations

$$\{f \le r\} = \bigcap_{n=1}^{\infty} \{f < r + 1/n\} \qquad \text{and} \qquad \{f < r\} = \bigcup_{n=1}^{\infty} \{f \le r - 1/n\},$$

(see Exercise 10.E) and the fact that the intersection and union of a sequence of measurable sets are measurable, as was seen in Theorem 10.2(a).

(d) $\Leftrightarrow$ (e) This equivalence follows from the relations

$$\{f > r\} = I - \{f \le r\} \qquad \text{and} \qquad \{f \le r\} = I - \{f > r\}.$$

Thus the theorem is proved. Q.E.D.

10.5 Definition. A function is said to be a **simple** function if it has only a finite number of values.

Thus a function is simple if and only if it is a finite linear combination of characteristic functions. While every step function is a simple function, the Dirichlet function introduced in Example 2.3(a) is an example of a simple function that is not a step function.

• **10.6 Corollary.** *A nonnegative function f on $I := [a, b]$ is in $\mathcal{M}(I)$ if and only if it is the limit of an increasing sequence of measurable simple functions.*

Proof. ($\Leftarrow$) Apply the Measurable Limit Theorem 9.2.

($\Rightarrow$) If f is nonnegative and measurable, then the argument given in Theorem 10.4(b) $\Rightarrow$ (a) applies. Q.E.D.

The next result assures that certain important operations on sequences of measurable functions yield measurable functions.

• **10.7 Theorem.** *Let $(f_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{M}(I)$ such that for every $x \in I$, the set $\{f_n(x) : n \in \mathbb{N}\}$ is bounded in $\mathbb{R}$. If we define the functions f, F, f^* and F^* on I by*

$$f(x) := \inf\{f_n(x) : n \in \mathbb{N}\} \qquad \text{and} \qquad F(x) := \sup\{f_n(x) : n \in \mathbb{N}\},$$

$$f^*(x) := \liminf_{n\to\infty} f_n(x) \qquad \text{and} \qquad F^*(x) := \limsup_{n\to\infty} f_n(x),$$

then f, F, f^* and F^* belong to $\mathcal{M}(I)$.

Proof. We observe (see Exercise 10.H) that if $r \in \mathbb{R}$, then

$$\{f \geq r\} = \bigcap_{n=1}^{\infty}\{f_n \geq r\} \qquad \text{and} \qquad \{F > r\} = \bigcup_{n=1}^{\infty}\{f_n > r\},$$

so that f and F are in $\mathcal{M}(I)$ when the functions f_n $(n \in \mathbb{N})$ are. Since

$$f^*(x) = \sup_{n\geq 1}\left\{\inf_{m\geq n} f_m(x)\right\} \qquad \text{and} \qquad F^*(x) = \inf_{n\geq 1}\left\{\sup_{m\geq n} f_m(x)\right\},$$

the measurability of f^* and F^* is also established. Q.E.D.

Relativization of the Integral

The reader will recall that if $f \in \mathcal{R}^*(I)$ and $E \in \mathbb{M}(I)$, then in Definition 6.16 we defined f to be **integrable on** E in case the product $f \cdot \mathbf{1}_E \in \mathcal{R}^*(I)$ and, in that case, we write $f \in \mathcal{R}^*(E)$ and

$$(10.\kappa) \qquad \int_E f := \int_I f \cdot \mathbf{1}_E.$$

Theorem 6.18 implies that if $f \in \mathcal{L}(I)$ and $E \in \mathbb{M}(I)$, then $f \in \mathcal{L}(E)$ (that is, both f and $|f|$ belong to $\mathcal{R}^*(I)$) and we readily see that if we define

$$\|f\|_E := \int_E |f| = \int_I |f| \cdot \mathbf{1}_E, \tag{10.λ}$$

then it follows from Corollary 3.3 that

$$\|f\|_E \le \|f\|_I. \tag{10.μ}$$

Also, if $f \in \mathcal{M}(I)$, if $|f| \le K$ and if $E \in \mathbb{M}(I)$, then $f \in \mathcal{L}(E)$ and

$$\|f\|_E \le K|E|. \tag{10.ν}$$

It is a consequence of Corollary 3.8 that if J is any *subinterval* of I, then every $f \in \mathcal{R}^*(I)$ belongs to $\mathcal{R}^*(J)$, and it is easy to see that this implies that $f \in \mathcal{R}^*(E)$ whenever E is a *finite union* of intervals in I. Example 6.17(b) shows that if $f \in \mathcal{R}^*(I)$ and if E is the union of a *countable sequence* of pairwise disjoint subintervals in I, then f may not belong to $\mathcal{R}^*(E)$.

The following question arises: What functions $f \in \mathcal{M}(I)$ have the property that $f \in \mathcal{R}^*(E)$ for *every* set $E \in \mathbb{M}(I)$? This question is easily answered.

• **10.8 Relativization Theorem.** *A function $f \in \mathcal{M}(I)$ has the property that $f \in \mathcal{R}^*(E)$ for every set $E \in \mathbb{M}(I)$ if and only if $f \in \mathcal{L}(I)$.*

Proof. ($\Rightarrow$) Suppose $f \in \mathcal{M}(I)$ has the stated property and let $E^+ := \{f \ge 0\}$ and $E^- := \{f < 0\}$. Then $f^+ = f \cdot \mathbf{1}_{E^+}$ and $f^- = -f \cdot \mathbf{1}_{E^-}$ belong to $\mathcal{R}^*(I)$. Theorem 7.11 then implies that $f \in \mathcal{L}(I)$.

($\Leftarrow$) If $f \in \mathcal{L}(I)$, then Theorem 6.18 applies. Q.E.D.

The Indefinite Integrals

If $f \in \mathcal{R}^*(I)$, where $I := [a, b]$ is a compact interval, we have introduced (in Definition 4.1(c)) the **indefinite integral** with base point a as a function $F : I \to \mathbb{R}$ given by

$$F(x) := \int_a^x f \qquad \text{for} \quad x \in I.$$

Sometimes it is useful to introduce the **indefinite integral** ν_f to be the function defined on a set $E \in \mathbb{M}(I)$ for which $f \in \mathcal{R}^*(E)$ by

$$\nu_f(E) := \int_E f = \int_I f \cdot \mathbf{1}_E.$$

However, note that the indefinite integral ν_f is defined for *all sets* $E \in \mathbb{M}(I)$ only when $f \in \mathcal{L}(I)$.

For the moment, we will consider only the indefinite integral of $|f|$, when $f \in \mathcal{L}(I)$. Thus, if we fix $f \in \mathcal{L}(I)$ and consider the set $E \in \mathbb{M}(I)$ as variable, we have the mapping $E \mapsto \|f\|_E$ already defined in (10.λ). We will establish a few properties of this function.

• **10.9 Theorem.** *Let f belong to $\mathcal{L}(I)$.*

(a) *If $(B_k)_{k=1}^{\infty}$ is an increasing sequence in $\mathbb{M}(I)$ and if $B := \bigcup_{k=1}^{\infty} B_k$, then we have $\|f\|_B = \lim \|f\|_{B_k}$.*

(b) *If $(D_k)_{k=1}^{\infty}$ is a decreasing sequence in $\mathbb{M}(I)$ and if $D := \bigcap_{k=1}^{\infty} D_k$, then we have $\|f\|_D = \lim \|f\|_{D_k}$.*

(c) *If $(E_k)_{k=1}^{\infty}$ is a pairwise disjoint sequence in $\mathbb{M}(I)$ and if we have $E := \bigcup_{k=1}^{\infty} E_k$, then $\|f\|_E = \sum_{k=1}^{\infty} \|f\|_{E_k}$.*

Proof. (a) By hypothesis, the sequence $(|f| \cdot \mathbf{1}_{B_k})_{k=1}^{\infty}$ is an increasing sequence in $\mathcal{L}(I)$ with $|f| \cdot \mathbf{1}_{B_k} \le |f|$. It follows from the Dominated Convergence Theorem 8.8 that its limit function $|f| \cdot \mathbf{1}_B$ also belongs to $\mathcal{L}(I)$ and the conclusion holds.

(b) The proof of this assertion is similar to that of (a).

(c) Let $B_n := E_1 \cup \cdots \cup E_n$ so that (B_n) is an increasing sequence of sets with $E = \bigcup_{k=1}^{\infty} E_k = \bigcup_{n=1}^{\infty} B_n = B$. Since $|f| \cdot \mathbf{1}_{B_n} = \sum_{k=1}^{n} |f| \cdot \mathbf{1}_{E_k}$ for all $n \in \mathbb{N}$, it follows from (a) that $\|f\|_E = \|f\|_B = \lim_{n\to\infty} \|f\|_{B_n} = \sum_{k=1}^{\infty} \|f\|_{E_k}$. Q.E.D.

• **10.10 Theorem.** *Let f belong to $\mathcal{L}(I)$.*

(a) *Given $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $E \in \mathbb{M}(I)$ and $|E| \le \eta_\varepsilon$, then $\|f\|_E \le \varepsilon$.*

(b) *If $H_k := \{|f| \ge k\}$ for $k \in \mathbb{N}$, then $\lim_{k\to\infty} |H_k| = 0$.*

(c) *If H_k is as in (b), then $\lim_{k\to 0} \int_{H_k} |f| = 0$.*

Proof. (a) Given $\varepsilon > 0$, then the Density Theorem 9.15(a) implies that there exists a step function $s \in \mathcal{M}(I)$ such that $\|f - s\| \le \frac{1}{2}\varepsilon$. Since s is a step function, there exists a constant M_s such that $|s| \le M_s$ on I. It follows from Corollary 3.3 and Theorem 3.1 that

$$\Big| \int_E |f| - \int_E |s| \Big| = \Big| \int_I (|f| - |s|) \cdot \mathbf{1}_E \Big| \le \int_I |f - s| = \|f - s\| \le \tfrac{1}{2}\varepsilon.$$

Therefore, if $|E| \le \varepsilon/2M_s$, another application of Corollary 3.3 implies that

$$\int_E |s| \le M_s \int_I \mathbf{1}_E = M_s |E| \le \tfrac{1}{2}\varepsilon,$$

so that we have $\|f\|_E \le \int_E |s| + \frac{1}{2}\varepsilon \le \varepsilon$.

(b) Since $k \cdot \mathbf{1}_{H_k} \le |f|$, we have $k|H_k| \le \|f\|$, whence it follows that $\lim_k |H_k| = 0$.

(c) If $\varepsilon > 0$, part (b) implies that there exists K_ε such that if $k \ge K_\varepsilon$, then $|H_k| \le \eta_\varepsilon$ so that, by part (a), we have $\int_{H_k} |f| = \|f\|_{H_k} \le \varepsilon$. Q.E.D.

The property in 10.10(a) is often called the **absolute continuity property** of the indefinite integral $\nu_{|f|} : E \mapsto \int_E |f|$. It is closely related to the absolute continuity property of the function $F(x) = \int_a^x |f|$, where $f \in \mathcal{L}(I)$, which will be discussed in Section 14.

Multipliers

We now turn to a related problem. If f belongs to $\mathcal{R}^*(I)$, for what functions $m \in \mathcal{R}^*(I)$ is it true that the product $f \cdot m$ belongs to $\mathcal{R}^*(I)$? In general, the answer depends strongly on the function f; for example, if f vanishes on a subinterval of I, then it is clear that we can take m quite arbitrarily on that subinterval.

We now show that only functions $f \in \mathcal{L}(I)$ have the property that their product with an *arbitrary* bounded measurable function belongs to $\mathcal{R}^*(I)$.

• **10.11 Theorem.** *Suppose that $f \in \mathcal{R}^*(I)$. Then $f \in \mathcal{L}(I)$ if and only if, for every function $m \in \mathcal{M}(I)$ that is bounded on I, we have $f \cdot m \in \mathcal{R}^*(I)$.*

Proof. ($\Rightarrow$) If $f \in \mathcal{L}(I)$ and if $m \in \mathcal{M}(I)$ is bounded on I by M, then $-M|f| \le f \cdot m \le M|f|$ a.e. on I, whence the Integrability Theorem 9.1 implies that $f \cdot m \in \mathcal{R}^*(I)$.

($\Leftarrow$) Given $f \in \mathcal{R}^*(I)$, let $E^+ := \{f \ge 0\}$ and $E^- := \{f < 0\}$, so that both E^+ and E^- belong to $\mathbb{M}(I)$. We let $m(x) := 1$ if $x \in E^+$ and $m(x) := -1$ if $x \in E^-$. If $|f| = f \cdot m \in \mathcal{R}^*(I)$, then $f \in \mathcal{L}(I)$. Q.E.D.

We will show that the product of a function in $\mathcal{R}^*(I)$ and an arbitrary function of bounded variation is integrable. (In a sense, this is an integral version of Abel's Test for nonabsolutely convergent series — see Exercise 10.S.) Surprisingly, a key to proving this fact is the Riemann-Stieltjes integral! The results about the Riemann-Stieltjes integral that we will need are established in Appendix H. (For another proof of this result, see Gordon [G-3; p. 197].)

◇ **10.12 Multiplier Theorem.** *If $f \in \mathcal{R}^*(I)$ and $\varphi \in BV(I)$, then the product $f \cdot \varphi$ belongs to $\mathcal{R}^*(I)$ and*

$$(10.\xi) \qquad \int_a^b f\varphi = \int_a^b \varphi\, dF = F(b)\varphi(b) - \int_a^b F\, d\varphi,$$

where F is the indefinite integral $F(x) := \int_a^x f$ of f on $I := [a, b]$, and the second and third integrals are Riemann-Stieltjes integrals.

Proof. Since F is continuous on I, Theorems H.3 and H.5 imply that the latter two integrals exist. Thus, given $\varepsilon > 0$ there exists $\zeta_\varepsilon > 0$ such that if $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of I with mesh $\leq \zeta_\varepsilon$, then

$$\Big|\sum_{i=1}^n \varphi(t_i)\big[F(x_i) - F(x_{i-1})\big] - \int_a^b \varphi\, dF\Big| \leq \varepsilon.$$

Now φ is bounded, and we let $M \geq |\varphi(x)|$ for all $x \in I$. Since $f \in \mathcal{R}^*(I)$, there exists a gauge δ_ε on I such that if $\dot{\mathcal{P}} \ll \delta_\varepsilon$, then

$$\Big|\sum_{i=1}^n \Big\{f(t_i)(x_i - x_{i-1}) - \big[F(x_i) - F(x_{i-1})\big]\Big\}\Big| \leq \varepsilon/2M,$$

and it follows from Corollary 5.4 of the Saks-Henstock Lemma that

$$\sum_{i=1}^n \Big|f(t_i)(x_i - x_{i-1}) - \big[F(x_i) - F(x_{i-1})\big]\Big| \leq \varepsilon/M.$$

Since we may assume that $\delta_\varepsilon(x) \leq \zeta_\varepsilon$ for all $x \in I$, we conclude that

$$\begin{aligned}
&\Big|\sum_{i=1}^n f(t_i)\varphi(t_i)(x_i - x_{i-1}) - \int_a^b \varphi\, dF\Big| \\
&\quad\leq \Big|\sum_{i=1}^n f(t_i)\varphi(t_i)(x_i - x_{i-1}) - \sum_{i=1}^n \varphi(t_i)\big[F(x_i) - F(x_{i-1})\big]\Big| \\
&\qquad\qquad + \Big|\sum_{i=1}^n \varphi(t_i)\big[F(x_i) - F(x_{i-1})\big] - \int_a^b \varphi\, dF\Big| \\
&\quad\leq M \sum_{i=1}^n \Big|f(t_i)(x_i - x_{i-1}) - \big[F(x_i) - F(x_{i-1})\big]\Big| + \varepsilon \\
&\quad\leq M \cdot \big(\frac{\varepsilon}{M}\big) + \varepsilon = 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then $f\varphi \in \mathcal{R}^*(I)$ and the first equality in (10.ξ) holds. The second equality follows from the Integration by Parts Theorem H.5. Q.E.D.

Remark. It is evident that if a function $\varphi_1 : I \to \mathbb{R}$ is a.e. equal to a function $\varphi \in BV(I)$, then $f \cdot \varphi_1 \in \mathcal{R}^*(I)$ for every $f \in \mathcal{R}^*(I)$. The converse statement is also true: If $f \cdot \varphi_1 \in \mathcal{R}^*(I)$ for every $f \in \mathcal{R}^*(I)$, then φ_1 is

a.e. equal to a function in $BV(I)$. For a proof of this assertion, see Sargent [Sr].

We now show that if f has a c-primitive F on I and if $\varphi \in BV(I)$, then the product $f \cdot \varphi$ also has a c-primitive on I. This result is due to Richard J. Fleissner, and the proof given here is due to Michael W. Botsko.

$\diamond$ **10.13 Theorem.** *If f has a c-primitive F on $I := [a,b]$ and if $\varphi \in BV(I)$, then $f \cdot \varphi$ has a c-primitive on I given by*

$$(10.o) \qquad \Pi(x) := \int_a^x \varphi\, dF = \varphi(x)F(x) - \int_a^x F\, d\varphi \qquad \text{for} \quad x \in I.$$

Proof. It suffices to consider the case where φ is increasing on I. Since φ is monotone, there exists a countable set C_φ in I off which φ is continuous, and there exists a countable set C_f off which $F'(x) = f(x)$.

Now let $x \in (a,b) - (C_f \cup C_\varphi)$; we will show that Π is differentiable at x and that $\Pi'(x) = f(x) \cdot \varphi(x)$. By the formula (10.$\xi$) applied to $[a,x]$ and $[a, x+h]$, we have

$$\Pi(x+h) - \Pi(x) = \int_x^{x+h} \varphi\, dF = \varphi F \Big|_x^{x+h} - \int_x^{x+h} F\, d\varphi.$$

The Mean Value Theorem H.6 for a Riemann-Stieltjes integral implies there exists ξ between x and $x+h$ with

$$\begin{aligned}\Pi(x+h) - \Pi(x) &= \big[\varphi(x+h) \cdot F(x+h) - \varphi(x) \cdot F(x)\big] \\ &\qquad - F(\xi) \cdot \big[\varphi(x+h) - \varphi(x)\big] \\ &= \varphi(x+h) \cdot \big[F(x+h) - F(x)\big] \\ &\qquad - \big[\varphi(x+h) - \varphi(x)\big] \cdot \big[F(\xi) - F(x)\big].\end{aligned}$$

The continuity of Π results from this equation. Moreover, we have

$$\begin{aligned}\frac{\Pi(x+h) - \Pi(x)}{h} &= \varphi(x+h) \cdot \Big[\frac{F(x+h) - F(x)}{h}\Big] \\ &\qquad - \big[\varphi(x+h) - \varphi(x)\big] \cdot \Big[\frac{F(\xi) - F(x)}{\xi - x}\Big] \cdot \Big[\frac{\xi - x}{h}\Big].\end{aligned}$$

Now, since ξ lies between x and $x+h$, then $|(\xi - x)/h| \le 1$. Since

$$\lim_{h\to 0}\big[\varphi(x+h) - \varphi(x)\big] = 0 \qquad \text{and} \qquad \lim_{h\to 0}\Big[\frac{F(\xi) - F(x)}{\xi - x}\Big] = F'(x),$$

it follows that

$$\Pi'(x) = \varphi(x) \cdot F'(x) = \varphi(x) \cdot f(x).$$

The same argument applies to one-sided derivatives at the endpoints, if they do not belong to $C_f \cup C_\varphi$. Therefore Π is a c-primitive of $\varphi \cdot f$ on I with exceptional set $C_f \cup C_\varphi$. Q.E.D.

◇ **10.14 Corollary.** *If f has a primitive F on I and $\varphi \in BV(I)$ is continuous, then $f \cdot \varphi$ has a primitive Π on I given by* (10.o).

Proof. It follows from Exercises 7.J and 7.M that φ is the difference of two increasing continuous functions. Since both C_f and C_φ are empty, the result follows directly from the theorem. Q.E.D.

Exercises

10.A Use induction arguments to establish the finite subadditivity and finite additivity properties stated before 10.2.

10.B Let $(E_k)_{k=1}^{\infty}$ be a sequence in $\mathbf{M}(I)$. Let $E := \bigcap_{k=1}^{\infty} E_k$ and let $F := \bigcup_{k=1}^{\infty} E_k$. Prove that $\mathbf{1}_E = \inf\{\mathbf{1}_{E_k} : k \in \mathbb{N}\}$ and $\mathbf{1}_F = \sup\{\mathbf{1}_{E_k} : k \in \mathbb{N}\}$.

10.C Let $(E_k)_{k=1}^{\infty}$ be a sequence of subsets of a set X.

(a) Show that $\emptyset \subseteq \liminf_k E_k \subseteq \limsup_k E_k \subseteq X$.

(b) Give an example of a sequence (E_k) with $\liminf_k E_k = \emptyset$ and such that $\limsup_k E_k = X$.

(c) Give an example of a sequence (E_k) that is not monotone but such that $\liminf_k E_k = \limsup_k E_k$.

10.D Let $(E_k)_{k=1}^{\infty}$ be a sequence of subsets of $I := [a, b]$, let $E_* := \liminf_k E_k$ and let $E^* := \limsup_k E_k$.

(a) Show that $\mathbf{1}_{E_*} = \liminf_k \mathbf{1}_{E_k}$ and $\mathbf{1}_{E^*} = \limsup_k \mathbf{1}_{E_k}$.

(b) Use Exercise 8.K to show that $\limsup_k |E_k| \leq |E^*|$ for $E_k \in \mathbf{M}(I)$.

10.E If $f : I \to \mathbb{R}$ and $r \in \mathbb{R}$, prove that

$$\{f \leq r\} = \bigcap_{n=1}^{\infty} \{f < r + 1/n\} \qquad \text{and} \qquad \{f < r\} = \bigcup_{n=1}^{\infty} \{f \leq r - 1/n\}.$$

10.F Prove that if $f \in \mathcal{M}(I)$ is bounded on I and $f \geq 0$, then the increasing sequence (ψ_n) constructed in the proof of Theorem 10.4(b) converges

to f *uniformly* on I. Conclude that a bounded function in $\mathcal{M}(I)$ is the *uniform* limit of a sequence of simple measurable functions.

10.G Prove that if $f \in \mathcal{M}(I)$, where $I := [a, b]$, then f is "nearly bounded" in the sense that given $\gamma > 0$, there exists a set $E_\gamma \in \mathbb{M}(I)$ with $|E_\gamma| < \gamma$ such that f is bounded on $I - E_\gamma$.

10.H Let (f_n) be a sequence of functions on $I \to \mathbb{R}$ such that for each $x \in I$, the set $\{f_n(x) : n \in \mathbb{N}\}$ is bounded in $\mathbb{R}$. If $f(x) := \inf\{f_n(x) : n \in \mathbb{N}\}$ and $F(x) := \sup\{f_n(x) : n \in \mathbb{N}\}$ and $r \in \mathbb{R}$, prove that

$$\{f \geq r\} = \bigcap_{n=1}^{\infty} \{f_n \geq r\} \qquad \text{and} \qquad \{F > r\} = \bigcup_{n=1}^{\infty} \{f_n > r\}.$$

10.I Suppose that (E_n) is a sequence of sets in $\mathbb{M}(I)$, where $I := [a, b]$, and that $(\mathbf{1}_{E_n})$ is a mean Cauchy sequence in $\mathcal{L}(I)$. Prove that this sequence of functions converges in mean to $\mathbf{1}_E$ for some set $E \in \mathbb{M}(I)$.

10.J If $E, F \in \mathbb{M}(I)$, where $I := [a, b]$, define $\rho(E, F) := |E \triangle F|$, where $E \triangle F := E \cup F - E \cap F$ is the **symmetric difference** of the sets E and F.

(a) Show that $E \triangle F = (E - F) \cup (F - E)$, that $E \triangle F = F \triangle E$ and that $E \triangle \emptyset = E$ and $E \triangle E = \emptyset$ for all sets E, F. Also show that $E \triangle (F \triangle G) = (E \triangle F) \triangle G$ for all sets E, F, G. Conclude that $\mathbb{M}(I)$ is an abelian group under the operation of the symmetric difference, with $\emptyset$ as the identity element.

(b) Show that $\rho(E, F) = \int_I |\mathbf{1}_E - \mathbf{1}_F|$ for $E, F \in \mathbb{M}(I)$.

(c) Show that ρ is a semimetric on $\mathbb{M}(I)$ and $\mathbb{M}(I)$ is complete under ρ.

10.K Let (E_n) be a sequence in $\mathbb{M}([0, 1])$ with $|E_n| \geq r > 0$ for all $n \in \mathbb{N}$.

(a) Show that there exist points that belong to infinitely many of the E_n.

(b) Show that $|\limsup E_n| \geq r$. (This is sometimes called the **Arzelà-Young Theorem.**)

10.L Let $f \in \mathcal{M}(I)$ be such that $f \geq 0$.

(a) Let $(\psi_n)_{n=1}^{\infty}$ be the increasing sequence defined in the proof of Theorem 10.4(b), and let $\psi_0 := 0$. Show that $f(x) = \sum_{n=1}^{\infty} \{\psi_n(x) - \psi_{n-1}(x)\}$, where the series is absolutely convergent on I.

(b) Since each $\psi_n - \psi_{n-1}$ is a nonnegative simple function, show that f has the form

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \mathbf{1}_{A_n}(x) \qquad \text{for all} \quad x \in I,$$

where $\alpha_n \geq 0$ and $A_n \in \mathbb{M}(I)$ for $n \in \mathbb{N}$.

10.M Let $f \in \mathcal{L}(I), f \geq 0$ and $r > 0$. Let $E_r := \{r \leq f\} \in \mathbb{M}(I)$. Establish the **Chebyshev (= Čebyšev = Tchebycheff) Inequality**:

$$|E_r| \leq \frac{1}{r}\int_{E_r} f \leq \frac{1}{r}\|f\|.$$

More generally, if $0 < r < s$ and $F_{rs} := \{r \leq f \leq s\}$, show that $r|F_{rs}| \leq \int_{F_{rs}} f \leq s|F_{rs}|$.

10.N (a) If $f \in \mathcal{L}(I)$ and $\int_E f = 0$ for every $E \in \mathbb{M}(I)$, prove $f = 0$ a.e.

(b) If $f \in \mathcal{R}^*(I)$ and $\int_J f = 0$ for every interval $J \subseteq I$, prove $f = 0$ a.e.

(c) If $f \in \mathcal{R}^*(I)$ and $\int_J f = 0$ for every interval $J \subseteq I$ having rational endpoints, prove $f = 0$ a.e.

10.O Suppose that $f \geq 0$ is such that $f^n \in \mathcal{L}([0,1])$ and $\int_0^1 f^n = \int_0^1 f$ for all $n \in \mathbb{N}$. Prove that there exists a set $E \in \mathbb{M}([0,1])$ such that $f = \mathbf{1}_E$ a.e.

10.P If $f \in \mathcal{M}(I)$, it follows that $\operatorname{sgn} \circ f \in \mathcal{M}(I)$.

(a) Prove this assertion by using the Characterization Theorem 10.2.

(b) Prove this assertion by using the Measurable Limit Theorem 9.2. [Hint: Consider $\varphi_n(x) := (2/\pi)\operatorname{Arctan}(nf(x))$ for $x \in [a,b]$.]

10.Q Let l^1 be the vector space of real sequences $u = (u_n)$ generating **absolutely convergent series** with the norm $N_1(u) := \sum_{n=1}^{\infty} |u_n|$ that was introduced in Exercise 9.L. Also let l^∞ be the vector space of **bounded sequences** $v = (v_n)$ with the norm $N_\infty(v) := \sup\{|v_n| : n \in \mathbb{N}\}$ that was introduced in Exercise 9.K. Both of these spaces were seen to be complete normed spaces (i.e., they are Banach spaces) under these norms.

(a) If $u = (u_n) \in l^1$ and $v = (v_n) \in l^\infty$, show that the sequence $u \cdot v := (u_n v_n)$ belongs to l^1 and that $N_1(u \cdot v) \leq N_1(u)N_\infty(v)$.

(b) If $v \in l^\infty$, show that $N_\infty(v) = \sup\{|\sum_{n=1}^{\infty} u_n v_n| : u \in l^1, N_1(u) \le 1\}$.

(c) If $u \in l^1$, show that $N_1(u) = \sup\{|\sum_{n=1}^{\infty} u_n v_n| : v \in l^\infty, N_\infty(v) \le 1\}$.

(d) If w is a sequence and if $w \notin l^\infty$, show that there exists $\tilde{u}$ in l^1 such that $\tilde{u} \cdot w \notin l^1$. [Hint: There exists a subsequence $(w_{n(k)})$ of w such that $k^2 < |w_{n(k)}|$. Now define $\tilde{u}$.]

(e) Show that if w is a sequence with $u \cdot w \in l^1$ for all $u \in l^1$, then $w \in l^\infty$.

(f) Show that if w is a sequence with $w \cdot v \in l^1$ for all $v \in l^\infty$, then $w \in l^1$.

10.R A measurable function $f : I \to \mathbb{R}$ is said to be **essentially bounded** on $I := [a, b]$ if there exists $M > 0$ such that $|f(x)| \le M$ a.e. on I. In this case we define the **essential supremum** to be

$$\|f\|_\infty := \inf\{M : |f(x)| \le M \text{ a.e.}\}.$$

(a) Show that the collection of all essentially bounded measurable functions on I, which is denoted by $\mathcal{L}^\infty(I)$, is a vector space under the operations:

$$(cf)(x) := cf(x) \quad \text{and} \quad (f+g)(x) := f(x) + g(x) \quad \text{for all } x \in I.$$

(b) Show that $\|\cdot\|_\infty$ is a seminorm on $\mathcal{L}^\infty(I)$ and that $|f(x)| \le \|f\|_\infty$ a.e.

(c) Show that $\mathcal{L}^\infty(I)$ is complete under the seminorm $\|\cdot\|_\infty$ (in the sense that every Cauchy sequence converges to some function in $\mathcal{L}^\infty(I)$).

(d) If $f \in \mathcal{L}^\infty(I)$ and $g \in \mathcal{L}^1(I)$, show that $fg \in \mathcal{L}^1(I)$ and that $\|fg\|_1 \le \|f\|_\infty \cdot \|g\|_1$. (Here, as elsewhere, $\mathcal{L}^1(I) = \mathcal{L}(I)$ and $\|g\|_1 = \|g\|$.)

10.S We use the notations introduced in the preceding exercise.

(a) If $f \in \mathcal{L}^\infty(I)$ and $a < \|f\|_\infty$, show that $A := \{a < |f|\}$ is in $\mathbb{M}(I)$ and that $|A| > 0$. If $g_a := (\operatorname{sgn} f/|A|)1_A$, show that $g_a \in \mathcal{L}^1(I)$, $\|g_a\|_1 = 1$ and $a \le \int_I f g_a$. [Hint: Use Exercise 10.P.]

(b) If $f \in \mathcal{L}^\infty(I)$, show that $\|f\|_\infty = \sup\{|\int_I fg| : g \in \mathcal{L}^1(I), \|g\|_1 \le 1\}$. [Hint: Use Exercise 10.R(d) and part (a).]

(c) If $g \in \mathcal{L}^1(I)$, show that $\|g\|_1 = \sup\{|\int_I fg| : f \in \mathcal{L}^\infty(I), \|f\|_\infty \le 1\}$.

(d) If $f \in \mathcal{M}(I)$, but $f \notin \mathcal{L}^\infty(I)$, show that there exists a strictly increasing sequence (b_n) in $\mathbb{N}$ with $\sum_{n=1}^\infty 1/b_n < \infty$ such that $B_n := \{b_n < |f| \le b_{n+1}\}$ satisfies $|B_n| > 0$. Let $\tilde{g} := \sum_{n=1}^\infty 1/(b_n|B_n|) \cdot \mathbf{1}_{B_n}$ and show that $\tilde{g} \in \mathcal{L}^1(I)$ but that $f\tilde{g} \notin \mathcal{L}^1(I)$.

(e) If $f \in \mathcal{M}(I)$ and $fg \in \mathcal{L}^1(I)$ for all $g \in \mathcal{L}^1(I)$, show that $f \in \mathcal{L}^\infty(I)$.

(f) If $g \in \mathcal{M}(I)$ and $fg \in \mathcal{L}^1(I)$ for all $f \in \mathcal{L}^\infty(I)$, show that $g \in \mathcal{L}^1(I)$.

10.T This exercise gives some convergence tests for real series that are often attributed to Abel and Dirichlet.

(a) If (u_k) and (v_k) are sequences, let $s_0 := 0$ and $s_n := v_1 + \cdots + v_n$. If $m > n$, establish **Abel's partial summation formula**:

$$\sum_{k=n}^{m} u_k v_k = \sum_{k=n}^{m-1} (u_k - u_{k+1}) s_k + (u_m s_m - u_n s_{n-1}).$$

(b) (Dirichlet's Test, I) Suppose that (s_k) is bounded, that (u_k) converges to 0, and that $\sum_1^\infty |u_k - u_{k+1}|$ converges. Then the series $\sum_1^\infty u_k v_k$ converges to $\sum_1^\infty (u_k - u_{k+1}) s_k$.

(c) (Dirichlet's Test, II) If (s_k) is bounded and (u_k) is monotone and converges to 0, then $\sum_1^\infty u_k v_k$ converges to $\sum_1^\infty (u_k - u_{k+1}) s_k$.

(d) If $\sum_1^\infty |u_k - u_{k+1}|$ converges, then the sequence (u_k) is convergent.

(e) (Abel's Test, I) If $\sum_1^\infty |u_k - u_{k+1}|$ and $\sum_1^\infty v_k$ converge, then $\sum_1^\infty u_k v_k$ converges to $\sum_1^\infty (u_k - u_{k+1}) s_k + (\lim u_k)(\lim s_k)$. [Hint: To estimate $u_m s_m - u_n s_{n-1}$, subtract and add $u_m s + u_n s$, where $s := \lim s_k$.]

(f) (Abel's Test, II) If (u_k) is a bounded monotonic sequence and the series $\sum_1^\infty v_k$ converges, then the series $\sum_1^\infty u_k v_k$ converges to $\sum_1^\infty (u_k - u_{k+1}) s_k + (\lim u_k)(\lim s_k)$.

10.U Let cs denote the collection of all **convergent series** (that is, real sequences $v = (v_k)_{k=1}^\infty$ such that $\sum_{k=1}^\infty v_k$ is convergent). Also let bv denote the collection of all real sequences $u = (u_k)_{k=1}^\infty$ with $\sum_{k=1}^\infty |u_k - u_{k+1}| < \infty$, which are called the sequences with **bounded variation**.

(a) Show that cs is a vector space under the pointwise operations:

$$\alpha(v_k) := (\alpha v_k) \qquad \text{and} \qquad (v_k) + (w_k) := (v_k + w_k),$$

and with the 0-sequence as zero vector.

(b) Let $s_n := \sum_{k=1}^{n} v_k$ and define $N_{cs}(v_k) := \sup\{|s_n| : n \in \mathbb{N}\}$. Show that N_{cs} is a norm and that cs is complete under this norm.

(c) Show that bv is a vector space with the same pointwise operations as in (a) and with the 0-sequence as zero vector.

(d) Define $N_{bv}(u_k) := \sup\{\sum_{k=1}^{m-1} |u_k - u_{k+1}| + |u_m| : m \in \mathbb{N}\}$. Show that N_{bv} is a norm and that bv is complete under this norm.

(e) If $v = (v_k) \in cs$ and $u = (u_k) \in bv$, show that $u \cdot v := (u_k v_k)$ belongs to cs and that $N_{cs}(u \cdot v) \le N_{bv}(u) N_{cs}(v)$. [Hint: Use Abel's partial summation formula 10.T(a).]

(f) Show that there exists $\bar{u} \in bv$ with $N_{bv}(\bar{u}) = 1$ such that $\bar{u} \cdot v = v$ for all $v \in cs$. Therefore, $N_{cs}(v) = \sup\{N_{cs}(u \cdot v) : u \in bv, N_{bv}(u) \le 1\}$.

(g) Show that $N_{bv}(u) = \sup\{N_{cs}(u \cdot v) : v \in cs,\ N_{cs}(v) \le 1\}$. [Hint: Given $u \in bv$ and $\varepsilon > 0$, show that there exists $\bar{v} \in cs$ with $N_{cs}(\bar{v}) \le 1$ such that $N_{bv}(u) - \varepsilon \le N_{cs}(u \cdot \bar{v})$.]

(h) If $v \notin cs$, show there exists $\bar{u} \in bv$ with $N_{bv}(\bar{u}) = 1$ and $\bar{u} \cdot v \notin cs$. Also, given $u \notin bv$, show there exists a sequence $\bar{v}_n \in cs$ with $N_{cs}(\bar{v}_n) = 1$ and $N_{cs}(u \cdot \bar{v}_n) \to \infty$.

Modes of Convergence

In this section we will study a number of modes of convergence that are of importance in analysis and in the theory of probability. We will examine in some detail the relations between these types of convergence. We finish with a pair of necessary and sufficient conditions for a sequence of functions in $\mathcal{L}(I)$ to be convergent in mean. Throughout this section we will suppose that I is a compact interval in $\mathbb{R}$. While most of the results presented here have extensions to unbounded intervals (see Section 20), some additional hypotheses may be needed in that case.

Almost Uniform Convergence

We have already discussed uniform convergence, pointwise convergence and a.e. convergence. We now introduce another mode of convergence of functions in $\mathcal{M}(I)$ that is often useful. Intuitively, almost uniform convergence of a sequence in $\mathcal{M}(I)$ means that, outside of certain subsets of I having arbitrarily small measure, one has *uniform convergence*.

WARNINGS. (a) This is *not* the same thing as saying that one has uniform convergence outside of a null set. See Exercises 11.C and 11.D.

(b) This use of the word "almost" is in slight conflict with the "almost everywhere" terminology. However, we will use it because it is quite standard.

• **11.1 Definition.** **(a)** A sequence (f_n) in $\mathcal{M}(I)$ is said to be **almost uniformly convergent** to a function f on $I := [a, b]$ if for every $\gamma > 0$ there exists a measurable set $E_\gamma \subseteq I$ with $|E_\gamma| \leq \gamma$ such that (f_n) converges to

f uniformly on the set $I - E_\gamma$. In this case we sometimes write

$$f_n \to f \quad \text{[a.u.] on} \quad I.$$

(b) We say that a sequence (f_n) in $\mathcal{M}(I)$ is **almost uniformly Cauchy** on I if for every $\gamma > 0$ there exists a measurable set $E_\gamma \subseteq I$ with $|E_\gamma| \le \gamma$ such that (f_n) is a uniform Cauchy sequence on $I - E_\gamma$. [This means that for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that if $m > n \ge N(\varepsilon)$ and $x \in I - E_\gamma$, then $|f_m(x) - f_n(x)| \le \varepsilon$.]

It is an easy exercise to show that if a sequence (f_n) in $\mathcal{M}(I)$ is almost uniformly convergent to f on I, then it is almost uniformly Cauchy on I. Moreover, this sequence converges a.e. to f on I so that $f \in \mathcal{M}(I)$. We now establish a result in the converse direction.

• **11.2 Lemma.** *If a sequence (f_n) in $\mathcal{M}(I)$ is almost uniformly Cauchy on I, then there exists a function $f \in \mathcal{M}(I)$ such that (f_n) converges almost uniformly and almost everywhere to f.*

Proof. If $k \in \mathbb{N}$, let $E_k \in \mathcal{M}(I)$ be such that $|E_k| \le 1/2^k$ and (f_n) is uniformly Cauchy and therefore uniformly convergent on $I - E_k$. Let $F := \limsup_k E_k$, so by the Borel-Cantelli Lemma 10.3(b), we have $|F| = 0$. It follows from the definition of F that if $x \in I - F$, then there exists k_x such that $x \in I - E_k$ for all $k \ge k_x$. Therefore, $\lim_n f_n(x)$ exists for all $x \in I - F$ and we define f on I by $f(x) := \lim_n f_n(x)$ for $x \in I - F$ and $f(x) := 0$ for $x \in F$. Therefore $f_n \to f$ a.e. and $f \in \mathcal{M}(I)$.

To see that the convergence is almost uniform, let $\gamma > 0$ be given and let K be such that $1/2^{K-1} \le \gamma$. If $F_K := \bigcup_{j=K}^{\infty} E_j$, then it follows from (10.ζ) that

$$|F_K| \le \sum_{j=K}^{\infty} |E_j| \le 1/2^{K-1} \le \gamma.$$

Since $I - F_K \subseteq I - E_K$, the sequence (f_n) is a uniform Cauchy sequence on $I - F_K$, whence it follows that (f_n) converges to f uniformly on $I - F_K$. Q.E.D.

We now establish an important theorem, proved in 1911 by the Russian mathematician Dmitriĭ Fedorovich Egorov (= D.-Th. Egoroff). As this result is stated here, it is valid only for *compact* intervals. In Section 20 we will give a formulation of this result for unbounded intervals.

11.3 Egorov's Theorem. *Let I be a compact interval and let (f_n) be a sequence in $\mathcal{M}(I)$ that converges almost everywhere to $f \in \mathcal{M}(I)$ on I. Then the sequence (f_n) converges almost uniformly to f on I.*

Proof. We suppose without loss of generality that (f_n) converges to f at every point of I. If $m, n \in \mathbb{N}$, we let

$$E_n(m) := \bigcup_{k=n}^{\infty} \Big\{ |f_k - f| \geq 1/m \Big\},$$

so that $E_n(m) \in \mathbb{M}(I)$ and $E_{n+1}(m) \subseteq E_n(m)$. Since $f_n \to f$ on I, then

$$\bigcap_{n=1}^{\infty} E_n(m) = \emptyset.$$

Therefore (10.δ) implies that $|E_n(m)| \to 0$ as $n \to \infty$, for each $m \in \mathbb{N}$. If $\gamma > 0$ is given, for each $m \in \mathbb{N}$, we let $k_m \in \mathbb{N}$ be such that $|E_{k_m}(m)| \leq \gamma/2^m$ and set $E_\gamma := \bigcup_{m=1}^{\infty} E_{k_m}(m)$. Therefore $E_\gamma \in \mathbb{M}(I)$ and $|E_\gamma| \leq \gamma$ by (10.ζ). We note that if $x \notin E_\gamma$, then $x \notin E_{k_m}(m)$ for every $m \in \mathbb{N}$, so that

$$|f_k(x) - f(x)| < 1/m$$

for all $k \geq k_m$. Therefore (f_n) is uniformly convergent to f on $I - E_\gamma$. Q.E.D.

As an application of Egorov's Theorem, we will establish one form of a remarkable theorem, proved in 1912 by Nikolaĭ Nikolaevich Luzin (= Lusin) (1883–1950).

• **11.4 Luzin's Theorem.** *If f belongs to $\mathcal{M}(I)$, then given $\gamma > 0$ there exists a measurable set $E_\gamma \subseteq I$ with $|E_\gamma| \leq \gamma$ such that the restriction of f to $F_\gamma := I - E_\gamma$ is continuous on F_γ.*

Proof. Since $f \in \mathcal{M}(I)$, it follows from Theorem 6.7 that there exists a sequence (h_k) of continuous functions that converges to f a.e. on I. In view of Egorov's Theorem, for each $\gamma > 0$ there exists a set $E_\gamma \in \mathbb{M}(I)$ with $|E_\gamma| \leq \gamma$ such that (h_n) converges to f uniformly on $F_\gamma := I - E_\gamma$. Of course, the restriction $h_n|F_\gamma$ is continuous on F_γ and since this sequence converges to $f|F_\gamma$ uniformly on F_γ, we conclude that the restriction $f|F_\gamma$ is continuous. Q.E.D.

Remarks. **(a)** One should not misunderstand the assertion in Luzin's Theorem. It is *not* being claimed that f is continuous at any point of F_γ. Note that if f is Dirichlet's function from Example 2.3(a), then there is a null set Z such that the restriction of f to $[0, 1] - Z$ is continuous; however, f is not continuous at any point of $[0, 1]$.

(b) Another form of Luzin's Theorem is that if $f \in \mathcal{M}(I)$, then for every $\gamma > 0$, there exists a continuous function g on I such that $|\{f \neq g\}| \leq \gamma$. (See Theorem 20.18.)

Convergence in Measure

There is another mode of convergence for measurable functions that is particularly important in probability theory. First we note that if $f_n, f \in \mathcal{M}(I)$, then (by Theorem 6.1) the function $|f_n - f|$ is also measurable and therefore (by Theorem 10.4) the set $\{|f_n - f| \geq \alpha\}$ is a measurable set in I.

• **11.5 Definition.** **(a)** A sequence $(f_n) \in \mathcal{M}(I)$ **converges in measure** (or **converges in probability**) to $f \in \mathcal{M}(I)$ if for every $\alpha > 0$, we have

$$\lim_{n\to\infty} \left|\left\{|f_n - f| \geq \alpha\right\}\right| = 0. \tag{11.α}$$

In this case we sometimes write

$$f_n \to f \quad [\text{meas}] \text{ on } \quad I.$$

(b) A sequence $(f_n) \in \mathcal{M}(I)$ is **Cauchy in measure** if for every $\alpha > 0$, we have

$$\lim_{m,n\to\infty} \left|\left\{|f_m - f_n| \geq \alpha\right\}\right| = 0. \tag{11.β}$$

It seems reasonable, but is not obvious, that a sequence that converges in measure is also Cauchy in measure, and that the limit of a sequence that converges in measure is unique a.e. We now state these results formally.

• **11.6 Lemma.** **(a)** *If (f_n) converges in measure to f, then (f_n) is Cauchy in measure.*

(b) *If (f_n) converges in measure to f and also to g, then $f = g$ a.e.*

Proof. (a) It follows from the Triangle Inequality that

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)|,$$

whence we infer that

$$\left\{|f_m - f_n| \geq \alpha\right\} \subseteq \left\{|f_m - f| \geq \tfrac{1}{2}\alpha\right\} \cup \left\{|f - f_n| \geq \tfrac{1}{2}\alpha\right\}.$$

Since the measures of the sets on the right side approach 0 as $m, n \to \infty$, the statement follows from (11.α) and the fact that $|A \cup B| \leq |A| + |B|$.

The proof of (b) is similar and is an exercise. Q.E.D.

We need to relate the notions of convergence in measure [= meas] with those of almost uniform [= a.u.] convergence and convergence in mean [= mean].

• **11.7 Lemma.** **(a)** *If a sequence $(f_n) \in \mathcal{M}(I)$ converges almost uniformly to $f \in \mathcal{M}(I)$ on I, then it converges in measure to f on I.*

(b) *If a sequence $(f_n) \in \mathcal{R}^*(I)$ converges in mean to $f \in \mathcal{R}^*(I)$, then it converges in measure to f on I.*

Proof. (a) Let $\alpha > 0$ be given. By hypothesis, for every $m \in \mathbb{N}$ there exists $E_m \in \mathbb{M}(I)$ with $|E_m| \leq 1/m$ such that (f_n) converges uniformly to f on $I - E_m$. Consequently, there exists $N(\alpha, m) \in \mathbb{N}$ such that if $n \geq N(\alpha, m)$, then the set $\{|f_n - f| \geq \alpha\} \subseteq E_m$ so that

$$\left|\{|f_n - f| \geq \alpha\}\right| \leq |E_m| \leq 1/m.$$

Since m is arbitrary, we conclude that $f_n \to f$ [meas] on I.

(b) Let $\alpha > 0$ be given and let $F_n := \{|f_n - f| \geq \alpha\} \in \mathbb{M}(I)$. Since we have $\alpha \cdot \mathbf{1}_{F_n} \leq |f_n - f|$, it follows from Corollary 3.3 or from Chebyshev's Inequality (Exercise 10.M) that

$$\alpha |F_n| \leq \int_I |f_n - f| = \|f_n - f\|.$$

But since $\|f_n - f\| \to 0$, we conclude that $|F_n| \to 0$ as $n \to \infty$ for each fixed $\alpha > 0$. Thus $f_n \to f$ [meas] on I. Q.E.D.

The next examples show that there are some drastic differences between convergence in measure and the other modes of convergence.

11.8 Examples. **(a)** Let $h_n := n \cdot \mathbf{1}_{(0,1/n]}$ on the interval $I := [0, 1]$ for $n \in \mathbb{N}$. It is an exercise to show that the sequence (h_n) converges everywhere (and hence almost everywhere), almost uniformly, and in measure to the zero function on I. However, this sequence does *not* converge in mean.

(b) Let (f_k) be the sequence of functions defined in Exercise 8.F. It was shown that this sequence converges in mean to the zero function. Therefore, it follows from Theorem 11.7(b) that it converges in measure. However, it was also seen in that exercise that this sequence does not converge at *any point* of I, so it does not converge a.e. or a.u. on I.

It was seen in the proof of the Completeness Theorem 9.12 that a sequence that is Cauchy in mean has a subsequence that is a.e. convergent. We will now show that a sequence that is Cauchy in measure has a subsequence that is a.e. convergent. (This result is due to F. Riesz.)

• **11.9 Riesz Subsequence Theorem.** *If $(f_n) \in \mathcal{M}(I)$ is Cauchy in measure, then there exist a subsequence (f_{n_k}) and $f \in \mathcal{M}(I)$ such that $f_{n_k} \to f$ almost everywhere, almost uniformly, and in measure on $I = [a, b]$.*

In fact, the entire sequence (f_n) converges in measure to f.

Proof. If (f_n) is Cauchy in measure, it is an exercise to show that for every $\alpha > 0$ there exists $N(\alpha) \in \mathbb{N}$ such that if $m > n \geq N(\alpha)$, then

$$\left|\{|f_m - f_n| \geq \alpha\}\right| \leq \alpha.$$

We let $n_1 := N(1/2)$ and inductively define $n_{k+1} := \max\{n_k + 1, N(1/2^k)\}$. Now set $g_k := f_{n_k}$ to obtain a subsequence of (f_n) with the property that if $E_k := \{|g_{k+1} - g_k| \geq 1/2^k\}$, then $|E_k| \leq 1/2^k$.

We now let $F := \limsup_k E_k$, so by the Borel-Cantelli Lemma $|F| = 0$. If $x \in I - F$, there exists k_x such that $x \in I - E_k$ for all $k \geq k_x$. Thus, if $j > i \geq k_x$, we have

$$\begin{aligned} |g_j(x) - g_i(x)| &\leq |g_j(x) - g_{j-1}(x)| + \cdots + |g_{i+1}(x) - g_i(x)| \\ &\leq 1/2^{j-1} + \cdots + 1/2^i \\ &< 1/2^{i-1}. \end{aligned} \tag{11.γ}$$

Therefore it follows that $(g_k(x))$ converges for each $x \in I - F$. We define $f(x) := \lim_k g_k(x)$ for $x \in I - F$ and $f(x) := 0$ for $x \in F$. Consequently $g_k \to f$ [a.e.] on I and $f \in \mathcal{M}(I)$.

It follows from Egorov's Theorem 11.3 that $g_k \to f$ [a.u.] and from Lemma 11.7(a) that $g_k \to f$ [meas] on I.

It remains to show that the original sequence (f_n) converges in measure to f on I. Indeed, an argument similar to that in Lemma 11.6 shows that

$$\left|\{|f - f_n| \geq \alpha\}\right| \leq \left|\{|f - f_{n_k}| \geq \tfrac{1}{2}\alpha\}\right| + \left|\{|f_{n_k} - f_n| \geq \tfrac{1}{2}\alpha\}\right|.$$

Now the first term on the right approaches 0 since the subsequence (f_{n_k}) converges in measure to f, and the second term approaches 0 since the sequence (f_n) is Cauchy in measure. Q.E.D.

A Diagram

It is useful to summarize the implications that have been established concerning the various modes of convergence. Following M. E. Munroe, we do so in the following diagram. Here a solid arrow signifies implication, a dashed arrow signifies that a subsequence converges in the indicated mode, and the absence of an arrow indicates that a counterexample can be established. (Thus Egorov's Theorem is indicated by the solid arrow from [a.e.] to [a.u.], and one of the assertions in the Riesz Subsequence Theorem is indicated by the dashed arrow from [meas] to [a.u.]. Similarly, convergence in measure does not imply convergence in mean, in general.)

The reader should verify the implications that are indicated in this diagram, and show that no other implications are valid without additional hypotheses.

$$
\begin{tikzcd}
{[\text{a.e.}]} \arrow[rr, shift left] \arrow[ddrr] & & {[\text{a.u.}]} \arrow[ll, shift left] \arrow[dd, shift right] \\
& & \\
{[\text{mean}]} \arrow[uu, dashed] \arrow[rr] \arrow[uurr, dashed] & & {[\text{meas}]} \arrow[uu, dashed, shift right] \arrow[uull, dashed]
\end{tikzcd}
$$

◇ **Diagram 11.1 Compact interval.**

Mean Convergence

We have seen from Diagram 11.1 that none of the solid arrows ends at [mean]. Therefore, we will insert several results here that have mean convergence as a conclusion. The first one is a version of the Mean Convergence Theorem 8.9 for a sequence that is Cauchy in measure and satisfies a domination condition.

• **11.10 Dominated Convergence Theorem.** *Suppose that $(f_n) \subset \mathcal{R}^*(I)$ is Cauchy in measure on I and $\alpha, \omega \in \mathcal{R}^*(I)$ are such that for each $n \in \mathbb{N}$,*

$$(11.\delta) \qquad \alpha(x) \le f_n(x) \le \omega(x) \qquad \text{for a.e. } x \in I.$$

Then there exists $f \in \mathcal{R}^*(I)$ such that $\|f - f_n\| \to 0$.

Proof. By the Riesz Subsequence Theorem 11.9, there exists $f \in \mathcal{M}(I)$ such that (f_n) converges to f in measure. If (f_n) does not converge in mean to f, there exist $\varepsilon_0 > 0$ and a subsequence (h_k) of (f_n) such that $\|h_k - f\| \ge \varepsilon_0$. Since (f_n) converges in measure to f, so does its subsequence (h_k). Hence by the Riesz Theorem and 11.6(b) there is a further subsequence $(h_{k(r)})$ that converges a.e. to f. By the Mean Convergence Theorem 8.9, the subsequence $(h_{k(r)})$ converges in mean to f, which contradicts that $\|h_k - f\| \ge \varepsilon_0$ for all $k \in \mathbb{N}$. Q.E.D.

Our next result shows that the mean convergence of a sequence (f_n) in $\mathcal{R}^*(I)$ to a function f takes place when the sequence (f_n) converges in measure to f and is Cauchy in mean.

• **11.11 Theorem.** *Suppose that $(f_n) \subset \mathcal{R}^*(I)$ is Cauchy in mean and converges in measure to a function f. Then $f \in \mathcal{R}^*(I)$ and $\|f - f_n\| \to 0$.*

Proof. Since $\|f_m - f_n\| \to 0$ as $m, n \to \infty$, given $\varepsilon > 0$, there exists N_ε such that if $m > n \geq N_\varepsilon$, then

$$\int_I |f_m - f_n| \leq \varepsilon. \tag{11.ε}$$

Since $f_n \to f$ [meas] on I, it follows from the Riesz Subsequence Theorem 11.9 that there exist a subsequence (g_k) of (f_n) and $g \in \mathcal{M}(I)$, such that $g_k \to g$ [a.e.] and [meas] on I. Since (g_k) is a subsequence of (f_n), Lemma 11.6(b) implies that $g = f$ a.e. Now replace f_m in (11.ε) by g_k for k sufficiently large and apply Fatou's Lemma 8.7 to conclude that

$$\int_I |f - f_n| \leq \liminf_{k \to \infty} \int_I |g_k - f_n| \leq \varepsilon$$

for all $n \geq N_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\|f - f_n\| \to 0$. Q.E.D.

The Vitali Convergence Theorems

We conclude this section with two theorems that characterize mean convergence for a sequence in $\mathcal{L}(I)$. First, it will be seen that if $f_n \to f$ [a.e.] on I, then $f_n \to f$ [mean] if and only if the mapping $E \mapsto \|f_n\|_E$ defined in equation (10.λ) satisfies either of the uniformity conditions that we now define.

11.12 Definition. (a) A collection $\mathcal{F} \subset \mathcal{L}(I)$ is **uniformly absolutely continuous** if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $f \in \mathcal{F}$ and $E \in \mathbb{M}(I), |E| \leq \delta_\varepsilon$, then

$$\|f\|_E := \int_E |f| \leq \varepsilon.$$

(b) A collection $\mathcal{F} \subset \mathcal{L}(I)$ is **uniformly integrable** if for every $\varepsilon > 0$ there exists $K = K_\varepsilon \in \mathbb{N}$ such that if $f \in \mathcal{F}$ and $H_{f,K} := \{|f| \geq K\}$, then

$$\|f\|_{H_{f,K}} := \int_{H_{f,K}} |f| \leq \varepsilon.$$

Intuitively, uniform absolute continuity requires that the integrals $\int_E |f|$, $f \in \mathcal{F}$, are uniformly small when $|E|$ is small. Similarly, uniform integrability requires that the integrals of $f \in \mathcal{F}$ over the sets where $|f|$ is large are uniformly small.

11.13 Vitali Convergence Theorem, I. *Let $I := [a, b]$ be a compact interval and let (f_n) be a sequence in $\mathcal{L}(I)$ with $f_n \to f$ [a.e.] on I.*

Then the following statements are equivalent:

(a) $f \in \mathcal{L}(I)$ *and* $\|f_n - f\| \to 0$ *as* $n \to \infty$.

(b) *The set* $\{f_n : n \in \mathbb{N}\}$ *is uniformly absolutely continuous.*

(c) *The set* $\{f_n : n \in \mathbb{N}\}$ *is uniformly integrable.*

Proof. (a) $\Rightarrow$ (b) Given $\varepsilon > 0$ there exists n_1 such that if $n > n_1$, then $\|f_n\|_E - \|f\|_E \le \|\mathbf{1}_E f_n - \mathbf{1}_E f\| \le \|f_n - f\| \le \frac{1}{2}\varepsilon$, whence it follows that

$$\|f_n\|_E \le \tfrac{1}{2}\varepsilon + \|f\|_E \qquad \text{for} \quad n > n_1,\ E \in \mathbb{M}(I).$$

If we let $g := \max\{|f_1|, \cdots, |f_{n_1}|, |f|\}$, then $g \in \mathcal{L}(I)$ so it follows from Theorem 10.10(a) that there exists $\delta_\varepsilon > 0$ such that if $|E| \le \delta_\varepsilon$, then $\|g\|_E \le \frac{1}{2}\varepsilon$. Therefore we infer that

$$\|f_n\|_E \le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon \qquad \text{for all} \quad n \in \mathbb{N}.$$

(b) $\Rightarrow$ (a) Given $\varepsilon > 0$ let $\delta_\varepsilon > 0$ be such that if $|E| \le \delta_\varepsilon, n \in \mathbb{N}$, then $\|f_n\|_E \le \varepsilon$. Since $|f_n| \to |f|$ [a.e.] on I and hence on E, Fatou's Lemma 8.7 implies that $f \in \mathcal{L}(I)$ and

$$\|f\|_E \le \liminf_{n\to\infty} \|f_n\|_E \le \varepsilon.$$

Egorov's Theorem 11.3 implies that there exists a set $B \in \mathbb{M}(I)$ with $|B| \le \delta_\varepsilon$ such that $f_n \to f$ uniformly on $I - B$. Therefore

$$\begin{aligned} \|f_n - f\| &\le \|f_n - f\|_{I-B} + \|f_n\|_B + \|f\|_B \\ &\le \|f_n - f\|_{I-B} + 2\varepsilon. \end{aligned}$$

Further, there exists n_0 such that if $n \ge n_0$ and $x \in I - B$, then $|f_n(x) - f(x)| \le \varepsilon/|I|$, whence we conlude that

$$\|f_n - f\|_{I-B} \le \big(\varepsilon/|I|\big) \cdot |I - B| \le \varepsilon.$$

Consequently we have $\|f_n - f\| \le 3\varepsilon$ whenever $n \ge n_0$. Since $\varepsilon > 0$ is arbitrary, assertion (a) follows.

(b) $\Rightarrow$ (c) Given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $|E| \le \delta_\varepsilon$ and $n \in \mathbb{N}$, then $\|f_n\|_E \le \varepsilon$. Let $\{I_1, \cdots, I_M\}$ be a partition of I into nonoverlapping intervals with length $\le \delta_1$. Then

$$\int_a^b |f_n| = \sum_{j=1}^{M} \int_{I_j} |f_n| \le \sum_{j=1}^{M} 1 = M$$

for $n \in \mathbb{N}$. Now let $K = K_\varepsilon > M/\delta_\varepsilon$. If $H_{n,K} := \{|f_n| \geq K\}$, then since $K \cdot \mathbf{1}_{H_{n,K}} \leq |f_n|$, it follows from the inequality

$$K|H_{n,K}| \leq \|f_n\| \leq M$$

that $|H_{n,K}| \leq M/K \leq \delta_\varepsilon$, whence (b) implies that $\|f_n\|_{H_{n,K}} \leq \varepsilon$ for $n \in \mathbb{N}$.

(c) $\Rightarrow$ (b) Given $\varepsilon > 0$ let K be such that $\|f_n\|_{H_{n,K}} \leq \frac{1}{2}\varepsilon$ for all $n \in \mathbb{N}$. Now let $\delta_\varepsilon := \varepsilon/2K$, so that if $|E| \leq \delta_\varepsilon$ and $n \in \mathbb{N}$, then

$$\begin{aligned}\|f_n\|_E &= \|f_n\|_{E\cap H_{n,K}} + \|f_n\|_{E-H_{n,K}} \\ &\leq \|f_n\|_{H_{n,K}} + K|E| \leq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.\end{aligned}$$

Thus (b) is proved. Q.E.D.

We now obtain a version of the Vitali theorem replacing the hypothesis that $f_n \to f$ [a.e.] on I by the hypothesis that $f_n \to f$ [meas] on I.

11.14 Vitali Convergence Theorem, II. *Let $I := [a, b]$ be a compact interval and let (f_n) be a sequence in $\mathcal{L}(I)$ with $f_n \to f$ in measure on I.*

Then the statements **(a)**, **(b)**, *and* **(c)** *in* 11.13 *are equivalent.*

Proof. The proofs of (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (b) make no reference to the convergence of the sequence and do not need any change. However, the proof of (b) $\Rightarrow$ (a) given above uses a.e. convergence and needs to be modified.

If the sequence (f_n) does not converge in mean to f, there exists $\varepsilon_0 > 0$ and a subsequence $(f_{n'})$ such that $\|f_{n'} - f\| \geq \varepsilon_0 > 0$ for all n'. Since the subsequence $(f_{n'})$ converges in measure to f, the Riesz Subsequence Theorem implies that it has a further subsequence $(f_{n''})$ that converges a.e. to f. Hence, by the Vitali Convergence Theorem I, the sequence $(f_{n''})$ converges in mean to f, which contradicts the above inequality. Q.E.D.

Exercises

11.A Show that the following sequences do not converge uniformly on the indicated intervals, but that they do converge a.e. and hence a.u. Given $\gamma > 0$, find a set E_γ with $|E_\gamma| \leq \gamma$ such that the convergence is uniform on $I - E_\gamma$.

(a) $f_n(x) := nx/(1 + nx)$ on $[0, 1]$.

(b) $g_n(x) := x^n/(1+x^n)$ on $[0,2]$.

(c) $h_n(x) := 1/(1+x^n)$ on $[0,1]$.

(d) $\varphi_n(x) := 1/(1+x^n)$ on $[0,2]$.

11.B Same as in Exercise 11.A. Here the functions are 0 at $x = 0$.

(a) $f_n(x) := 1/\sqrt{x}(1+x^n)$ for $x \in (0,1]$

(b) $g_n(x) := 1/\sqrt{x}(2-x^n)$ for $x \in (0,1]$.

11.C If $n \geq 2$, let $f_n := \mathbf{1}_{[1/n,2/n]}$ and $f := 0$ on the interval $I := [0,1]$.

(a) Show that $f_n \to f$ everywhere (and therefore a.e.) on I, in mean and in measure.

(b) If $\gamma > 0$ is given, show that $f_n \to f$ uniformly on $[\gamma, 1]$, so that $f_n \to f$ almost uniformly.

(c) Show that there does not exist a null set Z such that $f_n \to f$ uniformly on $I - Z$.

(d) Show that $\|f_n - f\|_2 \to 0$.

11.D Let $g_n := \sqrt{n} f_n = \sqrt{n}$ on $[1/n, 2/n]$ and $:= 0$ elsewhere, and let $g := 0$.

(a) Show that $g_n \to g$ everywhere (and therefore a.e.) on I, in mean and in measure.

(b) If $\gamma > 0$ is given, show that $g_n \to g$ uniformly on $[\gamma, 1]$, so that $g_n \to g$ almost uniformly.

(c) Show that there does not exist a null set Z such that $g_n \to g$ uniformly on $I - Z$.

(d) Show that $\|g_n - g\|_2 \not\to 0$.

11.E Let $E_n \in \mathbb{M}([a,b])$ for $n \in \mathbb{N}$.

(a) Show that $(\mathbf{1}_{E_n})$ converges to 0 uniformly on $[a,b]$ if and only if $E_n = \emptyset$ for sufficiently large n.

(b) Show that $(\mathbf{1}_{E_n})$ converges to 0 everywhere on $[a,b]$ if and only if $\limsup_{n\to\infty} E_n = \emptyset$.

(c) Show that $(\mathbf{1}_{E_n})$ converges to 0 almost everywhere on $[a,b]$ if and only if $\limsup_{n\to\infty} E_n$ is a null set.

(d) When does $(\mathbf{1}_{E_n})$ converge to 0 almost uniformly?

(e) Show that $(\mathbf{1}_{E_n})$ converges to 0 in measure on $[a,b]$ if and only if $\lim_{n\to\infty} |E_n| = 0$.

(f) When does $(\mathbf{1}_{E_n})$ converge to 0 in mean?

11.F (a) If (f_n) in $\mathcal{M}([a,b])$ converges to f in measure, show that any subsequence also converges to f in measure.

(b) If (f_n) in $\mathcal{M}([a,b])$ is Cauchy in measure, show that any subsequence of (f_n) is also Cauchy in measure.

11.G Let $f_n, f \in \mathcal{M}([a,b])$ for $n \in \mathbb{N}$. Show that the following statements are equivalent:

(a) The sequence (f_n) converges to f in measure.

(b) For any $\alpha > 0$, $\varepsilon > 0$ there exists $N_{\alpha,\varepsilon} \in \mathbb{N}$ such that if $n \geq N_{\alpha,\varepsilon}$, then we have $|\{|f_n - f| \geq \alpha\}| \leq \varepsilon$.

(c) For any $\alpha > 0$, $\varepsilon > 0$ there exists $M_{\alpha,\varepsilon} \in \mathbb{N}$ such that if $n \geq M_{\alpha,\varepsilon}$, then we have $|\{|f_n - f| > \alpha\}| \leq \varepsilon$.

(d) For any $\alpha > 0$ there exists $P_\alpha \in \mathbb{N}$ such that if $n \geq P_\alpha$, then we have $|\{|f_n - f| \geq \alpha\}| \leq \alpha$.

(e) For any $\alpha > 0$ there exists $Q_\alpha \in \mathbb{N}$ such that if $n \geq Q_\alpha$, then we have $|\{|f_n - f| > \alpha\}| \leq \alpha$.

11.H Show that the sequence (f_n) in $\mathcal{M}([a,b])$ does not converge in measure to f if and only if there exist $\alpha > 0$ and a subsequence (f_{n_k}) such that $|\{|f_{n_k} - f| \geq \alpha\}| > \alpha$ for all $k \in \mathbb{N}$.

11.I Let $f_n, f \in \mathcal{M}([a,b])$ and $\alpha > 0$. Further let $F_n := \{|f_n - f| \geq \alpha\}$, let $G_n := \bigcup_{k=n}^{\infty} F_k$ and let $D := \{x : f_n(x) \not\to f(x)\}$.

(a) Show that $\limsup_{n\to\infty} F_n \subseteq D$.

(b) Show that $\lim_{n\to\infty} |G_n| \leq |D|$.

(c) Conclude (without using Egorov's Theorem) that almost everywhere convergence on $[a,b]$ implies convergence in measure.

11.J Let $f_n, f \in \mathcal{M}([a,b])$ for $n \in \mathbb{N}$. Show that (f_n) converges to f in measure if and only if every subsequence of (f_n) has a further subsequence that converges a.e. to f.

11.K Suppose that f_n, f, g_n, g are in $\mathcal{M}([a,b])$ for $n \in \mathbb{N}$ and that $f_n \to f$ and $g_n \to g$ in measure.

(a) If $k \in \mathbb{R}$, show that $kf_n \to kf$ in measure.

(b) Show that $f_n + g_n \to f + g$ in measure.

(c) Show that $f_n \cdot g_n \to f \cdot g$ in measure. [Hint: Use Exercise 11.J.]

(d) Show that $|f_n| \to |f|$ in measure.

(e) Show that $\max\{f_n, g_n\} \to \max\{f, g\}$ and $\min\{f_n, g_n\} \to \min\{f, g\}$ in measure.

11.L Show that $f : [a,b] \to \mathbb{R}$ belongs to $\mathcal{M}([a,b])$ if and only if f is the limit in measure of a sequence (s_n) of step functions.

11.M Show that the sequence (s_n) in Exercise 11.L is "nearly uniformly bounded" in the sense that for every $\gamma > 0$ there exists a set E_γ with $|E_\gamma| \leq \gamma$ and an $M_\gamma > 0$ such that $|s_n(x)| \leq M_\gamma$ for all $x \in [a,b] - E_\gamma$, $n \in \mathbb{N}$.

11.N Show that $f : [a,b] \to \mathbb{R}$ belongs to $\mathcal{L}([a,b])$ if and only if there exists a sequence (s_n) of step functions such that (i) $s_n \to f$ in measure, and (ii) (s_n) is Cauchy in mean.

11.O Establish the following version of Fatou's Lemma: Let $f_n, \alpha \in \mathcal{R}^*(I)$, where $I := [a,b]$, $n \in \mathbb{N}$, be such that $\alpha \leq f_n$ a.e. on I and $\liminf \int_I f_n < \infty$, and let $f_n \to f$ in measure. Show that $f \in \mathcal{R}^*(I)$ and that $\int_I f \leq \liminf \int_I f_n$.

11.P Let $f_n, f \in \mathcal{L}(I)$, $I := [a,b]$ with $f_n \geq 0$ a.e., and let $\int_I f_n \to \int_I f$.

(a) If $f_n \to f$ [a.e.], show that $f_n \to f$ [mean].

(b) If $f_n \to f$ [meas], show that $f_n \to f$ [mean].

(c) Show that (a) and (b) may fail unless $f_n \geq 0$ a.e.

11.Q For $f, g \in \mathcal{M}([a,b])$, we define $\sigma(f,g) := \int_a^b \frac{|f-g|}{1+|f-g|}$. Show that σ is well defined and that it satisfies $\sigma(f,g) \geq 0$, $\sigma(f,f) = 0$, $\sigma(f,g) = \sigma(g,f)$ and $\sigma(f,g) \leq \sigma(f,h)+\sigma(h,g)$ for all $f,g,h \in \mathcal{M}([a,b])$. (Thus, σ is a **semimetric** on $\mathcal{M}([a,b])$ in the sense of Appendix G.) [Hint: If $a, b \in \mathbb{R}$, then $|a+b|/(1+|a+b|) \leq |a|/(1+|a|) + |b|/(1+|b|)$.]

11.R Use the notation in Exercise 11.Q.

(a) Show that $f_n \to f$ in measure if and only if $\sigma(f_n, f) \to 0$.

(b) Show that (f_n) is Cauchy in measure if and only if it is Cauchy with respect to σ; that is, $\sigma(f_m, f_n) \to 0$ as $m, n \to \infty$.

(c) Show that $\mathcal{M}([a,b])$ is **complete** with respect to σ in the sense that every Cauchy sequence with respect to σ converges to an element of $\mathcal{M}([a,b])$.

11.S Let $\mathcal{F} \subset \mathcal{L}([a,b])$. Show that $\mathcal{F}$ is uniformly absolutely continuous if and only if it is uniformly integrable (in the sense of Definition 11.12).

11.T If $0 \leq \omega \in \mathcal{L}([a,b])$, let $\mathcal{F}_\omega := \{f \in \mathcal{L}([a,b]) : |f| \leq \omega \text{ a.e.}\}$.

(a) Show directly that $\mathcal{F}_\omega$ is uniformly absolutely continuous.

(b) Show directly that $\mathcal{F}_\omega$ is uniformly integrable.

(c) Give an example of a set $\mathcal{F} \subset \mathcal{L}([a,b])$ that is uniformly absolutely continuous, but which is not bounded by any $0 \leq \omega \in \mathcal{L}([a,b])$.

11.U (a) If $\mathcal{F} \subset \mathcal{L}([a,b])$ is uniformly integrable on $[a,b]$, show directly that there exists a $B > 0$ such that $\|f\| \leq B$ for all $f \in \mathcal{F}$.

(b) Give an example of a bounded set in $\mathcal{L}([a,b])$ that is not uniformly absolutely continuous.

11.V Let $\Phi : [0,\infty) \to \mathbb{R}$ be nonnegative and increasing with $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$. Let $\mathcal{F} \subset \mathcal{L}([a,b])$ satisfy $\int_a^b \Phi(|f|) \leq B$ for some $B > 0$ and all $f \in \mathcal{F}$.

(a) Show that $\mathcal{F}$ is uniformly integrable.

(b) Show that $\mathcal{F}$ is uniformly absolutely continuous.

(c) Let $\mathcal{F}_A$ be the set of all $f \in \mathcal{L}([a,b])$ such that $\|f\|_2 = (\int_a^b |f|^2)^{1/2} \leq A$ for some $A > 0$ and all $f \in \mathcal{F}_A$. Show that $\mathcal{F}_A$ is uniformly integrable.

11.W In this and the next exercise, we will define another semimetric on $\mathcal{M}([a,b])$.

(a) If $f \in \mathcal{M}([a,b])$, show that $r \mapsto |\{|f| > r\}|$ is a decreasing function on $[0,\infty)$ and that $\lim_{r\to\infty} |\{|f| > r\}| = 0$. [Hint: Use Theorem 10.2(c).]

(b) Show that the set $Q(f) := \{r \in \mathbb{R} : |\{|f| > r\}| \leq r\}$ is a nonempty interval in $[0,\infty)$ and let $\rho(f) := \inf Q(f)$.

(c) Show that $\rho(f)$ belongs to $Q(f)$, so that $|\{|f| > \rho(f)\}| \leq \rho(f)$. [Hint: Let (r_n) decrease to $\rho(f)$ and use Theorem 10.2(b).] This means that $|f(x)| \leq \rho(f)$ for all x outside of a set $A(f)$ with $|A(f)| \leq \rho(f)$.

(d) Prove that $\rho(f+g) \leq \rho(f) + \rho(g)$ for $f, g \in \mathcal{M}([a,b])$.

(e) Show that if $|f| \leq |g|$ a.e., then $\rho(f) \leq \rho(g)$.

(f) If $c > 0$, $E \in \mathbf{M}([a,b])$, show that $\rho(c\mathbf{1}_E) = \min\{c, |E|\}$.

(g) If $0 < c \leq 1$, show that $\rho(cf) \leq \rho(f)$. If $1 < c$, show that $\rho(f) \leq \rho(cf) \leq c\rho(f)$. Give an example where $\rho(2f) \neq 2\rho(f)$.

11.X We continue from the preceding exercise.

(a) Show that $f_n \to 0$ [meas] if and only if $\rho(f_n) \to 0$. Conclude that $f_n \to f$ [meas] if and only if $\rho(f_n - f) \to 0$.

(b) Show that $\tau(f,g) := \rho(f-g)$ is a semimetric on $\mathcal{M}([a,b])$. Also show that convergence with respect to τ is equivalent to convergence in measure.

(c) Show that $\mathcal{M}([a,b])$ is complete with respect to the semimetric τ.

(d) Let (g_n) be a sequence in $\mathcal{M}([a,b])$ such that $\sum_{k=1}^{\infty} \rho(g_{n+1} - g_k)$ is convergent. Show that there exists $g \in \mathcal{M}([a,b])$ such that $g_n \to g$ [a.e.]; moreover, $g_n \to g$ [meas].

11.Y We recall from Theorem 6.3(d) that if $f \in \mathcal{M}([a,b])$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, then $\varphi \circ f \in \mathcal{M}([a,b])$.

(a) If φ is continuous and if $f_n \to f$ everywhere [respectively, almost everywhere, almost uniformly, in measure] then $\varphi \circ f_n \to \varphi \circ f$ everywhere [resp., a.e., a.u., meas].

(b) Conversely, if φ is not continuous on $\mathbb{R}$, then there exists a sequence (f_n) in $\mathcal{M}([a,b])$ that converges uniformly to f, and therefore a.e., a.u., in measure and in mean, but such that $(\varphi \circ f_n)$ does not converge at any point or in measure to $\varphi \circ f$.

(c) If φ is uniformly continuous on $\mathbb{R}$ and if $f_n \to f$ uniformly on $[a,b]$, show that $\varphi \circ f_n \to \varphi \circ f$ uniformly on $[a,b]$.

(d) If φ is continuous but not uniformly continuous, show that there exists a sequence such that $f_n \to f$ uniformly on $[a,b]$, but such that $(\varphi \circ f_n)$ does not converge unifomly to $\varphi \circ f$.

11.Z (a) If $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$(*) \qquad |\varphi(t)| \le P(1 + |t|) \qquad \text{for all} \quad t \in \mathbb{R},$$

for some $P > 0$, show that $\varphi \circ f \in \mathcal{L}([a,b])$ for all $f \in \mathcal{L}([a,b])$.

(b) If $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $(*)$, show that if $(f_n) \subset \mathcal{L}([a,b])$ and $f_n \to f$ [mean], then $\varphi \circ f_n \to \varphi \circ f$ [mean].

(c) If φ is continuous, but does not satisfy $(*)$, show that there exists $f \in \mathcal{L}([0,1])$ such that $\varphi \circ f \notin \mathcal{L}([0,1])$.

(d) If φ is not continuous at some point of $\mathbb{R}$, show that there exists a sequence (f_n) in $\mathcal{L}([0,1])$ with $f_n \to f$ [mean], but $\varphi \circ f_n \not\to \varphi \circ f$ [mean].

(e) If φ is continuous but does not satisfy $(*)$, show that there exists a sequence (f_n) in $\mathcal{L}([0,1])$ with $f_n \to f$ [mean], but $\varphi \circ f_n \not\to \varphi \circ f$ [mean].

Applications to Calculus

We will now apply the theory of the (generalized Riemann) integral that we have developed to obtain a number of results that are familiar from calculus, except that the hypotheses are much weaker than customary.

This section is divided into four parts. In the first part, we will obtain very general versions of the Integration by Parts formula. We then apply these results to obtain various versions of the Mean Value Theorems. The third part is concerned with a theorem due to Hake that shows that the generalized Riemann integral does not admit (neither does it need) an extension analogous to the "improper integral" that is familiar from calculus. This result can also be viewed as providing a method for the *evaluation* of integrals. In the final part of this section we will consider integrands that depend on a parameter, and obtain some results that can be used in handling such integrals.

In this section we will limit our discussion to the case of a compact interval $I := [a,b]$. It will be seen later that most of these results can be extended to unbounded intervals.

Integration by Parts

This familiar result is a consequence of the "Product Rule" for differentiation. It will be convenient to use the notation

$$H\big|_{\alpha}^{\beta} := H(\beta) - H(\alpha),$$

where H is a function defined on an interval that contains the points α, β.

We first consider the case where the functions $f, g \in \mathcal{R}^*(I)$ have c-primitives F, G, since this is the case most commonly encountered in elementary applications, and since the proof is very easy. We will show that the product FG is a c-primitive of the function $Fg + fG$ and that $Fg \in \mathcal{R}^*(I)$ if and only if $fG \in \mathcal{R}^*(I)$. In this case the familiar formula holds.

• **12.1 Integration by Parts.** *If $f, g \in \mathcal{R}^*(I)$ have c-primitives F, G on an interval $I := [a, b]$, then $Fg + fG$ has a c-primitive FG and therefore belongs to $\mathcal{R}^*(I)$, and*

$$\int_a^b (Fg + fG) = FG\Big|_a^b. \tag{12.α}$$

Moreover, Fg belongs to $\mathcal{R}^*(I)$ if and only if fG belongs to $\mathcal{R}^*(I)$, in which case

$$\int_a^b Fg = FG\Big|_a^b - \int_a^b fG. \tag{12.β}$$

Proof. By hypothesis, F and G are continuous on I and there exist countable sets C_f and C_g of I such that $F'(x) = f(x)$ for $x \in I - C_f$ and $G'(x) = g(x)$ for $x \in I - C_g$. Let $C := C_f \cup C_g$, so that C is a countable set. The Product Rule for differentiation implies that

$$(FG)'(x) = F(x)G'(x) + F'(x)G(x) = F(x)g(x) + f(x)G(x) \tag{12.γ}$$

for $x \in I - C$. The Fundamental Theorem 4.7 implies that $Fg + fG = (FG)'$ belongs to $\mathcal{R}^*(I)$ and has integral $FG\big|_a^b$, proving (12.α).

Theorem 3.1 implies that $Fg \in \mathcal{R}^*(I)$ if and only if $fG \in \mathcal{R}^*(I)$. Equation (12.β) now follows from (12.α). Q.E.D.

We now present a theorem that gives a definitive form of the Integration by Parts formula in terms of indefinite integrals $F(x) := \int_c^x f$ and $G(x) := \int_c^x g$ of $f, g \in \mathcal{R}^*(I)$, rather than c-primitives of these functions. This proof, taken from [P-1; p. 110] is considerably more involved than that of 12.1.

• **12.2 Integration by Parts*.** *Let $f, g \in \mathcal{R}^*(I)$ and let F, G be their respective indefinite integrals with base point $c \in I$.*

(a) *Then $Fg + fG$ belongs to $\mathcal{R}^*(I)$ and has FG as indefinite integral with base point c. Therefore equation (12.α) holds.*

(b) *Moreover, Fg belongs to $\mathcal{R}^*(I)$ if and only if fG belongs to $\mathcal{R}^*(I)$, in which case equation (12.β) holds.*

Proof. We will treat the case where $c = a$, leaving the general case as an exercise.

Theorem 4.11 implies that the indefinite integrals F, G are continuous and hence bounded on $I := [a,b]$ and we let $M \geq b-a > 0$ be such that $|F(x)| \leq M$ and $|G(x)| \leq M$ for $x \in I$. Given $\varepsilon > 0$, we conclude from the continuity of F and G that there exists a gauge δ_ε on I such that if $x \in I$, $|x-t| \leq \delta_\varepsilon(t)$, then

$$(12.\delta) \quad |f(t)| \cdot |G(t) - G(x)| \leq \varepsilon/4M \quad \text{and} \quad |g(t)| \cdot |F(t) - F(x)| \leq \varepsilon/4M.$$

Since $f, g \in \mathcal{R}^*(I)$, we may also assume that the gauge δ_ε is such that if $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a δ_ε-fine partition of I, then

$$|S(f;\dot{\mathcal{P}}) - F(b)| \leq \varepsilon/8M \qquad \text{and} \qquad |S(g;\dot{\mathcal{P}}) - G(b)| \leq \varepsilon/8M.$$

It follows from Corollary 5.4 of the Saks-Henstock Lemma that

$$(12.\varepsilon_1) \qquad \sum_{i=1}^n \Big| f(t_i)(x_i - x_{i-1}) - \big[F(x_i) - F(x_{i-1})\big] \Big| \leq \varepsilon/4M,$$

$$(12.\varepsilon_2) \qquad \sum_{i=1}^n \Big| g(t_i)(x_i - x_{i-1}) - \big[G(x_i) - G(x_{i-1})\big] \Big| \leq \varepsilon/4M.$$

But, since we have

$$\begin{aligned} F(x_i)G(x_i) - &F(x_{i-1})G(x_{i-1}) \\ &= F(x_i)\big[G(x_i) - G(x_{i-1})\big] + G(x_{i-1})\big[F(x_i) - F(x_{i-1})\big] \end{aligned}$$

and $F(x_0) = 0 = G(x_0)$, we can expand $F(b)G(b)$ in a telescoping sum:

$$\begin{aligned} F(b)G(b) &= \sum_{i=1}^n \big[F(x_i)G(x_i) - F(x_{i-1})G(x_{i-1})\big] \\ &= \sum_{i=1}^n \Big\{ F(x_i)\big[G(x_i) - G(x_{i-1})\big] + G(x_{i-1})\big[F(x_i) - F(x_{i-1})\big] \Big\}. \end{aligned}$$

Using the above expression, we obtain

$$\begin{aligned} \Big| S(Fg + &fG;\dot{\mathcal{P}}) - F(b)G(b) \Big| \\ &= \Big| \sum_{i=1}^n \big[F(t_i)g(t_i) + f(t_i)G(t_i)\big](x_i - x_{i-1}) - F(b)G(b) \Big| \end{aligned}$$

$$(12.\zeta_1) \qquad \leq \sum_{i=1}^n \big| F(t_i)g(t_i)(x_i - x_{i-1}) - F(x_i)\big[G(x_i) - G(x_{i-1})\big] \big|$$

$$(12.\zeta_2) \qquad + \sum_{i=1}^n \Big| f(t_i)G(t_i)(x_i - x_{i-1}) - G(x_{i-1})\big[F(x_i) - F(x_{i-1})\big] \Big|.$$

But, since $F(t_i) = F(x_i) + \left[F(t_i) - F(x_i)\right]$, the term (12.$\zeta_1$) is dominated by

$$(12.\eta_1) \qquad \sum_{i=1}^{n} |F(x_i)| \cdot \left|g(t_i)(x_i - x_{i-1}) - [G(x_i) - G(x_{i-1})]\right|$$

$$(12.\eta_2) \qquad + \sum_{i=1}^{n} |g(t_i)| \cdot |F(t_i) - F(x_i)| \cdot (x_i - x_{i-1}).$$

We now use the fact that $|F(x_i)| \le M$ and (12.ε_2) in (12.η_1), and the second inequality in (12.δ) in (12.η_2) to conclude that (12.ζ_1) is dominated by

$$M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon}{4M} \cdot (b - a) \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Using $G(t_i) = G(x_{i-1}) + \left[G(t_i) - G(x_{i-1})\right]$, a similar argument shows that (12.ζ_2) is also dominated by $\frac{1}{2}\varepsilon$. Therefore, we infer that if $\dot{\mathcal{P}} \ll \delta_\varepsilon$, then

$$\left|S(Fg + fG; \dot{\mathcal{P}}) - F(b)G(b)\right| \le \varepsilon.$$

But since $\varepsilon > 0$ is arbitrary, we conclude that $Fg + fG$ belongs to $\mathcal{R}^*(I)$ with integral $F(b)G(b)$.

Since b can be replaced by an arbitrary point $x \in I$, it follows that $Fg + fG$ has FG as indefinite integral with base point a.

The assertion in part (b) follows as before. Q.E.D.

Sometimes it is convenient to write formula (12.β) in the "calculus form":

$$(12.\theta) \qquad \int_a^b F(x)g(x)\,dx = FG\Big|_a^b - \int_a^b f(x)G(x)\,dx.$$

We recall that in calculus courses one often uses the notation

$$u(x) := F(x), \qquad dv(x) := g(x)\,dx = G'(x)\,dx,$$

so that we have

$$du(x) = F'(x)\,dx = f(x)\,dx, \qquad v(x) = G(x).$$

Hence equation (12.θ) takes the form

$$\int_a^b u(x)\,dv(x) = u(x)v(x)\Big|_a^b - \int_a^b v(x)\,du(x),$$

which is often abbreviated as

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

The reader is certainly familiar with the technique of integrating by parts, so we will not give any routine examples here. Our first example shows that formula (12.β) does not hold for the Lebesgue integral unless it is assumed that *both* Fg and fG belong to $\mathcal{L}(I)$. The second example shows that (12.β) does not hold unless *at least one* (and hence both) of Fg and fG belongs to $\mathcal{R}^*(I)$. The third example applies Theorem 12.2 to an interesting integral.

12.3 Examples. (a) It is possible that $Fg \in \mathcal{L}(I)$ but $fG \notin \mathcal{L}(I)$.

Let $F(x) := x$ and let $G(x) := x\cos(\pi/x^2)$ for $x \in (0,1]$ and $G(0) := 0$. Then F is a primitive of $f(x) := 1$ and G is a c-primitive on $I := [0,1]$ with exceptional set $\{0\}$ of the function

$$g(x) := \cos(\pi/x^2) + (2\pi/x^2)\sin(\pi/x^2) \qquad \text{for} \quad x \in (0,1]$$

and $g(0) := 0$. Moreover, the product FG is a primitive of the function

$$(FG)'(x) = 2x\cos(\pi/x^2) + (2\pi/x)\sin(\pi/x^2) \qquad \text{for} \quad x \in (0,1]$$

and $(FG)'(0) = 0$. In this case the product fG belongs to $\mathcal{L}(I)$.

However, Fg has the form

$$(Fg)(x) = x\cos(\pi/x^2) + (2\pi/x)\sin(\pi/x^2) \qquad \text{for} \quad x \in (0,1].$$

Now the first term belongs to $\mathcal{L}(I)$, but it is seen in Exercise 12.C that the second term does not belong to $\mathcal{L}(I)$. Therefore $Fg \notin \mathcal{L}(I)$; however, both Fg and fG belong to $\mathcal{R}^*(I)$.

(b) It is possible that neither of the functions Fg and fG belongs to $\mathcal{R}^*(I)$.

Let F, G be defined on $I := [0,1]$ by $F(x) := 0 =: G(0)$ and

$$F(x) := x^{1/2}\sin(\pi/x), \qquad G(x) := x^{1/2}\cos(\pi/x) \qquad \text{for} \quad x \in (0,1].$$

Then both F, G are continuous on $I := [0,1]$; moreover

$$F'(x) = \tfrac{1}{2}x^{-1/2}\sin(\pi/x) - \pi x^{-3/2}\cos(\pi/x) \qquad \text{for} \quad x \in (0,1]$$

so that F is a c-primitive of $f := F'$ (where $f(0) := 0$) with exceptional set $\{0\}$. Therefore, if $x \in (0, 1]$, then we have

$$\begin{aligned} f(x)G(x) &= \tfrac{1}{2}\sin(\pi/x)\cos(\pi/x) - (\pi/x)\cos^2(\pi/x) \\ &= \tfrac{1}{4}\sin(2\pi/x) - \tfrac{1}{2}(\pi/x)\cos(2\pi/x) - \tfrac{1}{2}(\pi/x), \end{aligned}$$

where we have used the identities $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$.

Now the first term is bounded and measurable and hence is in $\mathcal{L}(I)$, and the second term is seen to be in $\mathcal{R}^*(I)$ as in Example 6.13(b) (or Exercise 6.T). However, the last term does not belong to $\mathcal{R}^*(I)$ so $fG \notin \mathcal{R}^*(I)$. The fact that $Fg \notin \mathcal{R}^*(I)$ follows from this fact, or can be proved in the same way. (See Exercise 12.D.)

Thus the products fG and Fg do not belong to $\mathcal{R}^*(I)$ even though their sum does.

(c) If $f \in \mathcal{R}^*([a,b])$, we will show that

$$(12.\iota) \qquad \frac{1}{n}\int_a^b f(x)\cos nx\,dx \to 0 \qquad \text{as} \quad n \to \infty,$$

and similarly if $\cos nx$ is replaced by $\sin nx$ (see Exercise 12.E). To prove the first assertion, we will make use of the Riemann-Lebesgue Lemma 9.17 that if $\varphi \in \mathcal{L}([a,b])$, then

$$\int_a^b \varphi(x)\sin nx\,dx \to 0 \qquad \text{as} \quad n \to \infty,$$

and similarly if $\sin nx$ is replaced by $\cos nx$.

To prove $(12.\iota)$, we let $F(x) := \int_a^x f(x)\,dx$ and $G(x) := \int_a^x \sin nx\,dx$. Since F is continuous and $g(x) := G'(x) = \sin nx$, then Fg is continuous and belongs to $\mathcal{R}^*([a,b])$. Theorem 12.2 then implies that $fG \in \mathcal{R}^*([a,b])$ and that

$$\int_a^b f(x)G(x)\,dx = F(b)G(b) - \int_a^b F(x)\sin nx\,dx.$$

But since $G(x) = (1/n)[\cos na - \cos nx]$, a straightforward calculation yields

$$\frac{1}{n}\int_a^b f(x)\cos nx\,dx = \frac{1}{n}F(b)\cos nb + \int_a^b F(x)\sin nx\,dx.$$

The Riemann-Lebesgue Lemma applied to $\varphi = F$ now gives $(12.\iota)$.

Mean Value Theorems

We will now establish the important Mean Value Theorems in a high level of generality. In order to prove the Second Mean Value Theorem, we need to know that certain products of functions are integrable. In particular, we will use the Multiplier Theorem 10.12, which asserts that the product of a function in $\mathcal{R}^*(I)$ and a function in $BV(I)$ is integrable. For information concerning the history of these results, see Hobson [Hb-1; p. 616 ff.].

• **12.4 First Mean Value Theorem.** *If f is continuous on $I := [a, b]$ and if $p \in \mathcal{R}^*(I)$ does not change sign on I, then there exists $\xi \in I$ such that*

$$\int_a^b fp = f(\xi) \int_a^b p. \tag{12.κ}$$

Proof. In fact, $p \in \mathcal{L}(I)$ so that $fp \in \mathcal{L}(I)$. If $p \geq 0$, then $mp \leq fp \leq Mp$, where $m := \inf\{f(x) : x \in I\}$ and $M := \sup\{f(x) : x \in I\}$, so that

$$m \int_a^b p \leq \int_a^b fp \leq M \int_a^b p.$$

If $\int_I p = 0$, the result is trivial; if not, it follows immediately from the Bolzano Intermediate Value Theorem. If $p \leq 0$, the argument is similar. Q.E.D.

◇ **12.5 Second Mean Value Theorem.** *If $f \in \mathcal{R}^*(I)$ and g is monotone on $I := [a, b]$, then there exists $\xi \in I$ such that*

$$\int_a^b fg = g(a) \int_a^\xi f + g(b) \int_\xi^b f. \tag{12.λ}$$

Proof. It follows from the Multiplier Theorem 10.12 that $fg \in \mathcal{R}^*(I)$, and from the Integration by Parts formula $(H.\gamma)$ for the Riemann-Stieltjes integral that

$$\int_a^b fg = \int_a^b g\, dF = gF\Big|_a^b - \int_a^b F\, dg.$$

If we apply the Mean Value Theorem for the Riemann-Stieltjes integral (Theorem H.6), we conclude that there exists $\xi \in I$ such that the term on the right equals

$$\begin{aligned} gF\Big|_a^b - F(\xi)\cdot g\Big|_a^b &= g(a)\big[F(\xi) - F(a)\big] + g(b)\big[F(b) - F(\xi)\big]. \\ &= g(a)\int_a^\xi f + g(b)\int_\xi^b f. \end{aligned}$$

If we combine these expressions, we obtain (12.λ). Q.E.D.

Proofs of special cases of this theorem not using the Riemann-Stieltjes integral are outlined in the exercises.

12.6 Bonnet's Mean Value Theorem. *If $f \in \mathcal{R}^*(I)$ and $g \geq 0$ is increasing on I, then there exists $\xi \in I$ such that*

$$\int_a^b fg = g(b) \int_\xi^b f. \tag{12.μ}$$

Proof. Define $g_1 : I \to \mathbb{R}$ by $g_1(a) := 0$ and $g_1(x) := g(x)$ for $x \in (a, b]$. Now apply the Second Mean Value Theorem 12.5. Q.E.D.

There are analogous forms of Bonnet's Theorem for decreasing and for negative functions (see the exercises).

12.7 Examples. (a) If f is not continuous, then the First Mean Value Theorem 12.4 may fail.

Let $f(x) := -1$ for $x \in [-1, 0)$ and $f(x) := 1$ for $x \in [0, 1]$ and let $p := 1$ on $I := [-1, 1]$. Then f is not continuous at 0, but $p > 0$ and $fp \in \mathcal{R}^*(I)$. However, $\int_I fp = 0$ and $\int_I p = 2$, so that (12.κ) does not hold.

(b) If p changes sign, then the First Mean Value Theorem 12.4 may fail.

Indeed, let $f(x) := x =: p(x)$ on $I := [-1, 1]$, so that f is continuous on I and $p, fp \in \mathcal{R}^*(I)$. However, $\int_I fp = 2/3$ and $\int_I p = 0$, so that (12.κ) fails.

(c) If g is not monotone, the Second Mean Value Theorem 12.5 may fail.

Let $f(x) := x^2 - 1 =: g(x)$ on $I := [-1, 1]$. Then $f \in \mathcal{R}^*(I)$ and, although g is not monotone, $g \in BV(I)$. However, $\int_I fg = \int_I (x^2 - 1)^2 \, dx = 16/15$ while $g(-1) = g(1) = 0$, so that (12.λ) does not hold.

(d) If g is not increasing, then Bonnet's Theorem 12.6 may fail.

Let f and g be as in (c).

(e) If $0 < a < b$, then $\left| \int_a^b x^{-1} \sin x \, dx \right| \leq 2(1/a + 1/b)$.

If $g(x) := x^{-1}$ on $I := [a, b]$, then g is monotone on I, so the Second Mean Value Theorem 12.5 implies there exists ξ such that $\int_a^b x^{-1} \sin x \, dx = (1/a)[\cos a - \cos \xi] + (1/b)[\cos \xi - \cos b]$, whence the inequality follows.

The inequality in (e) (and the Cauchy Condition for the limit) establishes the existence of the important limit:

$$\lim_{T\to\infty}\int_0^T \frac{\sin x}{x}\,dx = \int_0^\infty \frac{\sin x}{x}\,dx. \tag{12.ν}$$

Improper Integrals

We will now prove a remarkable result that was established for the Perron integral by Heinrich Hake in 1921. In effect it asserts that there is no such thing as an "improper integral" for the generalized Riemann integral. By this we mean that any function that has an "improper integral" is already integrable. However, the limiting procedure may be useful in *evaluating* the integral, as we will see.

We will state this result only for the case of the right endpoint. We leave it to the reader to formulate the statement for the left endpoint, or where the difficulty occurs at an interior point of the interval. In Section 16 we will establish Hake's Theorem for infinite intervals in $\mathbb{R}$.

• **12.8 Hake's Theorem.** *Let $I := [a,b]$ and $f : I \to \mathbb{R}$. Then the function $f \in \mathcal{R}^*(I)$ if and only if there exists $A \in \mathbb{R}$ such that for every $c \in (a,b)$ the restriction of f to $[a,c]$ is integrable and*

$$\lim_{c\to b-}\int_a^c f = A. \tag{12.ξ}$$

In this case, $A = \int_a^b f$.

Proof. ($\Rightarrow$) It follows from Corollary 3.8 that if $c \in (a,b)$, then the restriction of f to $[a,c]$ is integrable. Moreover, by Theorem 4.11 (see also Theorem 5.6), the indefinite integral of f with base point a is continuous at b, so that

$$\int_a^b f = \lim_{c\to b-}\int_a^c f.$$

Hence the statement follows with $A := \int_I f$.

($\Leftarrow$) Suppose there exists $A \in \mathbb{R}$ such that for every $c \in (a,b)$ the restriction of f belongs to $\mathcal{R}^*([a,c])$ and (12.ξ) holds. Now let $(c_k)_{k=0}^\infty$ be a strictly increasing sequence with $a = c_0$ and $b = \lim_k c_k$. Given $\varepsilon > 0$, let $r \in \mathbb{N}$ be such that $b - c_r \le \varepsilon/(|f(b)|+1)$ and such that if $t \in [c_r, b)$, then

$$\left|\int_a^t f - A\right| \le \varepsilon.$$

If $k \in \mathbb{N}$, let δ_k be a gauge on $I_k := [c_{k-1}, c_k]$ such that if $\dot{\mathcal{P}}_k$ is any δ_k-fine partition of I_k, then

$$\left|S(f;\dot{\mathcal{P}}_k) - \int_{I_k} f\right| \le \varepsilon/2^k.$$

Without loss of generality, we may assume that

$$\begin{array}{ll} \text{(i)} & \delta_1(c_0) \le \tfrac{1}{2}(c_1 - c_0), \qquad \text{and if} \quad k \ge 1, \quad \text{that} \\ \text{(ii)} & \delta_{k+1}(c_k) \le \min\{\delta_k(c_k), \tfrac{1}{2}(c_k - c_{k-1}), \tfrac{1}{2}(c_{k+1} - c_k)\}. \\ \text{(iii)} & \delta_k(t) \le \min\{\tfrac{1}{2}(t - c_{k-1}), \tfrac{1}{2}(c_k - t)\} \quad \text{for} \quad t \in (c_{k-1}, c_k). \end{array}$$

We now define δ on I by:

$$\delta(t) := \begin{cases} \delta_k(t) & \text{if} \quad t \in [c_{k-1}, c_k),\ k \in \mathbb{N}, \\ b - c_r & \text{if} \quad t = b. \end{cases}$$

Thus δ is a gauge on I and we let $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a δ-fine partition of I. Since b does not belong to any interval I_k, the last subinterval $[x_{n-1}, b]$ in $\dot{\mathcal{P}}$ must have its tag $t_n = b$. But since $\dot{\mathcal{P}} \ll \delta$, this implies that

$$c_r = b - \delta(b) \le x_{n-1}.$$

Now let $s \in \mathbb{N}$ be the smallest positive integer such that $x_{n-1} \le c_s$, so that $r \le s$. If $k = 1, \cdots, s-1$, then condition (iii) implies that the point c_k must be a tag for any subinterval in $\dot{\mathcal{P}}$ that contains c_k. Using the right-left procedure, we may assume that the points $c_0, \cdots, c_{s-1}$ are also points in $\dot{\mathcal{P}}$. We let

$$\dot{\mathcal{Q}}_1 := \dot{\mathcal{P}} \cap [c_0, c_1], \quad \cdots, \quad \dot{\mathcal{Q}}_{s-1} := \dot{\mathcal{P}} \cap [c_{s-2}, c_{s-1}], \quad \dot{\mathcal{Q}}_s := \dot{\mathcal{P}} \cap [c_{s-1}, x_{n-1}].$$

Since each $\dot{\mathcal{Q}}_k$ ($k = 1, \cdots, s-1$) is a δ_k-fine partition of I_k, then

$$\left|S(f;\dot{\mathcal{Q}}_k) - \int_{I_k} f\right| \le \varepsilon/2^k \qquad \text{for} \quad k = 1, \cdots, s-1.$$

Since $\dot{\mathcal{Q}}_s$ is a δ_s-fine subpartition of I_s, the Saks-Henstock Lemma 5.3 implies that

$$\left|S(f;\dot{\mathcal{Q}}_s) - \int_{c_{s-1}}^{x_{n-1}} f\right| \le \varepsilon/2^s.$$

If $\dot{\mathcal{Q}}^b := \{([x_{n-1}, b], b)\}$, then $S(f;\dot{\mathcal{Q}}^b) = f(b)(b - x_{n-1})$, whence it follows that $|S(f;\dot{\mathcal{Q}}^b)| \le |f(b)|(b - x_{n-1}) \le \varepsilon$. Since $\dot{\mathcal{P}} = \dot{\mathcal{Q}}_1 \cup \cdots \cup \dot{\mathcal{Q}}_{s-1} \cup \dot{\mathcal{Q}}_s \cup \dot{\mathcal{Q}}^b$, we have

$$\begin{aligned} |S(f;\dot{\mathcal{P}}) - A| &= \left|\sum_{i=1}^{s} S(f;\dot{\mathcal{Q}}_i) + S(f;\dot{\mathcal{Q}}^b) - A\right| \\ &\le \left|\sum_{i=1}^{s} S(f;\dot{\mathcal{Q}}_i) - \int_a^{x_{n-1}} f\right| + |S(f;\dot{\mathcal{Q}}^b)| \\ &\quad + \left|\int_a^{x_{n-1}} f - A\right| \le 3\varepsilon. \end{aligned}$$

But since $\varepsilon > 0$ is arbitrary, then $f \in \mathcal{R}^*(I)$ with integral A. Q.E.D.

We now give some rather straightforward applications of Hake's Theorem, where the limit is taken at the left endpoint.

12.9 Examples. **(a)** Consider the integral $\int_0^1 x^r\,dx$ for $r \in \mathbb{R}$.

Let $g_r(x) := x^r$ for $x \in (0,1]$ and $g_r(0) := 0$. If $r+1 > 0$, then g_r has the function $x \mapsto x^{r+1}/(r+1)$ as a c-primitive with exceptional set either $\{0\}$ or $\emptyset$. Hence $g_r \in \mathcal{R}^*([0,1])$ and

$$(12.o) \qquad \int_0^1 x^r\,dx = x^{r+1}/(r+1)\Big|_0^1 = 1/(r+1) \qquad \text{for} \quad r > -1.$$

(b) Consider the integral $\int_0^1 x^{-r}\,dx$ for $r \geq 1$.

The function $x \mapsto x^{-1}$ has $x \mapsto \ln x$ as a primitive on any interval $[c,1]$ with $0 < c < 1$. But since

$$\int_c^1 x^{-1}\,dx = \ln x\Big|_c^1 = -\ln c \to \infty$$

as $c \to 0+$, we conclude from Hake's Theorem that $x^{-1} \notin \mathcal{R}^*([0,1])$.

Since $x^r < x$ when $r > 1$ and $x \in (0,1)$, we have $x^{-1} < x^{-r}$ so that $x^{-r} \notin \mathcal{R}^*([0,1])$ when $r > 1$.

(c) In Example 6.13(c), it was shown that $x \mapsto (1/x^2)\sin(\pi/x)$ does not belong to $\mathcal{R}^*([0,1])$. We give another proof of that fact here.

Indeed, $(1/x^2)\sin(\pi/x) = [(1/\pi)\cos(\pi/x)]'$ for $x \in [c,1]$ with $0 < c < 1$. Thus we have

$$\int_c^1 (1/x^2)\sin(\pi/x)\,dx = (1/\pi)\big[\cos\pi - \cos(\pi/c)\big].$$

Since $\cos(\pi/c)$ does not have a limit as $c \to 0+$, it follows from Hake's Theorem that $(1/x^2)\sin(\pi/x) \notin \mathcal{R}^*([0,1])$.

(d) Consider the integral $\int_0^1 x^r \ln x\,dx$ for $r > -1$.

If $s := r+1 > 0$, the integration by parts formula leads us to find that

$$F(x) := s^{-1}\big[x^s \ln x - s^{-1}x^s\big] \qquad \text{for} \quad x \in (0,1],$$

and $F(0) := 0$ is a c-primitive with exceptional set $\{0\}$, where we have used L'Hospital's Rule to show that F is continuous at $x = 0$ when $s > 0$. A calculation shows that if $0 < c < 1$, then

$$\int_c^1 x^r \ln x\,dx = s^{-2}[c^s - 1] - s^{-1}c^s \ln c.$$

Another application of L'Hospital's Rule shows that, if $r > -1$, there is a limit as $c \to 0+$, so $x^r \ln x$ belongs to $\mathcal{R}^*([0,1])$ and $\int_0^1 x^r \ln x\, dx = -(r+1)^{-2}$.

(e) Consider the integral $\int_0^1 x^{-1} \ln x\, dx$.

The function $G(x) := \frac{1}{2}(\ln x)^2$ for $x \in (0,1]$ has the property that $G'(x) = x^{-1}\ln x$ for $x \in (0,1]$. Hence, if $0 < c < 1$, then

$$\int_c^1 x^{-1} \ln x\, dx = -\tfrac{1}{2}(\ln c)^2 \to -\infty \qquad \text{as} \quad c \to 0+.$$

We conclude from Hake's Theorem that the function $x^{-1}\ln x$ does not belong to $\mathcal{R}^*([0,1])$.

Integrands with a Parameter

We now consider integrals where the integrand depends on a parameter. For the sake of simplicity, we will treat the case where the domain of the parameter is a bounded interval $T := [c,d]$, but many of our results can be extended to a considerably more general parameter domain without difficulty.

The next several results make use of the following hypothesis:

• **12.10 Hypothesis (H).** Let the function $f : I \times T \to \mathbb{R}$ be such that, for each $t \in T$, the function $x \mapsto f(x,t)$ is measurable on $I := [a,b]$.

In the results to follow, additional hypotheses will be given that imply that for each $t \in T$, the function $x \mapsto f(x,t)$ is in $\mathcal{R}^*(I)$, in which case the function $F : I \to \mathbb{R}$ given by

$$F(t) := \int_a^b f(x,t)\, dx \tag{12.π}$$

is well defined. We want to show that various properties of $t \mapsto f(x,t)$ (e.g., limit, continuity, differentiability) carry over to similar properties of F.

The case where f (and $f_t = \partial f/\partial t$) are continuous on $I \times T$ is relatively familiar and will be outlined in the Exercises. However, it often happens that difficulties occur at the endpoints of the interval I. These difficulties are usually handled by assuming that the integral (12.π) converges uniformly with respect to $t \in T$. Arguments of this sort are to be found in many books dealing with this subject. In this section, we will treat the case where the hypotheses of joint continuity and uniform convergence are replaced by domination conditions.

• **12.11 Limit Theorem.** *Let $f : I \times T \to \mathbb{R}$ satisfy Hypothesis (H), and suppose that:*

(i) *There exists $\tau \in T$ such that $f(x,\tau) = \lim_{t\to\tau} f(x,t)$ for all $x \in I$.*

(j) *There exist functions $\alpha, \omega \in \mathcal{R}^*(I)$ such that*

$$\alpha(x) \le f(x,t) \le \omega(x) \qquad \text{for all} \quad x \in I,\ t \in T.$$

Then the function F in $(12.\pi)$ exists on T and $F(\tau) = \lim_{t\to\tau} F(t)$; that is:

$$(12.\rho) \qquad \int_a^b f(x,\tau) = \lim_{t\to\tau} \int_a^b f(x,t)\,dx.$$

Proof. Hypothesis (H), condition (j), and the Integrability Theorem 9.1 imply that $x \mapsto f(x,t)$ belongs to $\mathcal{R}^*(I)$ for each $t \in T$. Hence the function F given in $(12.\pi)$ is defined on T. Now let (t_n) be any sequence in T converging to τ. If we let $\tilde{f}_n(x) := f(x,t_n)$ and $\tilde{f}(x) := f(x,\tau)$ for $x \in I$, it follows from (i), (j), and the Dominated Convergence Theorem 8.8 that

$$F(\tau) = \int_a^b \tilde{f}(x)\,dx = \lim_{n\to\infty} \int_a^b \tilde{f}_n(x)\,dx = \lim_{n\to\infty} F(t_n).$$

But since (t_n) is an arbitrary sequence in T converging to τ, we infer that $F(\tau) = \lim_{t\to\tau} F(t)$. Q.E.D.

• **12.12 Continuity Theorem.** *Let $f : I \times T \to \mathbb{R}$ satisfy Hypothesis (H) and suppose that:*

(i′) *The function $t \mapsto f(x,t)$ is continuous on T for each $x \in I := [a,b]$.*

(j′) *There exist functions $\alpha, \omega \in \mathcal{R}^*(I)$ such that*

$$\alpha(x) \le f(x,t) \le \omega(x) \qquad \text{for all} \quad x \in I,\ t \in T.$$

Then the function $F : T \to \mathbb{R}$ given by $(12.\pi)$ is continuous on T.

Proof. We apply the Limit Theorem 12.11 to each point in T. Q.E.D.

We now obtain a result showing that the derivative of F can be found as an integral of the partial derivative $f_t := \partial f/\partial t$. Thus, one can differentiate F in $(12.\pi)$ by "differentiating under the integral sign", provided the partial derivative f_t is dominated by an integrable function.

• **12.13 Differentiation Theorem.** *Let $f : I \times T \to \mathbb{R}$ satisfy Hypothesis (H) and suppose that:*

(i″) *There exists $\tau \in T$ such that the function $x \mapsto f(x,\tau)$ is in $\mathcal{R}^*(I)$.*

(j″) *The partial derivative f_t exists on $I \times T$.*

(k″) *There exist $\alpha, \omega \in \mathcal{R}^*(I)$ such that*

$$\alpha(x) \le f_t(x,t) \le \omega(x) \qquad \text{for all} \quad x \in I,\ t \in T.$$

Then we conclude that:

(a) *The function $x \mapsto f(x,t)$ is in $\mathcal{R}^*(I)$ for each $t \in T$.*

(b) *The function $x \mapsto f_t(x,t)$ is in $\mathcal{R}^*(I)$ for each $t \in T$.*

(c) *The function F in (12.π) is defined and differentiable on T and*

$$F'(t) = \int_a^b f_t(x,t)\,dx \qquad \text{for all} \quad t \in T. \tag{12.σ}$$

Proof. Let τ be as in hypothesis (i″). If $x \in I$ and $t \in T$, $t \ne \tau$, are fixed, then it follows from (j″) and the Mean Value Theorem of calculus that there exists a point $s = s(x,t,\tau)$ between t and τ such that

$$f(x,t) - f(x,\tau) = (t-\tau) f_t(x,s).$$

Thus, if $t \ge \tau$, then (k″) implies that

$$f(x,\tau) + (t-\tau)\alpha(x) \le f(x,t) \le f(x,\tau) + (t-\tau)\omega(x),$$

while if $t \le \tau$, then (k″) implies that

$$f(x,\tau) + (t-\tau)\omega(x) \le f(x,t) \le f(x,\tau) + (t-\tau)\alpha(x).$$

From Hypothesis (H), the above inequalities, and the Integrability Theorem 9.1, we conclude that for each $t \in T$, the function $x \mapsto f(x,t)$ belongs to $\mathcal{R}^*(I)$. This is conclusion (a).

Now let $t \in T$ be fixed and let (t_n) be any sequence in T with $t_n \ne t$, $t_n \to t$. It follows from (j″) that

$$f_t(x,t) = \lim_{n\to\infty} \frac{f(x,t_n) - f(x,t)}{t_n - t} \qquad \text{for} \quad x \in I.$$

Hypothesis (H) and the Measurable Limit Theorem 9.2 imply that the function $x \mapsto f_t(x,t)$ is measurable on I. From (k″) and the Integrability Theorem 9.1, we conclude that $x \mapsto f_t(x,t)$ is in $\mathcal{R}^*(I)$. Since $t \in T$ is arbitrary, statement (b) follows.

If $t \in T$ is fixed and (t_n) is as before, assumption (j″) and another application of the Mean Value Theorem imply that

$$\frac{f(x,t_n) - f(x,t)}{t_n - t} = f_t(x,s_n),$$

where $s_n = s_n(x, t_n, t)$ lies between t_n and t, so (k$''$) implies that

$$\alpha(x) \le \frac{f(x,t_n) - f(x,t)}{t_n - t} \le \omega(x).$$

But since we have

$$\frac{F(t_n) - F(t)}{t_n - t} = \int_a^b \frac{f(x,t_n) - f(x,t)}{t_n - t}\, dx,$$

it follows from the Dominated Convergence Theorem 8.8 that

$$\lim_{n\to\infty} \frac{F(t_n) - F(t)}{t_n - t} = \int_a^b f_t(x,t)\, dx.$$

Since (t_n) is an arbitrary sequence converging to t with $t_n \neq t$, we infer that $F'(t)$ exists and is given by (12.σ). Q.E.D.

We now wish to establish a version of the familiar Leibniz formula concerning the differentiation of an integral, where the limits as well as the integrand depend on the parameter. If we recall that an indefinite integral is easily shown to be differentiable only at a point of continuity of the integrand, the hypotheses of the next theorem do not seem excessive. (While it is necessary that f be defined on $I^* \times T$ for a sufficiently large interval I^*, for the sake of simplicity we state this result for f defined on $\mathbb{R} \times T$.)

12.14 Leibniz' Formula. *Let $f : \mathbb{R} \times T \to \mathbb{R}$ satisfy:*

(i$'''$) *The function $(x,t) \mapsto f(x,t)$ is continuous on $\mathbb{R} \times T$.*

(j$'''$) *The partial derivative $(x,t) \mapsto f_t(x,t)$ exists and is continuous on $\mathbb{R} \times T$.*

(k$'''$) *The functions $u, v : T \to \mathbb{R}$ are differentiable on T.*

Then the function $G : T \to \mathbb{R}$, defined by

$$G(t) := \int_{u(t)}^{v(t)} f(x,t)\, dx, \tag{12.τ}$$

exists and is differentiable on T. Moreover, its derivative is given by

$$G'(t) = \int_{u(t)}^{v(t)} f_t(x,t)\, dx - f\big(u(t),t\big)\cdot u'(t) + f\big(v(t),t\big)\cdot v'(t). \tag{12.υ}$$

Proof. Let $I^* := [A, B]$ be an interval in $\mathbb{R}$ such that

$$A + 1 \le u(t), v(t) \le B - 1 \qquad \text{for} \quad t \in T.$$

It follows from (i''') that Hypothesis (H) and condition 12.13(i'') are satisfied. It follows from (j''') that 12.13(j'') is satisfied and that f_t is bounded on $I^* \times T$, whence 12.13(k'') is also satisfied.

For convenience, let $T^* := T \times I^* \times I^*$, and define Γ on T^* by

$$\Gamma(t, u, v) := \int_u^v f(x,t)\,dx.$$

For u, v fixed, Theorem 12.13 implies that the partial derivative Γ_t exists on T^* and equals

$$\Gamma_t(t, u, v) = \int_u^v f_t(x,t)\,dx.$$

An extension of Theorem 12.12 to two parameters, the boundedness of the partial derivative f_t on T^* and Theorem 4.11, imply that Γ_t is continuous in (t, u, v) in T^*. In addition, it follows from (i''') and Corollary 4.10 that the partial derivatives

$$\Gamma_u(t, u, v) = -f(u, t) \qquad \text{and} \qquad \Gamma_v(t, u, v) = f(v, t),$$

are also continuous in (t, u, v) on T^*. Consequently, we may apply the Chain Rule (see [B-2; p. 361]) to conclude that G is differentiable on T and that

$$\begin{aligned} G'(t) = \Gamma_t\big(t, u(t), v(t)\big) \cdot 1 + \Gamma_u\big(t, u(t), v(t)\big) \cdot u'(t) \\ + \Gamma_v\big(t, u(t), v(t)\big) \cdot v'(t), \end{aligned}$$

whence formula (12.v) follows. Q.E.D.

We conclude this discussion with a result concerning the interchange of the order of integration. We will be content with a result that includes the hypothesis that the integrand in (12.φ) below has a primitive in $t \in T = [c, d]$, for each $x \in I$.

• **12.15 Integration Theorem.** *Let $g, \gamma : I \times T \to \mathbb{R}$ be such that γ satisfies Hypothesis (H), and suppose that:*

(i) *There exists $\tau \in T$ such that the function $x \mapsto \gamma(x, \tau)$ is in $\mathcal{R}^*(I)$.*

(j) *The partial derivative $\gamma_t(x,t) = g(x,t)$ for all $x \in I$, $t \in T$.*

(k) *There exist $\alpha, \omega \in \mathcal{R}^*(I)$ such that*

$$\alpha(x) \le g(x,t) \le \omega(x) \qquad \text{for all} \quad x \in I,\ t \in T.$$

Then the function $x \mapsto g(x,t)$ belongs to $\mathcal{R}^*(I)$ for each $t \in T$ so that

$$G(t) := \int_a^b g(x,t)\,dx \tag{12.φ}$$

is defined on T. Moreover, $G \in \mathcal{R}^*(I)$ and

$$\int_c^d G(t)\,dt = \int_a^b \left\{ \int_c^d g(x,t)\,dt \right\} dx. \tag{12.χ}$$

Proof. We will apply the Differentiation Theorem 12.13 with f replaced by γ. The hypotheses given above concerning γ correspond exactly to the hypotheses in 12.13 concerning f. We conclude from 12.13(a) that the function $x \mapsto \gamma(x,t)$ is in $\mathcal{R}^*(I)$ for each $t \in T = [c,d]$ and we let

$$\Gamma(t) := \int_a^b \gamma(x,t)\,dx$$

for $t \in T$. It also follows from 12.13(b) that $x \mapsto g(x,t) = \gamma_t(x,t)$ is in $\mathcal{R}^*(I)$ for each $t \in T$ and we let $G : T \to \mathbb{R}$ be defined as in (12.φ) for $t \in T$. Finally, it follows from 12.13(c) that Γ is differentiable on T, and that $\Gamma'(t) = G(t)$ for $t \in T$. Consequently, Γ is a primitive of G on T so that $G \in \mathcal{R}^*(T)$ and

$$\int_c^d G(t)\,dt = \Gamma(d) - \Gamma(c). \tag{12.ψ}$$

On the other hand, since $g(x,t) = \gamma_t(x,t)$, the Fundamental Theorem 4.5 implies that $t \mapsto g(x,t)$ belongs to $\mathcal{R}^*(T)$ for each $x \in I$ and that

$$\int_c^d g(x,t)\,dt = \gamma(x,d) - \gamma(x,c).$$

Since $x \mapsto \gamma(x,t)$ is in $\mathcal{R}^*(I)$ for each $t \in T$, it follows from the preceding formula that $x \mapsto \int_c^d g(x,t)\,dt$ belongs to $\mathcal{R}^*(I)$. Moreover,

$$\int_a^b \left\{ \int_c^d g(x,t)\,dt \right\} dx = \int_a^b \left\{ \gamma(x,d) - \gamma(x,c) \right\} dx = \Gamma(d) - \Gamma(c).$$

If we combine the last equation with (12.ψ), we obtain (12.χ). Q.E.D.

Exercises

12.A Write out the details of the proof that $(12.\zeta_2)$ is dominated by $\frac{1}{2}\varepsilon$.

12.B Prove Theorem 12.2 for an arbitrary base point $c \in [a,b]$.

12.C Show that the function $k(x) := (2\pi/x)\sin(\pi/x^2)$ for $x \in (0,1]$ and $k(0) := 0$, arising in Example 12.3(a), does not belong to $\mathcal{L}([0,1])$. [Hint: Consider $H(x) := x^2\cos(\pi/x^2)$ for $x \in (0,1]$ and $H(0) := 0$.]

12.D Show directly that the product Fg in Example 12.3(b) does not belong to $\mathcal{R}^*([0,1])$.

12.E Show directly that the function $Fg + fG$ in Example 12.3(b) does belong to $\mathcal{R}^*([0,1])$.

12.F Let $F(x) := x^2\sin(1/x^4)$ and $G(x) := x^2\cos(1/x^4)$ for $x \in (0,1]$ and $F(0) := 0 =: G(0)$. Show that F and G are differentiable at every point of $[0,1]$ and that $FG' + F'G$ belongs to $\mathcal{R}^*([0,1])$, but that FG' and $F'G$ do not belong to $\mathcal{R}^*([0,1])$.

12.G (a) Show that $\int_a^b \varphi(x)\cos nx\,dx \to 0$ as $n \to \infty$ when $\varphi \in \mathcal{L}([a,b])$.

(b) Use the result in (a) to show that $(1/n)\int_a^b f(x)\sin nx \to 0$ as $n \to \infty$ when $f \in \mathcal{R}^*([a,b])$.

12.H If f is continuous on $I := [a,b]$ and $p \in \mathcal{R}^*(I)$ with $p(x) > 0$ a.e. on I, show that there exists $\xi \in (a,b)$ such that $\int_a^b fp = f(\xi)\int_a^b p$.

12.I Let $f, p : [a,b] \to \mathbb{R}$ be such that f' exists on $[a,b]$, that $f'p, p \in \mathcal{L}([a,b])$ and that $p > 0$ a.e. Then there exists $\xi \in (a,b)$ such that $\int_a^b f'p = f'(\xi)\int_a^b p$.

12.J (a) If $f \in \mathcal{R}^*([a,b])$, if g is increasing on $[a,b]$ and if $A \le g(a), g(b) \le B$, show that there exists $\xi \in [a,b]$ such that $\int_a^b fg = A\int_a^\xi f + B\int_\xi^b f$.

(b) Formulate and prove an analogous statement when g is decreasing on $[a,b]$.

12.K Let $f \in \mathcal{R}^*([a,b])$. Establish the following versions of Bonnet's Theorem.

(a) If $g \le 0$ is increasing on $[a,b]$, then there exists $\xi \in [a,b]$ such that $\int_a^b fg = g(a)\int_a^\xi f$.

(b) If $g \geq 0$ is decreasing on $[a,b]$, then there exists $\xi \in [a,b]$ such that $\int_a^b fg = g(a)\int_a^\xi f$.

(c) If $g \leq 0$ is decreasing on $[a,b]$, then there exists $\xi \in [a,b]$ such that $\int_a^b fg = g(b)\int_\xi^b f$.

12.L In this exercise, we establish a special case of 12.5 without using the Riemann-Stieltjes integral. We assume that $f \in \mathcal{R}^*([a,b])$, that g is continuous and monotone on $[a,b]$, and that $\gamma := g'$ exists c.e. on $[a,b]$. It will be seen that there exists $\xi \in [a,b]$ such that equation (12.λ) holds.

(a) Let $F(x) := \int_a^x f$. Show that $F\gamma \in \mathcal{R}^*([a,b])$ and that $\int_a^b fg = Fg|_a^b - \int_a^b F\gamma$.

(b) Apply Theorem 12.4 to the integral $\int_a^b F\gamma$ and rearrange the terms to obtain equation (12.λ).

12.M We sketch a proof of Theorem 12.5 when $f \in \mathcal{L}([a,b])$ without using the Riemann-Stieltjes integral.

(a) Let $h_n := (b-a)/2^n$ and $x_{n,k} := a+kh_n$ for $n \in \mathbb{N}$, $k = 0, 1, \cdots, 2^n$. Let g_n agree with g at each point $x_{n,k}$ and be linear on $[x_{n,k-1}, x_{n,k}]$. Show that g_n is increasing, continuous, and differentiable f.e.

(b) Show that $g_n(x) \to g(x)$ at every point of continuity of g; in particular, $g_n \to g$ c.e.

(c) Apply Exercise 12.L and take limits to obtain equation (12.λ).

12.N If $f(x) := 6x^2 + ix \in \mathbb{C}$ and $p(x) = g(x) := 2x$ on $[0,1]$, show that neither (12.κ) nor (12.λ) holds.

12.O If $0 < a < b$, show that $|\int_a^b x^{-1}\sin x|\, dx \leq (b-a)/a$.

12.P Let $h : [0,1] \to \mathbb{R}$ be the function defined in Example 2.7. Use Hake's Theorem to show that $h \in \mathcal{R}^*([0,1])$ if and only if the series $\sum_{k=1}^\infty a_k$ converges.

12.Q Use Hake's Theorem to establish the convergence (and the value) or the divergence of the following integrals:

(a) $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1-x}}$, (b) $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}}$,

(c) $\displaystyle\int_0^{1/2} \frac{dx}{x\sqrt{1-x^2}}$, (d) $\displaystyle\int_{1/2}^1 \frac{dx}{x\sqrt{1-x^2}}$.

12.R If $\lfloor x \rfloor := n$ when $n \le x < n+1$, discuss the convergence of the integrals:

(a) $\displaystyle\int_0^1 \frac{(-1)^{\lfloor 1/x \rfloor}}{x}\,dx$, (b) $\displaystyle\int_0^1 \frac{(-1)^{\lfloor 1/x \rfloor}}{x^2}\,dx$.

12.S (a) Sketch a graph of $f(x) := \sin(\csc x)$ for $x \in (0, \pi/2]$, $f(0) := 0$. Explain why $f \in \mathcal{L}([0,\pi/2])$.

(b) Sketch a graph of $g(x) := (\csc x)(\sin(\csc x))$ for $x \in (0,\pi/2]$, $g(0) := 0$. By making a suitable substitution (see Section 13) and using Exercise 12.K(b), show that if $\varepsilon > 0$ is given, there exists $\delta > 0$ such that if $0 < c_1 < c_2 \le \delta$, then $|\int_{c_1}^{c_2} g(x)\,dx| \le \varepsilon$. Explain why this means that $g \in \mathcal{R}^*([0,\pi/2])$.

12.T Let $I = T := [0,1]$, and let $f(x,t) := (1 - x^{1/t})(x^{1/t}/t)$ for $x \in [0,1]$, $t \in (0,1]$, and let $f(x,0) := 0$ for $x \in [0,1]$.

(a) Show that $t \mapsto f(x,t)$ is continuous on T for each $x \in I$, and that $x \mapsto f(x,t)$ is continuous on I for each $t \in T$.

(b) Show that $(x,t) \mapsto f(x,t)$ is not continuous at $(1,0)$.

(c) If $F(t) := \int_0^1 f(x,t)\,dx$, show that $F(t) = 1/(1+t)(2+t)$, while $F(0) = 0$. Thus $\lim_{t\to 0+} F(t) \ne F(0)$ and F is not continuous at $t = 0$.

12.U Let $I := [a,b]$ and $T := [c,d]$ and let $f : I \times T \to \mathbb{R}$.

(a) If f is continuous on $I \times T$, show that the function defined by

$$F(t) := \int_a^b f(x,t)\,dx \qquad \text{for} \quad t \in T,$$

is continuous on T. [Hint: Use the uniform continuity of f on $I \times T$.]

(b) Suppose that f and $\partial f/\partial t = f_t$ are continuous on $I \times T$. Show that, for each $t \in T$, then the derivative $F'(t)$ exists and

$$F'(t) = \int_a^b f_t(x,t)\,dx \qquad \text{for} \quad t \in T.$$

[Hint: If $t_0 \in [c,d)$ and $h > 0$ is sufficiently small, then $F(t_0+h) - F(t_0) = \int_a^b \bigl(\int_{t_0}^{t_0+h} f_t(x,t)\,dt\bigr)\,dx$.]

(c) If f is continuous on $I \times T$, show that

$$\int_c^d \Bigl(\int_a^b f(x,t)\,dx\Bigr)\,dt = \int_c^d F(t)\,dt = \int_a^b \Bigl(\int_c^d f(x,t)\,dt\Bigr)\,dx.$$

[Hint: In case $z \in T$, let $G(z) := \int_c^z \bigl(\int_a^b f(x,t)\,dx\bigr)\,dt$ and $H(z) := \int_a^b \Phi(x,z)\,dx$, where $\Phi(x,z) := \int_c^z f(x,t)\,dt$. Use Parts (a), (b), and

Corollary 4.10 to show that Φ_t is continuous on $I \times T$. Now show that $H'(z) = G'(z)$ for $z \in T$, whence $H(d) = G(d)$.]

12.V If F is defined on $[0,1]$ by the following integrals, find $F'(t)$.

(a) $F(t) := \int_{-1}^{1} \ln(1 + te^x)\, dx$.

(b) $F(t) := \int_0^{\pi} x^{-1} e^{tx} \sin x\, dx$.

(c) $F(t) := \int_0^1 x^{-1/2} \cos(tx)\, dx$.

(d) $F(t) := \int_0^1 x^{1/2} \cos(t/x)\, dx$.

12.W Let $y : [a,b] \to \mathbb{R}$ be defined by $y(t) := \int_0^{\pi/2} \cos(t \sin x)\, dx$. Show that y is a solution on $[a,b]$ of the Bessel differential equation

$$ty''(t) + y'(t) + ty(t) = 0.$$

Substitution Theorems

The substitution (or change of variables) theorems, which are consequences of the Chain Rule of calculus, are often useful in converting one integral into another. For example, the integral

$$\int_0^1 \frac{du}{\sqrt{1-u^2}} = \operatorname{Arcsin} 1 = \tfrac{1}{2}\pi$$

can be converted (by the substitutions $u = y^2$ and $u = \sqrt{x}$) into the integrals

$$\int_0^1 \frac{2y\,dy}{\sqrt{1-y^4}} \qquad \text{and} \qquad \tfrac{1}{2}\int_0^1 \frac{dx}{\sqrt{x-x^2}}.$$

Both of these substitutions are instances of the First Substitution Theorem, which can be summarized by the formula

$$\int_{\Phi(a)}^{\Phi(b)} f(u)\,du = \int_a^b f\big(\Phi(x)\big)\cdot\Phi'(x)\,dx,$$

which is valid under certain hypotheses that will be stated.

The transformation in the direction $u \to \Phi(x)$ is usually straightforward. But every equation can be read from right to left, as well as from left to right. So the First Substitution Theorem is also useful when we see that an integral has the form on the right side of the above equation for appropriate functions f and Φ. For instance, we note that the integral

$$\int_0^4 \frac{2x\,dx}{1+x^2}$$

has this form where $f(u) := (1+u)^{-1}$ and $u = \Phi(x) := x^2$, so the value of this integral can be seen to equal $\ln 17$. Usually, however, the appropriate substitution is not as obvious as in this case.

The First Substitution Theorem

We will now establish the validity of the above formula under some conditions that are useful in practice. Frequently, the differentiable function Φ is assumed to be a strictly monotone mapping of the interval $I := [a,b]$ onto the interval $\Phi(I)$, but for many applications that hypothesis is not satisfied, and we need to permit Φ to be many-to-one. Of course, it is necessary that f be defined on the interval $\Phi(I)$, which contains — but need not equal — the interval with endpoints $\Phi(a)$ and $\Phi(b)$.

We will first state a theorem in the case where $f : J \to \mathbb{R}$ has a c-primitive F on J, and $\varphi : I \to \mathbb{R}$ has a c-primitive Φ on I. We will also suppose that Φ is a *countable-to-one* mapping of I into J in the sense that $\Phi^{-1}(\{p\})$ is a countable set in I for each $p \in J$.

It is stressed that for the equality of the two integrals, it is not necessary to know the c-primitive F of f explicitly, but merely to know that F exists (as it does, for example, when f is regulated).

$\diamond$ **13.1 First Substitution Theorem, I.** *Let $I := [a,b]$ and $J := [c,d]$ and suppose that:*

(i) *$f : J \to \mathbb{R}$ has a c-primitive F on J.*

(j) *$\varphi : I \to \mathbb{R}$ has a c-primitive Φ on I and $\Phi(I) \subseteq J$.*

(k) *Φ is a countable-to-one mapping of I into J.*

Then $(f \circ \Phi)\cdot\varphi \in \mathcal{R}^*(I)$ and $f \in \mathcal{R}^*\big(\Phi(I)\big)$; moreover,

$$(13.\alpha)\qquad \int_a^b (f\circ\Phi)\cdot\varphi = \big(F\circ\Phi\big)\Big|_a^b = \int_{\Phi(a)}^{\Phi(b)} f.$$

Proof. By hypothesis (i), F is continuous on J and there exists a countable set $C_f \subset J$ such that $F'(u) = f(u)$ for all $u \in J - C_f$. By hypothesis (j), Φ is continuous on I and there exists a countable set $C_\varphi \subset I$ such that $\Phi'(x) = \varphi(x)$ for all $x \in I - C_\varphi$. Therefore $F \circ \Phi$ is continuous on I. Hypothesis (k) implies that $\Phi^{-1}(C_f)$ is a countable set in I, so $C := C_\varphi \cup \Phi^{-1}(C_f)$ is a countable set in I. The Chain Rule (see [B-S; p. 162]) implies that

$$(F\circ\Phi)'(x) = F'\big(\Phi(x)\big)\cdot\Phi'(x) = (f\circ\Phi)(x)\cdot\varphi(x)$$

for all $x \in I - C$. Therefore $F \circ \Phi$ is a c-primitive of $(f \circ \Phi) \cdot \varphi$, so that $(f \circ \Phi) \cdot \varphi$ belongs to $\mathcal{R}^*(I)$ and

$$\int_a^b (f \circ \Phi) \cdot \varphi = (F \circ \Phi)\Big|_a^b = F\big(\Phi(b)\big) - F\big(\Phi(a)\big).$$

It follows from (i) that $f \in \mathcal{R}^*(J)$. But since $\Phi(I)$ is a compact interval in J, Corollary 3.8 implies that f is integrable on $\Phi(I)$ and also on the compact interval with endpoints $\Phi(a), \Phi(b)$.

If $\Phi(a) \le \Phi(b)$, then the Fundamental Theorem 4.7 applied to the interval $[\Phi(a), \Phi(b)]$ implies that

$$\int_{\Phi(a)}^{\Phi(b)} f = F\Big|_{\Phi(a)}^{\Phi(b)} = F\big(\Phi(b)\big) - F\big(\Phi(a)\big).$$

If $\Phi(b) < \Phi(a)$, then we apply the Fundamental Theorem 4.7 to the interval $[\Phi(b), \Phi(a)]$ to obtain

$$\int_{\Phi(a)}^{\Phi(b)} f = -\int_{\Phi(b)}^{\Phi(a)} f = -F\Big|_{\Phi(b)}^{\Phi(a)} = F\big(\Phi(b)\big) - F\big(\Phi(a)\big).$$

Hence (13.α) holds in either case. Q.E.D.

Remarks. **(a)** If F is a primitive of f on J (so that $C_f = \emptyset$), then hypothesis (k) is not needed in 13.1.

(b) As mentioned at the beginning of this section, we sometimes use formula (13.α) in the opposite direction. That is, to integrate $\int_\alpha^\beta \tilde{f}(x)\,dx$, we sometimes find a substitution $x = \Omega(v)$ that makes $(\tilde{f} \circ \Omega)(v)$ a relatively simple function of v, and such that $\omega(v) := \Omega'(v)$ is also simple. In this case the formula (13.α) can be read as

$$(13.\alpha') \qquad \int_\alpha^\beta \tilde{f}(x)\,dx = \int_a^b (\tilde{f} \circ \Omega) \cdot \omega,$$

where a and b are numbers such that $\alpha = \Omega(a)$ and $\beta = \Omega(b)$. Of course, in using this approach we still have to verify that the hypotheses of Theorem 13.1 are satisfied.

13.2 Examples. **(a)** Consider the integral $\displaystyle\int_{-1}^3 2x\sqrt{1+x^2}\,dx$.

We note that if we put $f(u) := \sqrt{u}$ for $u \ge 0$, and $u = \Phi(x) := 1 + x^2$, then $\Phi'(x) = 2x$ for $x \in [-1, 3]$. Thus the integrand has the form

$(f \circ \Phi)(x) \cdot \Phi'(x)$, where f has the primitive $F(u) := u^{3/2}$ and $C_f = \emptyset$. Since $\Phi(-1) = 2$ and $\Phi(3) = 10$, Theorem 13.1 implies that

$$\int_{-1}^{3} 2x\sqrt{1+x^2}\,dx = \int_{2}^{10} \sqrt{u}\,du = \tfrac{2}{3}u^{3/2}\Big|_{u=2}^{u=10} = \tfrac{2}{3}(10\sqrt{10} - 2\sqrt{2}).$$

Note that $\Phi([-1,3]) = [0,10] \neq [\Phi(-1), \Phi(3)]$ and that Φ is not one-one on $[-1,3]$; however, it is at most two-to-one on this interval.

(b) Consider $\displaystyle\int_0^9 \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$, where the integrand equals 0 at 0.

The integrand becomes unbounded as $x \to 0+$; hence there is some doubt about the existence of the integral. But, since the function is (measurable and) dominated in absolute value by $1/\sqrt{x} \in \mathcal{L}([0,9])$, the integral certainly exists.

We let $u = \Phi(x) := \sqrt{x}$ on $[0,9]$ so that Φ is an f-primitive of $\varphi(x) := 1/(2\sqrt{x})$. If we put $f(u) := \cos u$, which has a primitive $F(u) = \sin u$, then the integrand has the form

$$f\big(\Phi(x)\big) \cdot \varphi(x) = \frac{\cos\sqrt{x}}{2\sqrt{x}} \qquad \text{for} \quad x \in (0,9].$$

Since $C_f = \emptyset$, condition (k) is satisfied. Theorem 13.1 now yields

$$\int_0^9 \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int_0^3 \cos u\,du = 2\sin u\Big|_0^3 = 2\sin 3.$$

(b′) We consider the same integral as in (b).

This time we notice that since the integrand involves $\sqrt{x}$, it would be simplified if we used the substitution $x = \Omega(v) := v^2$, for if $v \geq 0$ then $\sqrt{x} = \sqrt{v^2} = v$ and so the integrand

$$\tilde{f}(x) := \frac{\cos\sqrt{x}}{\sqrt{x}} \qquad \text{becomes} \qquad (\tilde{f} \circ \Omega)(v) = \frac{\cos v}{v},$$

at least for $v > 0$. Moreover $\omega(v) := \Omega'(v) = 2v$ so that

$$(\tilde{f} \circ \Omega)(v) \cdot \omega(v) = \frac{\cos v}{v} \cdot 2v = 2\cos v.$$

We also note that Ω is a one-one mapping of $[0,3]$ onto $[0,9]$, so we are led once more to the integral $\int_0^3 2\cos v\,dv$.

Now since $\tilde{f} \in \mathcal{R}^*([0,9])$ is continuous on $(0,9]$, it follows from Corollary 4.9 that it has an (unknown) f-primitive on $[0,9]$. Clearly Ω is a primitive of ω on $[0,3]$ and Ω is one-one. Thus the hypotheses of 13.1 are satisfied.

(c) Consider the integral $\displaystyle\int_0^1 x(1-x^2)^{-1}\,dx$.

The integrand is unbounded as $x \to 1-$, so we will first consider the integral over $I_b := [0,b]$, with $b \in (0,1)$. If we let $u = \Phi(x) := 1 - x^2$, then Φ is a primitive of $\varphi(x) := -2x$ on I_b. Further, Φ is a strictly decreasing map of I_b onto $J_b := [1-b^2, 1]$. Also, the function $f(u) := (2u)^{-1}$ for $u \in J_b$ has a primitive $F(u) := \frac{1}{2}\ln u$ on J_b. Hence

$$\int_0^b \frac{x\,dx}{1-x^2} = -\int_0^b \frac{-2x\,dx}{2(1-x^2)} = -F\circ\Phi\Big|_0^b = -\tfrac{1}{2}\ln(1-b^2).$$

Note that the limit of $-\frac{1}{2}\ln(1-b^2)$ does not exist in $\mathbb{R}$ as $b \to 1-$. Therefore, by Hake's Theorem 12.8, the integral $\int_0^1 x(1-x^2)^{-1}\,dx$ does not exist.

(d) Consider the integral $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$.

We will make use of $(13.\alpha')$. Our knowledge of the trigonometric functions suggests that we use the substitution $x = \Omega(v) := \sin v$. If $\tilde{f}(x) = \sqrt{1-x^2}$, then

$$(\tilde{f}\circ\Omega)(v) = \sqrt{1-\sin^2 v} = \sqrt{\cos^2 v} = |\cos v|.$$

Since $\tilde{f}$ is continuous on $[0,1]$, it has a primitive on this interval. We note that Ω maps the interval $[0, \frac{1}{2}\pi]$ in a one-one fashion onto $[0,1]$ and is the primitive of $\omega(v) = \cos v \geq 0$ on $[0, \frac{1}{2}\pi]$. Therefore equation $(13.\alpha')$ gives

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\,dx &= \int_0^{\frac{1}{2}\pi} \cos v \cdot \cos v\,dv = \int_0^{\frac{1}{2}\pi} \cos^2 v\,dv \\ &= \tfrac{1}{2}\big[\sin v\cos v + v\big]\Big|_0^{\frac{1}{2}\pi} = \tfrac{1}{2}\big[\sin(\tfrac{1}{2}\pi)\cos(\tfrac{1}{2}\pi) + \tfrac{1}{2}\pi\big] = \tfrac{1}{4}\pi.\end{aligned}$$

The Second Substitution Theorem

The way that substitutions are often approached in calculus courses is to observe that setting $u = \Phi(x)$ will make it possible to write part of the integrand in the form $f\circ\Phi$ and then hope that we can come up with the factor $\varphi = \Phi'$ somewhere. If we can "build up" the factor φ by adjusting the remaining part of the integrand by constants, then we can use Theorem 13.1; otherwise, we usually abandon this substitution and try another one.

However, there is another theorem that enables us to convert an integral involving $f \circ \Phi$ into an integral involving f and the derivative of the function $\Psi := \Phi^{-1}$ that is inverse to Φ. Since we require Φ to have an inverse function, we now assume that Φ is one-one.

First we recall that if $\Phi : I \to \mathbb{R}$ is continuous on $I = [a, b]$ and has a nonzero derivative on (a, b), then by the Darboux Intermediate Value Theorem (see [B-S; p. 174]) applied to compact subintervals of (a, b), we infer that the derivative $\varphi = \Phi'$ has constant sign on (a, b). Hence, by the Mean Value Theorem and the continuity of Φ, the function Φ is strictly monotone on $[a, b]$ and we therefore have either

$$\Phi\big([a, b]\big) = [\Phi(a), \Phi(b)] \qquad \text{or} \qquad \Phi\big([a, b]\big) = [\Phi(b), \Phi(a)],$$

depending on whether Φ is increasing or decreasing. Consequently [B-S; p. 152], the function Ψ inverse to Φ is strictly monotone and continuous on the compact interval $\Phi\big([a, b]\big)$. Moreover [B-S; p. 165], the function Ψ is differentiable at every point u in the open interval $\Phi\big((a, b)\big)$ and

$$\psi(u) := \Psi'(u) = \frac{1}{\Phi'\big(\Psi(u)\big)} = \frac{1}{\varphi\big(\Psi(u)\big)}.$$

If $\psi = \Psi'$ does not exist at one or both of the endpoints $\Phi(a), \Phi(b)$, we define it to equal 0 there.

We now state a version of the Second Substitution Theorem. Although it is not the most general theorem possible, it applies in very many circumstances. We have taken pains to allow the possibility that $\varphi = \Phi'$ vanishes (or does not exist) at the endpoints a, b, since this case often arises in applications.

$\diamond$ **13.3 Second Substitution Theorem, I.** *Let $I := [a, b]$ and $J := [c, d]$ and suppose that:*

(i′) $f : J \to \mathbb{R}$.

(j′) $\Phi : I \to \mathbb{R}$ *is continuous,* $\Phi(I) \subseteq J$ *and* Φ' *exists and is* $\neq 0$ *on* (a, b).

(k′) $f \circ \Phi$ *has a c-primitive* W *on* I.

Then $f \circ \Phi$ is integrable on I and $f \cdot \psi$ is integrable on $\Phi(I)$, where Ψ is the function inverse to Φ and $\psi = \Psi'$ on $\Phi\big((a, b)\big)$. Moreover

$$\int_a^b f \circ \Phi = W\Big|_a^b = \int_{\Phi(a)}^{\Phi(b)} f \cdot \psi. \tag{13.β}$$

Remarks. **(a)** A c-primitive W exists if f is regulated (and so, if it is either continuous or monotone) on J.

(b) Sometimes it is not easy to find a c-primitive W of $f \circ \Phi$, but one can find a c-primitive of $f \cdot \psi$.

(c) When these c-primitives are not known explicitly, one still has the equality of the integrals.

Proof of 13.3. We note that hypothesis (j′) implies that there exists a continuous strictly monotone function Ψ inverse to Φ and that $\psi(u) := \Psi'(u)$ exists for all $u \in \Phi\big((a,b)\big) \subset J$.

If W is a c-primitive of $f \circ \Phi$ on I, there exists a countable set $C_1 \subset I$ such that $W'(x) = f\big(\Phi(x)\big)$ for $x \in I - C_1$. We let $C_2 := \Phi(C_1) \cup \{\Phi(a), \Phi(b)\}$ so that C_2 is a countable set in the compact interval $\Phi(I)$. By the Chain Rule, if $u \in \Phi(I) - C_2$, then $u \notin \{\Phi(a), \Phi(b)\}$ and $\Psi(u) \notin C_1$, so that

$$(W \circ \Psi)'(u) = W'\big(\Psi(u)\big) \cdot \Psi'(u) = f\big(\Phi \circ \Psi(u)\big) \cdot \psi(u) = f(u) \cdot \psi(u).$$

Thus $W \circ \Psi$ is a c-primitive of $f \cdot \psi$ on $\Phi(I)$, so that $f \cdot \psi \in \mathcal{R}^*\big(\Phi(I)\big)$. Moreover,

$$\int_a^b f \circ \Phi = W\Big|_a^b = (W \circ \Psi)\Big|_{\Phi(a)}^{\Phi(b)} = \int_{\Phi(a)}^{\Phi(b)} f \cdot \psi.$$

Therefore equation (13.β) is established. Q.E.D.

13.4 Example. Consider the integral $\displaystyle\int_0^2 (1+\sqrt{x})^{-1}\,dx$.

We use the Second Substitution Theorem 13.3 with $f(u) := (1+u)^{-1}$ and $u = \Phi(x) = \sqrt{x}$. Here $\Phi'(x) = 1/(2\sqrt{x}) > 0$ for $x \in (0,2]$ and it is clear that $\Psi(u) = u^2$ so that $\psi(u) = 2u$. Note that f is continuous (and also monotone) on $\Phi([0,2]) = [0, \sqrt{2}]$, so we have

$$\begin{aligned}\int_0^2 \frac{dx}{1+\sqrt{x}} &= \int_0^2 (f \circ \Phi)(x)\,dx \\ &= \int_0^{\sqrt{2}} f(u) \cdot \psi(u)\,du = \int_0^{\sqrt{2}} \frac{1}{1+u} \cdot 2u\,du.\end{aligned}$$

If we let $u = (1+u) - 1$ on the right, we readily show that the value of this integral equals $2[\sqrt{2} - \ln(1+\sqrt{2})]$.

[Of course, in calculus we were taught to set $u = \sqrt{x}$ so that $x = u^2$ and hence $dx = 2u\,du$, whence

$$\frac{dx}{1+\sqrt{x}} = \frac{2u\,du}{1+u}.$$

Thus the somewhat mysterious juggling with differentials gives the correct result even though it gives no indication that the point $x = 0$ is a difficult point for the function Φ.]

(b) Consider the integral $\displaystyle\int_0^1 (1 - \sqrt{x})^{-1}\, dx$.

The integrand is unbounded as $x \to 1-$, so we will consider the integral over $[0, b]$ with $b \in (0, 1)$. As in (a), the substitution $u = \Phi(x) := \sqrt{x}$ gives

$$\begin{aligned}\int_0^b \frac{dx}{1-\sqrt{x}} = \int_0^{\sqrt{b}} \frac{2u\, du}{1-u} &= -2[u + \ln(1-u)]\Big|_0^{\sqrt{b}} \\ &= -2[\sqrt{b} + \ln(1-\sqrt{b})].\end{aligned}$$

But this last expression approaches $+\infty$ as $b \to 1-$, so Hake's Theorem 12.8 implies that the integral $\int_0^1 (1-\sqrt{x})^{-1}\, dx$ does not exist.

An Extension of Theorem 13.1

We now give a version of the First Substitution Theorem 13.1 that does not assume the existence of the c-primitive of the function f. Instead, it will be assumed that the substitution function $\Phi : I \to \mathbb{R}$ is continuous, strictly monotone, and differentiable except possibly on a countable set $C \subset I$.

◇ **13.5 First Substitution Theorem, II.** *Let $I := [a, b]$ and $J := [c, d]$ and let $f : J \to \mathbb{R}$. Let $\Phi : I \to \mathbb{R}$ be a continuous strictly monotone function with $\Phi(I) \subseteq J$ and suppose that there exists a* ***countable*** *set $C \subset I$ such that $\varphi(x) := \Phi'(x)$ exists for all $x \in I - C$ and let $\varphi(x) := 0$ for $x \in C$.*

(a) *Then f belongs to $\mathcal{R}^*\big(\Phi(I)\big)$ if and only if $(f \circ \Phi) \cdot \varphi$ belongs to $\mathcal{R}^*(I)$.*

(b) *Also f belongs to $\mathcal{L}\big(\Phi(I)\big)$ if and only if $(f \circ \Phi) \cdot \varphi$ belongs to $\mathcal{L}(I)$.*

In either case, we have

$$(13.\gamma) \qquad \int_{\Phi(a)}^{\Phi(b)} f = \int_a^b (f \circ \Phi) \cdot \varphi.$$

Remark. Formula $(13.\gamma)$ will be proved when Φ is a strictly increasing function, but it also remains true when Φ is strictly decreasing, as the reader may show. However, in that case, if $I := [a, b]$, then $\Phi(I) = [\Phi(b), \Phi(a)]$, so the integral on the left side is from a larger to a smaller value. Bearing in mind that the derivative $\varphi = \Phi'$ is ≤ 0 on I, we can write both the

increasing and the decreasing case in the form

$$(13.\gamma') \qquad \int_{\Phi(I)} f = \int_I (f \circ \Phi) \cdot |\varphi|,$$

which is consistent with the situation in higher dimensions.

The proof of Theorem 13.1 was based on the Chain Rule and was quite straightforward. The proof of Theorem 13.5 is more involved because we need to take a careful look at the Riemann sums of two functions on different intervals. This theorem was given by McLeod [McL; pp. 64–65] without a detailed proof, and proved by Marie Bielawski by carefully adjusting the gauges for the Riemann sums approximating the two integrals in (13.γ). We will modify her argument by the use of "interval-gauges", which provides an alternative approach to the generalized Riemann integral and so has some independent interest.

Interval-gauges

In Exercise 1.T we defined an **interval-gauge** on $I := [a, b]$ to be a mapping $t \mapsto \Delta(t)$ of points $t \in I$ into bounded closed intervals $\Delta(t) = [a(t), b(t)]$ such that $t \in (a(t), b(t))$ for all $t \in I$. We say that an interval-gauge Δ on I is **symmetric** if t is the midpoint of $\Delta(t)$ for all $t \in I$. It is clear that if δ is a (point) gauge on I, then we can define a symmetric interval-gauge

$$t \mapsto \Delta(t) := [t - \delta(t), t + \delta(t)]$$

corresponding to δ. Conversely, if Δ is an interval-gauge on I, then we can define a (point) gauge δ_Δ on I by

$$(13.\delta) \qquad \delta_\Delta(t) := \min\{t - a(t), b(t) - t\} \qquad \text{for} \quad t \in I.$$

As in Exercise 1.U, if Δ is an interval-gauge on I, we say that a tagged partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ of I is **Δ-fine** if $I_i \subseteq \Delta(t_i)$ for all $i = 1, \cdots, n$. If Δ is an interval-gauge and if we define δ_Δ as in (13.δ), then any partition $\dot{\mathcal{P}}$ of I that is δ_Δ-fine is certainly also Δ-fine. Thus, by Cousin's Theorem 1.4, for every interval-gauge Δ there exist tagged partitions that are Δ-fine.

We say that a function $g : I \to \mathbb{R}$ is **integrable** on I to a number $D \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists an interval-gauge Δ_ε on I such that if $\dot{\mathcal{P}}$ is any Δ_ε-fine partition of I, then $|S(g; \dot{\mathcal{P}}) - D| \leq \varepsilon$.

From the preceding remarks, we see that a function g is integrable in this sense if and only if it is integrable in the sense of Definition 1.7, and the values of the integrals are equal.

The reason why interval-gauges are useful is that they map nicely with respect to strictly monotone functions. In more detail, let $\Phi : I \to \mathbb{R}$ be a continuous strictly *increasing* function. We extend Φ to all of $\mathbb{R}$ by defining

$$\Phi(x) := \begin{cases} \Phi(a) + t - a & \text{for} \quad t < a, \\ \Phi(b) + t - b & \text{for} \quad t > b. \end{cases}$$

This extended function Φ is continuous and strictly increasing on $\mathbb{R}$; therefore, it is an *order-preserving* map that sends compact intervals to compact intervals, and sends the endpoints (and the interior points) of such intervals to the endpoints (and the interior points) of the images of these intervals.

Now, if Δ is an interval-gauge on I, then we define $\acute{\Delta}$ on $\Phi(I)$ by

$$\acute{\Delta}(\Phi(t)) := \Phi(\Delta(t)) \qquad \text{for} \quad t \in I.$$

The properties of Φ mentioned above show that $\acute{\Delta}$ is an interval-gauge on $\Phi(I)$. Moreover, if $s = \Phi(t)$ and $\Delta(t) = [a(t), b(t)]$, then

$$\acute{\Delta}(s) = \big[\Phi\big(a(t)\big), \Phi\big(b(t)\big)\big].$$

Similarly, if $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n}$ is a tagged partition of I, then we define

$$\acute{\mathcal{P}} := \big\{\big(\Phi(I_i), \Phi(t_i)\big)\big\}_{i=1}^{n}.$$

The properties of Φ imply that $\acute{\mathcal{P}}$ is a tagged partition of $\Phi(I)$. Moreover, it is clear that if $\dot{\mathcal{P}}$ is Δ-fine, then $\acute{\mathcal{P}}$ is $\acute{\Delta}$-fine.

On the other hand, if $\Psi : \mathbb{R} \to \mathbb{R}$ is the function inverse to Φ, then Ψ is also continuous and strictly increasing. Thus it maps an interval-gauge Γ on $\Phi(I)$ into an interval-gauge $\grave{\Gamma}$ on $I = \Psi(\Phi(I))$ given by

$$\grave{\Gamma}(t) := \Psi\big(\Gamma(\Phi(t)\big) \qquad \text{for} \quad t \in I.$$

Similarly, if $\dot{\mathcal{Q}} := \{(J_k, s_k)\}_{k=1}^{m}$ is a tagged partition of $\Phi(I)$, then

$$\grave{\mathcal{Q}} := \big\{\big(\Psi(J_k), \Psi(s_k)\big)\big\}_{k=1}^{m}$$

is a tagged partition of I and if $\dot{\mathcal{Q}}$ is a Γ-fine partition of $\Phi(I)$, then we see that $\grave{\mathcal{Q}}$ is a $\grave{\Gamma}$-fine partition of I. Further, if $\dot{\mathcal{Q}}$ is given and $\dot{\mathcal{P}} := \grave{\mathcal{Q}}$, then $\dot{\mathcal{Q}} = \acute{\mathcal{P}}$.

Remark. An observant reader (with good eyesight) will have noticed that we are using *ácute* accents to indicate the transformed interval-gauges and partitions from I into $\Phi(I)$, and *gràve* accents to indicate the transformed interval-gauges and partitions from $\Phi(I)$ to I.

The above discusion was for a strictly *increasing* function Φ. If Φ is continuous and strictly *decreasing*, then we extend Φ to all of $\mathbb{R}$ by defining it to have slope -1 outside of $[a, b]$, and obtain a continuous and strictly decreasing function on $\mathbb{R}$. This extended function is an *order-reversing* map of $\mathbb{R}$. Consequently, the endpoints of the intervals need to be reversed, but the preceding considerations are readily modified.

Proof of Theorem 13.5

We now show that the stated properties of Φ guarantee that a suitably fine partition of I gives rise to Riemann sums of f and $(f \circ \Phi) \cdot \varphi$ that are nearly equal. We will treat only the increasing case. The reader will note that the proof of this lemma is similar to that of the Fundamental Theorem 4.7.

13.6 Lemma. *Let $f : J \to \mathbb{R}$ and $\Phi, \varphi : I \to \mathbb{R}$ be as in Theorem 13.5. Given $\varepsilon > 0$, there exists an interval-gauge Δ_Φ on I such that if $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is any Δ_Φ-fine partition of I and if $\acute{\mathcal{P}} = \{(\Phi(I_i), \Phi(t_i))\}_{i=1}^n$, then*

$$(13.\varepsilon) \qquad \left|S(f; \acute{\mathcal{P}}) - S((f \circ \Phi) \cdot \varphi; \dot{\mathcal{P}})\right| \leq \varepsilon(b - a + 1).$$

Proof. Let $C = \{c_k\}$ be an enumeration of the set where the derivative $\Phi' = \varphi$ does not exist and let $\varepsilon > 0$ be given. If $t \in I - C$, then Φ is differentiable at t, so we may apply the Straddle Lemma 4.4 to find a compact interval $\Delta_\Phi(t)$ with midpoint t such that if $u, v \in \Delta_\Phi(t)$ and $u \leq t \leq v$, then

$$\left|\Phi(v) - \Phi(u) - \varphi(t)(v - u)\right| \leq \frac{\varepsilon(v - u)}{|f(\Phi(t))| + 1},$$

whence it follows that

$$(13.\zeta) \qquad \left|f(\Phi(t))[\Phi(v) - \Phi(u)] - f(\Phi(t)) \cdot \varphi(t)(v - u)\right| \leq \varepsilon(v - u).$$

Since Φ is continuous at $c_k \in I$, there exists a compact interval $\Delta_\Phi(c_k)$ with midpoint c_k such that if $u, v \in \Delta_\Phi(c_k)$ and $u \leq c_k \leq v$, then

$$\left|\Phi(v) - \Phi(u)\right| \leq \frac{\varepsilon}{2^k(|f(\Phi(c_k))| + 1)},$$

and since $\varphi(c_k) := 0$, we have

$$(13.\eta) \qquad \left|f(\Phi(c_k))[\Phi(v) - \Phi(u)] - f(\Phi(c_k)) \cdot \varphi(c_k)(v - u)\right| \leq \frac{\varepsilon}{2^k}.$$

Now let $I_i = [x_{i-1}, x_i]$, so that $\Phi(I_i) = [\Phi(x_{i-1}), \Phi(x_i)]$. It follows that

$$S(f; \acute{\mathcal{P}}) = \sum_{i=1}^{n} f\big(\Phi(t_i)\big)\big[\Phi(x_i) - \Phi(x_{i-1})\big] \qquad \text{and}$$

$$S\big((f \circ \Phi) \cdot \varphi; \dot{\mathcal{P}}\big) = \sum_{i=1}^{n} f\big(\Phi(t_i)\big) \cdot \varphi(t_i)(x_i - x_{i-1}).$$

Therefore, if we use (13.ζ) and (13.η), then we conclude that

$$\begin{aligned} \big|S(f; \acute{\mathcal{P}}) - S\big((f \circ \Phi) \cdot \varphi; \dot{\mathcal{P}}\big)\big| &< \varepsilon \sum_{i=1}^{n}(x_i - x_{i-1}) + \sum_{k=1}^{\infty} \varepsilon/2^k \\ &= \varepsilon(b-a) + \varepsilon, \end{aligned}$$

which is the stated inequality. Q.E.D.

We are now prepared to give the proof of Theorem 13.5. We will consider only the case of increasing Φ.

Proof of Theorem 13.5. (a) Suppose that $(f \circ \Phi) \cdot \varphi$ belongs to $\mathcal{R}^*(I)$ with integral A. Then, given $\varepsilon > 0$ there exists an interval-gauge Δ_ε on I such that if $\dot{\mathcal{P}}$ is a Δ_ε-fine partition of I, then $|S\big((f \circ \Phi) \cdot \varphi; \dot{\mathcal{P}}\big) - A| \le \varepsilon$. If Δ_Φ is the interval-gauge in Lemma 13.6, we let $\Lambda_\varepsilon := \Delta_\varepsilon \cap \Delta_\Phi$, so that if $\dot{\mathcal{P}}$ is a Λ_ε-fine partition of I, then it is both Δ_ε-fine and Δ_Φ-fine. Now let $\acute{\Lambda}_\varepsilon$ be the interval-gauge on $\Phi(I)$ that corresponds to Λ_ε. If $\dot{\mathcal{Q}}$ is a $\acute{\Lambda}_\varepsilon$-fine partition of $\Phi(I)$, then $\dot{\mathcal{P}} := \grave{\mathcal{Q}}$ is Λ_ε-fine and $\dot{\mathcal{Q}} = \acute{\mathcal{P}}$, so that $S(f; \dot{\mathcal{Q}}) = S(f; \acute{\mathcal{P}})$. Therefore we have

$$\begin{aligned} \big|S(f; \dot{\mathcal{Q}}) - A\big| &\le \big|S(f; \acute{\mathcal{P}}) - S\big((f \circ \Phi) \cdot \varphi; \dot{\mathcal{P}}\big)\big| + \big|S\big((f \circ \Phi) \cdot \varphi; \dot{\mathcal{P}}\big) - A\big| \\ &\le \varepsilon(b - a + 2). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we deduce that f is in $\mathcal{R}^*\big(\Phi(I)\big)$ with integral A.

Now suppose that f belongs to $\mathcal{R}^*\big(\Phi(I)\big)$ with integral B. Then, given $\varepsilon > 0$, there exists an interval-gauge Γ_ε on $\Phi(I)$ such that if $\dot{\mathcal{Q}}$ is any Γ_ε-fine partition of $\Phi(I)$, then $|S(f; \dot{\mathcal{Q}}) - B| \le \varepsilon$. Now let $\grave{\Gamma}_\varepsilon$ be the interval-gauge on I corresponding to Γ_ε and let $\Omega_\varepsilon(t) := \Delta_\Phi(t) \cap \grave{\Gamma}_\varepsilon(t)$ for $t \in I$ so that Ω_ε is an interval-gauge on I. Now, if $\dot{\mathcal{P}}$ is an Ω_ε-fine partition of I, then $\dot{\mathcal{P}}$ is Δ_ε-fine and $\acute{\mathcal{P}}$ is Γ_ε-fine, so that

$$\begin{aligned} \big|S\big((f \circ \Phi) \cdot \varphi; \dot{\mathcal{P}}\big) - B\big| &\le \big|S\big((f \circ \Phi) \cdot \varphi; \dot{\mathcal{P}}\big) - S(f; \acute{\mathcal{P}})\big| + \big|S(f; \acute{\mathcal{P}}) - B\big| \\ &\le \varepsilon(b - a + 2). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $(f \circ \Phi) \cdot \varphi$ belongs to $\mathcal{R}^*(I)$ with integral equal to B.

(b) If Φ is increasing, then $\varphi \geq 0$ and so $\big((f \circ \Phi) \cdot \varphi\big)^{\pm} = (f^{\pm} \circ \Phi) \cdot \varphi$. We now apply part (a) to these functions and to $f^{\pm}$, using Theorem 7.11. Q.E.D.

The Second Substitution Theorem

We now give another version of the Second Substitution Theorem. It is possible to give a proof that is quite parallel to the proof of Theorem 13.5, but we prefer to deduce it from that result.

$\diamond$ **13.7 Second Substitution Theorem, II.** *Let $I := [a, b]$ and $J := [c, d]$ and let $f : I \to \mathbb{R}$. Let $\Phi : I \to \mathbb{R}$ be a continuous strictly monotone function with $\Phi(I) \subseteq J$ and suppose that there exists a* ***countable*** *set $C \subset I$ such that $\varphi(x) := \Phi'(x) \neq 0$ for all $x \in I - C$. Let Ψ be the continuous strictly monotone function inverse to Φ so that*

$$\psi(y) := \Psi'(y) = 1/\varphi\big(\Psi(y)\big) \qquad \text{for} \quad y \in \Phi(I - C),$$

and let $\psi(y) := 0$ for $y \in \Phi(C)$.

(a) *Then $f \cdot \psi$ belongs to $\mathcal{R}^*\big(\Phi(I)\big)$ if and only if $f \circ \Phi$ belongs to $\mathcal{R}^*(I)$.*

(b) *Also $f \cdot \psi$ belongs to $\mathcal{L}\big(\Phi(I)\big)$ if and only if $f \circ \Phi$ belongs to $\mathcal{L}(I)$.*

In either case, we have

$$\int_{\Phi(a)}^{\Phi(b)} f \cdot \psi = \int_a^b f \circ \Phi. \tag{13.ι}$$

Proof. (a) We note that $\psi\big(\Phi(x)\big) \cdot \varphi(x) = 1$ for all $x \in I - C$. Therefore, if we let $f_1(y) := f(y) \cdot \psi(y)$ for $y \in \Phi(I)$, it follows that

$$(f_1 \circ \Phi)(x) \cdot \varphi(x) = f\big(\Phi(x)\big) \cdot \psi\big(\Phi(x)\big) \cdot \varphi(x) = (f \circ \Phi)(x)$$

for $x \in I - C$. Theorem 13.5 implies that $f_1 = f \cdot \psi$ belongs to $\mathcal{R}^*\big(\Phi(I)\big)$ if and only if $(f_1 \circ \Phi) \cdot \varphi = f \circ \Phi$ belongs to $\mathcal{R}^*(I)$. In that case

$$\int_{\Phi(a)}^{\Phi(b)} f \cdot \psi = \int_{\Phi(a)}^{\Phi(b)} f_1 = \int_a^b (f_1 \circ \Phi) \cdot \varphi = \int_a^b f \circ \Phi,$$

but this yields equation (13.ι).

(b) If Φ is increasing, then $(f \cdot \psi)^{\pm} = f^{\pm} \cdot \psi$ and $(f \circ \Phi)^{\pm} = f^{\pm} \circ \Phi$. Now apply part (a). Q.E.D.

Another Substitution Theorem

We now obtain another version of the First Substitution Theorem in which we do not assume that Φ is monotone; instead, we will assume that $\Phi'(x)$ exists and is $\neq 0$ except on a *finite* subset of I.

$\diamond$ **13.8 First Substitution Theorem, III.** *Let $I := [a,b]$ and $J := [c,d]$ and let $f : J \to \mathbb{R}$. Let $\Phi : I \to \mathbb{R}$ be continuous with $\Phi(I) \subseteq J$ and suppose that there exists a* ***finite*** *set $E \subset I$ such that $\varphi(x) := \Phi'(x) \neq 0$ for all $x \in I - E$ and we set $\varphi(x) := 0$ for $x \in E$.*

(a) *Then f belongs to $\mathcal{R}^*(\Phi(I))$ if and only if $(f \circ \Phi) \cdot \varphi$ belongs to $\mathcal{R}^*(I)$.*

(b) *Also f belongs to $\mathcal{L}(\Phi(I))$ if and only if $(f \circ \Phi) \cdot \varphi$ belongs to $\mathcal{L}(I)$.*

In either case we have

$$\int_{\Phi(a)}^{\Phi(b)} f = \int_a^b (f \circ \Phi) \cdot \varphi. \tag{13.κ}$$

Proof. (a) Order the points in $E \cup \{a,b\}$ by $a =: e_0 < e_1 < \cdots < e_m := b$ and let $I_k := [e_{k-1}, e_k]$ for $k = 1, \cdots, m$. Then Φ is continuous on each interval I_k and $\Phi'(x) \neq 0$ for $x \in (e_{k-1}, e_k)$. It follows from the Darboux Intermediate Value Theorem [B-S; p. 174] that $\Phi'(x)$ does not change sign on I_k. The Mean Value Theorem then implies that Φ is strictly monotone on I_k. Theorem 13.5 implies that f is integrable on $\Phi(I_k)$ if and only if $(f \circ \Phi) \cdot \varphi$ is integrable on I_k. It therefore follows from Theorem 3.7 and induction that f is integrable on $\Phi(I)$ if and only if $(f \circ \Phi) \cdot \varphi$ is integrable on I, in which case the additivity of the integral over subintervals implies that (13.κ) holds.

(b) The case of absolute integrability is handled similarly. Q.E.D.

Remark. We will not state a corresponding version of the Second Substitution Theorem, since it requires the existence of an inverse function in each of the finite subintervals. However, it is clear that in certain instances, it might be useful to break the interval I into a finite number of parts and calculate the appropriate inverse functions on each part.

More Examples

13.9 Examples. (a) We saw in Example 6.13(a) that the **Beta function**

$$B(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1}\, dx \tag{13.λ}$$

exists when $p > 0,\ q > 0$. If we introduce the substitution $x = \Phi(\theta) := (\sin\theta)^2$ for $\theta \in [0, \pi/2]$, an elementary calculation gives

$$(13.\mu) \qquad B(p, q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\, d\theta.$$

(b) Consider the integral $\displaystyle\int_\alpha^\beta \frac{\sin(\pi/x)}{x}\, dx$ where $0 < \alpha < \beta$.

Here we let $x = \Phi(u) := 1/u$ for $u > 0$, noting that Φ is strictly decreasing and that $\varphi(u) := \Phi'(u) = -1/u^2$. Thus Φ is an order-reversing map of the interval $[1/\beta, 1/\alpha]$ onto $[\alpha, \beta]$. If we let $f(x) := [\sin(\pi/x)]/x$, then we have $f \circ \Phi(u) = [\sin(\pi u)]u$ and $f \circ \Phi(u) \cdot \varphi(u) = -[\sin(\pi u)]/u$.

If we apply formula $(13.\gamma)$ and simplify, we obtain

$$(13.\nu) \qquad \int_\alpha^\beta \frac{\sin(\pi/x)}{x}\, dx = \int_{1/\beta}^{1/\alpha} \frac{\sin(\pi u)}{u}\, du,$$

which will be used in Example 13.10(b).

Infinite Integrals

In Section 16 we will discuss the integral of a function over infinite intervals, such as the interval $[a, \infty)$, and we will obtain a version of Hake's Theorem asserting that the generalized Riemann integral $\int_a^\infty f$ exists if and only if $\int_a^c f$ exists for all $c > a$ and the limit $\lim_{c\to\infty} \int_a^c f$ exists.

We will now show that the substitution theorems we have established in this section sometimes give useful information concerning limits of this type. Hence, the results often provide an effective method of establishing the existence of, and of evaluating, these infinite integrals.

13.10 Examples. (a) If we take $x = \Phi(u) := u/(1+u)$ for $u \in [0, b]$ in equation $(13.\lambda)$, we obtain (after an easy calculation):

$$\int_0^{b/(1+b)} x^{p-1}(1-x)^{q-1}\, dx = 2\int_0^b \frac{u^{p-1}\, du}{(1+u)^{p+q}}.$$

Since we have seen in Example 6.13(a) that if $p > 0,\ q > 0$, then the function $x \mapsto x^{p-1}(1-x)^{q-1}$ belongs to $\mathcal{R}^*([0,1])$, we conclude from Hake's Theorem 12.8 and the fact that $b/(1+b) \to 1$ as $b \to \infty$ that

$$B(p, q) = 2\lim_{b\to\infty} \int_0^b \frac{u^{p-1}\, du}{(1+u)^{p+q}}.$$

By the version of Hake's Theorem to be proved in Section 16, we will have

$$B(p,q) = 2\int_0^\infty \frac{u^{p-1}\,du}{(1+u)^{p+q}}. \tag{13.ξ}$$

(b) If we take $\beta = 1$ in formula (13.ν), we infer that

$$\int_\alpha^1 \frac{\sin(\pi/x)}{x}\,dx = \int_1^{1/\alpha} \frac{\sin(\pi u)}{u}\,du$$

for all $\alpha \in (0,1]$. Now, since the function $x \mapsto (1/x)\sin(\pi/x)$ was shown in Example 6.13(b) to be integrable on $[0,1]$, we conclude from Hake's Theorem 12.8 that

$$\int_0^1 \frac{\sin(\pi/x)}{x}\,dx = \lim_{c\to\infty}\int_1^c \frac{\sin(\pi u)}{u}\,du.$$

Since $u \mapsto (1/u)\sin(\pi u)$ is bounded on $[0,1]$ and continuous (if we define it to equal π at $x = 0$), there is no doubt about the existence of the integral $\int_0^1 (1/u)\sin(\pi u)\,du$. Moreover, if we use (13.ν) with $\alpha = 1 < \beta$, we obtain

$$\int_1^\beta \frac{\sin(\pi/x)}{x}\,dx = \int_{1/\beta}^1 \frac{\sin(\pi u)}{u}\,du,$$

whence it follows that

$$\lim_{\beta\to\infty}\int_1^\beta \frac{\sin(\pi/x)}{x}\,dx = \int_0^1 \frac{\sin(\pi u)}{u}\,du.$$

Combining these observations, we obtain the nonobvious formula

$$\int_0^\infty \frac{\sin(\pi/x)}{x}\,dx = \int_0^\infty \frac{\sin(\pi u)}{u}\,du. \tag{13.o}$$

Note that the existence of the integrals over $[1,\infty)$ on each side of (13.o) follows from the existence of the integrals over $[0,1]$ on the *other* side.

Exercises

Some of the following integrals are divergent, and others are convergent. When possible, evaluate the convergent integrals exactly, checking your result with the given approximate values. State which theorem(s) you use and identify the functions; you may also use Integration by Parts. Assume that $0 \le a < b$.

13.A (a) $\int_0^3 x\sqrt{4+x^2}\,dx \approx 12.957$, (b) $\int_{-1}^1 x\sqrt{4+x^2}\,dx$.

13.B (a) $\int_0^3 \frac{x\,dx}{1+x^2} \approx 1.151$, (b) $\int_a^b \frac{x\,dx}{1+x^2}$.

13.C (a) $\int_0^4 \frac{dx}{2+\sqrt{x}} \approx 1.227$, (b) $\int_a^b \frac{dx}{2+\sqrt{x}}$.

13.D (a) $\int_1^3 \frac{dx}{x\sqrt{x+1}} \approx 0.664$, (b) $\int_0^3 \frac{dx}{x\sqrt{x+1}}$.

13.E (a) $\int_1^5 x\sqrt{2x+3}\,dx \approx 37.498$, (b) $\int_a^b x\sqrt{2x+3}\,dx$.

13.F (a) $\int_1^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx \approx 3.157$, (b) $\int_0^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$.

13.G (a) $\int_1^2 \frac{\sqrt{x-1}}{x}\,dx \approx 0.429$, (b) $\int_0^2 \frac{\sqrt{x-1}}{x}\,dx$.

13.H (a) $\int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx \approx 0.386$, (b) $\int_a^b \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$.

13.I (a) $\int_0^4 \frac{dx}{\sqrt{x}(x+4)} \approx 0.785$, (b) $\int_a^b \frac{dx}{\sqrt{x}(x+4)}$.

13.J $\int_a^b \frac{\cos x\,dx}{2+\sin x} = \ln(2+\sin x)\Big|_a^b$. [Use Theorem 13.8.]

13.K $\int_a^b \frac{\cos x\,dx}{2-\cos^2 x} = \mathrm{Arctan}(\sin x)\Big|_a^b$.

13.L (a) $\int_3^8 \frac{dx}{x\sqrt{x+1}} \approx 0.405$, (b) $\int_0^1 \frac{dx}{\sqrt{x-x^2}} \approx 1.571$.

13.M (a) $\int_0^1 \sqrt{\frac{x}{1-x^3}}\,dx \approx 1.047$, (b) $\int_0^1 \frac{\sqrt{x}\,dx}{1+x^3} \approx 0.524$.

13.N (a) $\displaystyle\int_{-1}^{3} \frac{x\,dx}{1+x^4} \approx 0.337$, (b) $\displaystyle\int_{0}^{1} \frac{x\,dx}{\sqrt{x}+x^2} \approx 0.462$.

13.O (a) Show that $f_s(x) := |\ln x|^s$ is in $\mathcal{R}^*([0,1])$ for $s > -1$ but not for $s \leq 1$.

(b) Show that if $s > 0$, then

$$\int_0^1 |\ln x|^s\,dx = s\int_0^1 |\ln x|^{s-1}\,dx.$$

Consequently, if $n \in \mathbb{N}$, then $\int_0^1 |\ln|^n\,dx = n!$.

(c) With an appropriate substitution, show that if $s > -1$, then

$$\int_0^1 |\ln x|^s\,dx = \lim_{\substack{a\to 0+\\ b\to\infty}} \int_a^b y^s e^{-y}\,dy =: \int_0^\infty y^s e^{-y}\,dy,$$

which is the Gamma function evaluated at the point $s+1$.

13.P (a) $\displaystyle\int_0^1 \sqrt{x}\ln x\,dx = -\tfrac{4}{9}$, (b) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}\ln x}$.

13.Q (a) $\displaystyle\int_0^2 \frac{\ln x\,dx}{\sqrt{x}} = -4$, (b) $\displaystyle\int_0^1 \frac{dx}{x\ln x}$.

•13.R For what positive values of r, t is the integral $\displaystyle\int_0^1 \frac{dx}{x^r|\ln x|^t}$ convergent?

13.S If $t > 0$, show that $\displaystyle\int_0^1 x^s\sin(\pi/x^t)\,dx$ exists if and only if $s+t > -1$, in which case it equals $\displaystyle(1/t)\int_0^1 y^{(s-t+1)/t}\sin(\pi/y)\,dy$.

13.T When the integrand involves $\sin x$ and $\cos x$ and more elementary substitutions fail, the substitution $y := \Phi(x) = \tan\frac{1}{2}x$ can often be used. If $[a,b] \subset (-\pi,\pi)$, show that $x \mapsto T(\sin x, \cos x)$ belongs to $\mathcal{R}^*([a,b])$ if and only if $y \mapsto T\big(\frac{2y}{1+y^2}, \frac{1-y^2}{1+y^2}\big)\cdot\frac{2}{1+y^2}$ belongs to $\mathcal{R}^*([\Phi(a),\Phi(b)])$, in which case we have

$$\int_{\Phi(a)}^{\Phi(b)} T(\sin x,\cos x)\,dx = \int_a^b T\Big(\frac{2y}{1+y^2}, \frac{1-y^2}{1+y^2}\Big)\cdot\frac{2}{1+y^2}\,dy.$$

13.U Use the substitution in the preceding exercise to treat the following:

(a) $\displaystyle\int_0^{\pi} \frac{dx}{1+\sin x} = 2.$ (b) $\displaystyle\int_0^{\pi/2} \frac{dx}{1-\sin x}.$

13.V Evaluate the following integrals exactly.

(a) $\displaystyle\int_0^{\pi/2} \frac{dx}{2+\cos x} \approx 0.605.$

(b) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sin x+\cos x} \approx 0.693.$

13.W Let $f \in \mathcal{L}([0,1])$ be bounded.

(a) Show that $x \overset{\bullet}{\mapsto} f(\sin x)$ belongs to $\mathcal{L}([0,\pi])$.

(b) Show that $\displaystyle\int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\pi/2} f(\sin x)\,dx.$

(c) Show that $\displaystyle\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx.$

13.X Use the preceding exercise to evaluate $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx \approx 2.467$ exactly.

13.Y Use the substitution $x = \tan y$ and other manipulations to evaluate the integral $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx \approx 0.272$ exactly.

Absolute Continuity

In Definition 7.3 we defined what it means for a function $F : [a, b] \to \mathbb{R}$ to have **bounded variation** on $I := [a, b]$. We denoted the variation of F by $\text{Var}(F; I)$ and the collection of all functions having bounded variation on I by $BV(I)$. It was seen in Exercise 7.E that linear combinations and pointwise products of functions in $BV(I)$ also belong to $BV(I)$. Also, it was noted in Exercise 7.J that a function is in $BV(I)$ if and only if it is the difference of two increasing functions. For that reason, in establishing results about functions in $BV(I)$, it is frequently useful to consider the case of increasing functions.

The following basic theorem was proved by Henri Lebesgue in 1904. A detailed proof of it is given in Appendix E.

• **14.1 Lebesgue's Differentiation Theorem.** *If $F \in BV(I)$, then there exists a null set $Z \subset I$ such that the derivative $F'(x)$ exists for all $x \in I - Z$.*

In other words, a function F in $BV(I)$ is differentiable a.e. on I. However, the converse assertion is not true; indeed, it was seen in Example 7.6(c) that the function $G(x) := x^2 \cos(\pi/x^2)$ for $x \in (0, 1]$ and $G(0) := 0$ is differentiable at every point of I, but $G \notin BV(I)$.

The question arises as to whether the derivative F' of a function $F \in BV(I)$ is integrable. The answer is: "Yes — but ... " In fact, the derivative F' always belongs to $\mathcal{L}(I)$; however, the integral of F' over $[a, x]$ may not yield $F|_a^x$.

• **14.2 Theorem.** *If $F \in BV(I)$, then $F' \in \mathcal{L}(I)$. In addition, if F is increasing on $I := [a, b]$, then*

$$\int_a^x F' \leq F(x) - F(a) \qquad \text{for} \quad x \in I. \tag{14.α}$$

Proof. It is enough to treat the case where F is increasing on I.

We extend F to $[a, b+1]$ by setting $F(x) := F(b)$ for $x \in (b, b+1]$. If $n \in \mathbb{N}$, we define

$$f_n(x) := n\Big[F(x + 1/n) - F(x)\Big] \qquad \text{for} \quad x \in I.$$

Since F is increasing, it is measurable and f_n is also measurable and $f_n(x) \geq 0$ for $x \in I$. Also Lebesgue's Theorem 14.1 implies that $\lim_n f_n(x)$ exists a.e. and equals $F'(x)$. Moreover, we have that

$$\int_a^b f_n = n\Big[\int_a^b F(t + 1/n)\,dt - \int_a^b F(t)\,dt\Big]. \tag{14.β}$$

Theorem 3.21(a) or the First Substitution Theorem 13.1 imply that

$$\int_a^b F(t+1/n)\,dt = \int_{a+1/n}^{b+1/n} F(t)\,dt = \int_{a+1/n}^{b} F(t)\,dt + \int_b^{b+1/n} F(t)\,dt.$$

Consequently, equation (14.β) becomes

$$\int_a^b f_n = n\Big[\int_b^{b+1/n} F - \int_a^{a+1/n} F\Big].$$

But, $F(x) = F(b)$ for $x \in [b, b+1/n]$ and $F(a) \leq F(x)$ for $x \in [a, a+1/n]$, whence

$$\int_a^b f_n \leq n\Big[F(b)\cdot(1/n) - F(a)\cdot(1/n)\Big] = F(b) - F(a).$$

Since $f_n \geq 0$, Fatou's Lemma 8.7 implies that $F' \in \mathcal{L}(I)$ and

$$0 \leq \int_a^b F' \leq F(b) - F(a),$$

which is (14.α) with $x = b$. If we replace the interval $[a, b]$ by the interval $[a, x]$ for $x \in [a, b]$, we obtain (14.α). Q.E.D.

14.3 Examples. (a) The Cantor-Lebesgue singular function $\Lambda : [0,1] \to \mathbb{R}$ in Theorem 4.17 is increasing and so belongs to $BV([0,1])$. Moreover,

$\Lambda'(x) = 0$ when $x \in [0,1] - \Gamma$, where Γ is the Cantor set (see 4.15). Here we have $0 = \int_0^x \Lambda' < \Lambda(x) - \Lambda(0)$ when $x \in (0,1]$. Note also that Λ is continuous on $[0,1]$.

(b) There exists a *strictly increasing* continuous function f on $[0,1]$ with $f'(x) = 0$ a.e. (See [He-St; pp. 278–282] or [St; p. 210].)

Absolute Continuity

We will now define the important class of functions that are "absolutely continuous". According to Hawkins [Hw-1; p. 142 ff.], Axel Harnack (1851–1888) called attention to this notion as early as the 1880s, and this property was also considered by other mathematicians near the end of the eighteenth century. However, the name was introduced in 1905 by Giuseppe Vitali (1875–1932). The reader will recall that we used the phrase "absolute continuity property" in connection with 10.10(a). The next definition is essentially this property for sets that are the union of a *subpartition* of I (that is, a finite collection of nonoverlapping closed intervals in I).

• **14.4 Definition.** Let $I := [a,b]$ and let $F : I \to \mathbb{R}$. We say that F is **absolutely continuous** on I and write $F \in AC(I)$ if, for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $\{[u_j, v_j]\}_{j=1}^s$ is any *subpartition* of I such that

$$(14.\gamma) \qquad \sum_{j=1}^{s} |v_j - u_j| \le \eta_\varepsilon, \qquad \text{then} \qquad \sum_{j=1}^{s} |F(v_j) - F(u_j)| \le \varepsilon.$$

Note. It will be seen in an exercise that it is essential that the subintervals $\{[u_j, v_j]\}_{j=1}^s$ be nonoverlapping.

We now will establish some important properties of the class $AC(I)$.

◇ **14.5 Theorem.** *Let $I := [a,b]$ be a compact interval.*

(a) *If $F \in AC(I)$, then F is (uniformly) continuous on I.*

(b) *If $F \in AC(I)$, then $F \in BV(I)$.*

(c) *If $F, G \in AC(I)$ and $c \in \mathbb{R}$, then the functions*

$$cF, \quad |F|, \quad F+G, \quad F-G, \quad \text{and} \quad F \cdot G$$

also belong to $AC(I)$.

Proof. (a) If $\xi \in I$, given $\varepsilon > 0$, let $\eta_\varepsilon > 0$ be as in Definition 14.4. If $s \in I$ and $|s - \xi| \le \eta_\varepsilon$, it follows that $|F(s) - F(\xi)| \le \varepsilon$, so F is continuous

at an arbitrary point $\xi \in I$. Since η_ε does not depend on ξ, the function F is uniformly continuous on I.

(b) Let $\eta_1 > 0$ be as in Definition 14.4 corresponding to $\varepsilon = 1$. If J is any subinterval of I with length $l(J) \leq \eta_1$, then $\mathrm{Var}(F; J) \leq 1$. Now let $r \in \mathbb{N}$ with $r > (b-a)/\eta_1$, and divide I into r nonoverlapping intervals $I_1, \cdots, I_r$ with length $(b-a)/r < \eta_1$. Exercises 7.G and 7.H imply that

$$\mathrm{Var}(F; I) = \sum_{k=1}^{r} \mathrm{Var}(F; I_k) \leq r.$$

(c) It is trivial that a constant multiple of $F \in AC(I)$ belongs to $AC(I)$. Moreover, the inequality $\big||F(v_j)| - |F(u_j)|\big| \leq |F(v_j) - F(u_j)|$, implies that $|F| \in AC(I)$. Further, since

$$\big|(F \pm G)(v_j) - (F \pm G)(u_j)\big| \leq \big|F(v_j) - F(u_j)\big| + \big|G(v_j) - G(u_j)\big|,$$

we readily conclude that $F \pm G \in AC(I)$.

If $|F(x)|, |G(x)| \leq M$ for $x \in I$, then since

$$\begin{aligned}\big|(FG)(v_j) - (FG)(u_j)\big| &\leq \big|F(v_j) - F(u_j)\big| \cdot |G(v_j)| \\ &\quad + |F(u_j)| \cdot \big|G(v_j) - G(u_j)\big| \\ &\leq M\Big[\big|F(v_j) - F(u_j)\big| + \big|G(v_j) - G(u_j)\big|\Big],\end{aligned}$$

it is seen that

$$\sum_{j=1}^{s} \big|FG(v_j) - FG(u_j)\big| \leq M\Big[\sum_{j=1}^{s} \big|F(v_j) - F(u_j)\big| + \sum_{j=1}^{s} \big|G(v_j) - G(u_j)\big|\Big],$$

whence it follows that $FG \in AC(I)$. **Q.E.D.**

The reader should recall the notion of negligible variation introduced in Definition 5.11. We now show that a function in $AC(I)$ belongs to $NV_I(Z)$ for *any* null set $Z \subset I$.

◇ **14.6 Lemma.** *If $F \in AC(I)$ and $Z \subset I$ is a null set, then $F \in NV_I(Z)$.*

Proof. We will construct a gauge δ_ε on Z as in Definition 5.11. Given $\varepsilon > 0$, let $\eta_\varepsilon > 0$ be as in Definition 14.4. Since Z is a null set, there exists a sequence $\{J_k\}_{k=1}^{\infty}$ of open intervals such that $Z \subseteq \bigcup_{k=1}^{\infty} J_k$ and $\sum_{k=1}^{\infty} l(J_k) \leq \eta_\varepsilon$. If $t \in Z$, let $k(t)$ be the smallest index k such that $t \in J_k$ and choose $\delta_\varepsilon(t) > 0$ such that $[t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)] \subset J_{k(t)}$. Now let

$\dot{\mathcal{P}}_0 := \{(I_j, t_j)\}_{j=1}^s$ be a (δ_ε, Z)-fine subpartition of I. Then, for $j = 1, \cdots, s$, we have

$$t_j \in Z \qquad \text{and} \qquad I_j \subseteq [t_j - \delta_\varepsilon(t_j), t_j + \delta_\varepsilon(t_j)] \subseteq J_{k(t_j)}.$$

Thus, for each $k \in \mathbb{N}$, the intervals in $\dot{\mathcal{P}}_0$ with tags in $Z \cap J_k$ have total length $\leq l(J_k)$. Further, the sum of the lengths of the intervals $I_j = [u_j, v_j]$ in $\dot{\mathcal{P}}_0$ is $\leq \eta_\varepsilon$ so that $\sum_{j=1}^s |F(v_j) - F(u_j)| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies that F belongs to $NV_I(Z)$. Q.E.D.

It has already been noted (see Theorem 4.17) that the Cantor-Lebesgue function Λ is continuous and in $BV(I)$, where $I := [0, 1]$. However (see Exercise 5.P), it is not in $NV_I(\Gamma)$, where Γ is the Cantor set. Thus Λ is not in $AC(I)$.

Indefinite Integrals of Functions in $\mathcal{L}(I)$

It was seen in the Characterization Theorem 5.12 that F is an indefinite integral of a function in $\mathcal{R}^*(I)$ if and only if F is differentiable a.e. and has negligible variation on its set of nondifferentiability. We now show that F is an indefinite integral of a function in $\mathcal{L}(I)$ if and only if F is in $AC(I)$, or if and only if F belongs to $BV(I)$ and has negligible variation on its set of nondifferentiability.

This result is sometimes called a "descriptive characterization" (or "descriptive definition") of the Lebesgue integral, since it gives a necessary and sufficient condition for a function to be the indefinite integral of a function in $\mathcal{L}(I)$. In contrast, the process of evaluating the integral as limits of sums is referred to as the "constructive definition" of the integral.

$\diamond$ **14.7 Characterization Theorem.** *Let $I := [a, b]$ and let $F : I \to \mathbb{R}$. Then the following assertions are equivalent:*

(a) *F is an indefinite integral of a function in $\mathcal{L}(I)$.*

(b) *$F \in AC(I)$.*

(c) *$F \in BV(I)$ and if $Z \subset I$ is the set where the derivative $F'(x)$ does not exist, then $F \in NV_I(Z)$.*

Proof. (a) $\Rightarrow$ (b) Suppose that $F(x) := C + \int_a^x f$ for some $C \in \mathbb{R}$ and some $f \in \mathcal{L}(I)$. By Theorem 10.10(a), given $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $E \subseteq I$ is a measurable set with measure $|E| \leq \eta_\varepsilon$, then $\int_E |f| \leq \varepsilon$. It follows that if $\{I_j\}_{j=1}^s = \{[u_j, v_j]\}_{j=1}^s$ is any subpartition of I such that $\sum_{j=1}^s |v_j - u_j| \leq \eta_\varepsilon$ and if $E := \bigcup_{j=1}^s I_j$, then $|E| \leq \eta_\varepsilon$. Therefore we have

$$(14.\delta) \qquad \sum_{j=1}^{s} |F(v_j) - F(u_j)| = \sum_{j=1}^{s} \Big| \int_{u_j}^{v_j} f \Big| \leq \sum_{j=1}^{s} \int_{u_j}^{v_j} |f| = \int_E |f| \leq \varepsilon.$$

Therefore $F \in AC(I)$.

(b) $\Rightarrow$ (c) If $F \in AC(I)$, then we have seen in Theorem 14.5 that F is continuous on I and $F \in BV(I)$. The Lebesgue Differentiation Theorem 14.1 implies that the set $Z \subseteq I$ where $F'(x)$ does not exist is a null set. Lemma 14.6 now implies that $F \in NV_I(Z)$.

(c) $\Rightarrow$ (a) Let Z be the set of nondifferentiability of the function F and let $f(x) := F'(x)$ for $x \in I - Z$ and $f(x) := 0$ for $x \in Z$. The Characterization Theorem 5.12 implies that $f \in \mathcal{R}^*(I)$ and $F(x) - F(a) = \int_a^x f$ for all $x \in I$, so that F is an indefinite integral of f. Q.E.D.

The next corollary complements Lemma 14.6.

$\diamond$ **14.8 Corollary.** *Let $I := [a, b]$ and let $F : I \to \mathbb{R}$. Then $F \in AC(I)$ if and only if $F \in BV(I)$ and $F \in NV_I(Z)$ for every null set $Z \subset I$.*

Proof. ($\Rightarrow$) This is a consequence of Theorem 14.5(b) and Lemma 14.6.

($\Leftarrow$) This follows from Theorems 14.1 and 14.7. Q.E.D.

The next result shows that, for an increasing function F, the equality holds in (14.α) with $x = b$ if and only if $F \in AC(I)$.

$\diamond$ **14.9 Corollary.** *Let $F : I \to \mathbb{R}$ be increasing on $I := [a, b]$. Then $F \in AC(I)$ if and only if*

$$\int_a^b F' = F(b) - F(a). \tag{14.ε}$$

Proof. ($\Rightarrow$) If $F \in AC(I)$, then Theorem 14.7 implies that there exist $C \in \mathbb{R}$ and $f \in \mathcal{L}(I)$ such that

$$F(x) = C + \int_a^x f \qquad \text{for} \quad x \in I.$$

Therefore $C = F(a)$ and $F(b) = F(a) + \int_a^b f$. The Differentiation Theorem 5.9 implies that $f = F'$ a.e. Therefore $\int_a^b f = \int_a^b F'$ and so (14.ε) holds.

($\Leftarrow$) Suppose (14.ε) holds and let $\xi \in (a, b)$ be arbitrary. We claim that

$$F(\xi) - F(a) = \int_a^\xi F'. \tag{14.ζ}$$

If not, then Theorem 14.2 applied to the interval $[a, \xi]$ implies that

$$F(\xi) - F(a) > \int_a^\xi F'.$$

If we apply Theorem 14.2 to the interval $[\xi, b]$, we infer that

$$F(b) - F(\xi) \geq \int_\xi^b F'.$$

If we add the last two inequalities, we conclude that $F(b) - F(a) > \int_a^b F'$, which contradicts (14.$\varepsilon$). Therefore the equation (14.ζ) holds for all $\xi \in [a, b]$, so that F is an indefinite integral of $F' \in \mathcal{L}([a, b])$. Therefore, Theorem 14.7 implies that $F \in AC(I)$. Q.E.D.

Note. It was seen in Example 4.18(a) that if Λ is the Cantor-Lebesgue function, then $0 = \int_0^1 \Lambda' < \Lambda(1) - \Lambda(0) = 1$. The preceding corollary provides another proof of the fact that, although Λ is continuous and in $BV([a, b])$, it is not in $AC([0, 1])$.

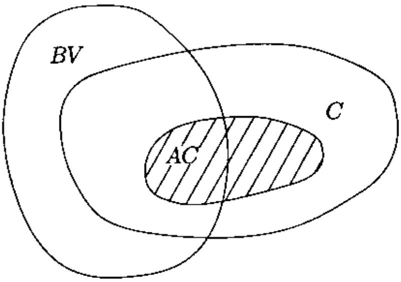

Diagram 14.1 The shaded set is $\int \mathcal{R}^*$.

Diagram 14.1 summarizes the inclusions between important classes of functions on a compact interval. In it, we denote the set of indefinite integrals of functions in $\mathcal{R}^*$ by $\int \mathcal{R}^*$ and shade this set. We denote the set of indefinite integrals of functions in $\mathcal{L}$ by $\int \mathcal{L}$, the set of functions having bounded variation by BV, the set of absolutely continuous functions by AC, and the set of continuous functions by C. In view of Theorems 7.5 and 14.7, we have $AC = \int \mathcal{L} = BV \cap \int \mathcal{R}^*$.

Singular Functions

We now introduce an important subset of the class of functions that are differentiable a.e.

• **14.10 Definition.** A function $F : I \to \mathbb{R}$ is said to be **singular on I** if its derivative $F'(x) = 0$ for a.e. $x \in I$.

The Cantor-Lebesgue singular function Λ considered in Theorem 4.17 is singular in the sense just defined. Although Λ belongs to $BV([0,1])$, we have seen that it does not belong to $AC([0,1])$ and is not constant.

The next result is a basic one concerning singular functions and we will give two proofs of it. The first one is short, but depends on a number of deep theorems. The second proof, due to Gordon [G-4; pp. 116–117] is longer, but entirely elementary.

◇ **14.11 Theorem.** *If $F \in AC(I)$ is singular on I, then F is a constant function.*

First Proof. Theorem 14.7 implies that $F(x) = F(a) + \int_a^x F'$ for all $x \in I$. Since $F'(x) = 0$ a.e., we conclude that $F(x) = F(a)$ for all $x \in I$.

Second Proof. Since $F \in AC(I)$, given $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ as in Definition 14.4. Now let Z be the set of all points $x \in I$ for which either $F'(x)$ does not exist or $F'(x) \neq 0$. Since Z is a null set, there exists a sequence $(I_k)_{k=1}^{\infty}$ of open intervals containing Z such that $\sum_{k=1}^{\infty} l(I_k) \leq \eta_\varepsilon$.

We now define a gauge δ_ε on $I := [a,b]$ as follows:

(i) If $t \notin Z$, then $F'(t) = 0$ and the Straddle Lemma 4.4 implies that there exists $\delta_\varepsilon(t) > 0$ such that if $u, v \in I$ satisfy $t - \delta_\varepsilon(t) \leq u \leq t \leq v \leq t + \delta_\varepsilon(t)$, then $|F(v) - F(u)| \leq \varepsilon(v - u)$.

(ii) If $t \in Z$, we let $k(t)$ be the smallest index k such that $t \in I_k$ and choose $\delta_\varepsilon(t) > 0$ so that $[t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)] \subset I_{k(t)}$.

Now let $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ be a δ_ε-fine partition of I and consider the sets of indices:

$$S_1 := \{i : t_i \notin Z\} \qquad \text{and} \qquad S_Z := \{i : t_i \in Z\}.$$

If $i \in S_1$, then we have $|F(x_i) - F(x_{i-1})| \leq \varepsilon |x_i - x_{i-1}|$. Further, if $i \in S_Z$, then $[x_{i-1}, x_i] \subset I_{k(t_i)}$, so that

$$\sum_{i \in S_Z} (x_i - x_{i-1}) \leq \sum_{k=1}^{\infty} l(I_k) \leq \eta_\varepsilon,$$

from which it follows that $\sum_{i \in S_Z} |F(x_i) - F(x_{i-1})| \leq \varepsilon$. Consequently, we have

$$\begin{aligned}
|F(b) - F(a)| &= \Big| \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \Big| \\
&\leq \sum_{i \in S_1} |F(x_i) - F(x_{i-1})| + \sum_{i \in S_Z} |F(x_i) - F(x_{i-1})| \\
&\leq \sum_{i \in S_1} \varepsilon(x_i - x_{i-1}) + \varepsilon \;\leq\; \varepsilon(b + a - 1).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $F(b) = F(a)$. But since the above argument applies to any subinterval $[a, x] \subseteq [a, b]$, we infer that $F(x) = F(a)$ for all $x \in [a, b]$. Q.E.D.

◇ **14.12 Lebesgue Decomposition Theorem.** *If $F \in BV(I)$, then F can be respresented as the sum*

$$F = F_a + F_s, \tag{14.η}$$

where $F_a \in AC(I)$ and $F_s \in BV(I)$ is singular on I. Moreover, this representation is unique up to a constant function.

Proof. We define F_a and F_s for $x \in I$ by

$$F_a(x) := \int_a^x F' \qquad \text{and} \qquad F_s(x) := F(x) - F_a(x).$$

Consequently, $F_s' = F' - F_a' = 0$ a.e. on I, so F_s is a singular function on I. Also, since F_a is the indefinite integral of $F' \in \mathcal{L}(I)$ with base point a, it follows from Theorem 14.7 that $F_a \in AC(I)$.

To establish the uniqueness, suppose that F also has the form $F = G_a + G_s$, where $G_a \in AC(I)$ and $G_s \in BV(I)$ is singular on I. Then

$$F_a - G_a = G_s - F_s$$

so that $F_a - G_a$ is both absolutely continuous and singular on I. Therefore, Theorem 14.11 implies that there exists a constant C such that $F_a = G_a + C$ and $F_s = G_s - C$. Q.E.D.

Mapping Properties of AC Functions

We will first show that functions in AC map null sets to null sets (and measurable sets to measurable sets). It will be seen in Theorem 14.15 that this property characterizes absolute continuity for a continuous function of bounded variation. In establishing these results, we will need a few facts proved in Section 18.

• **14.13 Theorem.** *Let $F \in AC(I)$ and let $Z \subset I := [a, b]$ be a null set. Then $F(Z)$ is a null set.*

Proof. Given $\varepsilon > 0$, let $\eta_\varepsilon > 0$ be as in Definition 14.4 and let $\{J_k\}_{k=1}^\infty$ be a countable collection of open intervals with

$$Z \subseteq \bigcup_{k=1}^\infty J_k \qquad \text{and} \qquad \sum_{k=1}^\infty l(J_k) \le \eta_\varepsilon.$$

Since the union $\bigcup_{j=1}^{\infty} J_k$ is an open set, it follows from [B-S; p. 315] that we may assume that the open intervals $J_k =: (u_k, v_k)$, $k \in \mathbb{N}$, are pairwise disjoint. For each $k \in \mathbb{N}$, the Maximum-minimum Theorem [B-S; p. 131] implies that there exist points a_k [respectively, b_k] in the compact interval $[u_k, v_k]$ where the restriction of F to this interval is minimized [respectively, maximized]. Therefore

$$(14.\theta) \qquad F(Z) \subseteq \bigcup_{k=1}^{\infty} F(J_k) \subseteq \bigcup_{k=1}^{\infty} [F(a_k), F(b_k)].$$

For each $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} |b_k - a_k| \leq \sum_{k=1}^{n} |v_k - u_k| \leq \eta_\varepsilon,$$

whence it follows that

$$\sum_{k=1}^{n} |F(b_k) - F(a_k)| \leq \varepsilon.$$

But, since $n \in \mathbb{N}$ is arbitrary, we conclude that

$$\sum_{k=1}^{\infty} |F(b_k) - F(a_k)| \leq \varepsilon.$$

Therefore, it follows from (14.θ) that the set $F(Z)$ is contained in the union of a countable collection of (closed) intervals with total length $\leq \varepsilon$. Consequently, $F(Z)$ is a null set. Q.E.D.

Remark. The condition that F sends null sets to null sets was called **Condition (N)** by Luzin (1915).

14.14 Theorem. *If $F \in AC(I)$ and if $E \subseteq I := [a, b]$ is a measurable set, then $F(E)$ is a measurable set.*

Proof. The proof is based on the fact (see Theorem 18.19) that if $E \subseteq I$ is a measurable set, then there exist a null set Z and a sequence $(K_n)_{n=1}^{\infty}$ of compact sets in I such that $E = Z \cup \bigcup_{n=1}^{\infty} K_n$. Since K_n is compact, then its image $F(K_n)$ is also compact and therefore (see Theorem 18.13) is measurable. By Theorem 14.13, the set $F(Z)$ is a null set and hence is measurable. Since

$$F(E) = F(Z) \cup \bigcup_{n=1}^{\infty} F(K_n),$$

Theorem 10.2(a) implies that $F(E)$ is measurable. Q.E.D.

We now state a theorem that was proved in 1925, independently, by Stefan Banach (1892–1945) and M. A. Zarecki. We will give a proof only for the case of an increasing function. The general case is treated in [He-St; pp. 288 ff., 303–304] and [G-3; p. 99].

14.15 Theorem (Banach-Zarecki). *If F is continuous on $I := [a,b]$, if F belongs to $BV(I)$ and if F sends null sets to null sets, then $F \in AC(I)$.*

Proof. Suppose that $F \notin AC(I)$, so that there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there is a subpartition of I into compact intervals $I_{kn} := [a_{kn}, b_{kn}]$, $k = 1, \cdots, s(n)$, with

$$\sum_{k=1}^{s(n)} (b_{kn} - a_{kn}) \leq 1/2^n, \tag{14.ι}$$

but such that

$$\sum_{k=1}^{s(n)} |F(b_{kn}) - F(a_{kn})| \geq \varepsilon. \tag{14.κ}$$

We now define $E_m := \bigcup_{n=m}^{\infty} \bigcup_{k=1}^{s(n)} I_{kn}$, so that E_m, being a countable union of compact intervals, is a measurable set. By inequality (10.ζ), we have

$$|E_m| \leq \sum_{n=m}^{\infty} \sum_{k=1}^{s(n)} (b_{kn} - a_{kn}) \leq \sum_{n=m}^{\infty} 1/2^n = 1/2^{m-1}.$$

Therefore, we conclude that $\lim_m |E_m| = 0$.

Since F is continuous, then $F(E_m)$ is also a countable union of compact intervals and so is measurable. Since $I \supseteq E_m \supseteq E_{m+1} \supseteq \cdots$, formula (10.$\delta$) implies that $E := \bigcap_{m=1}^{\infty} E_m$ is a null set. Therefore, by the hypothesis on F, the set $F(E)$ is a null set. But, since $F(I) \supseteq F(E_m) \supseteq F(E_{m+1}) \supseteq \cdots$, another application of (10.δ) implies that

$$\lim_{m \to \infty} |F(E_m)| = |F(E)| = 0. \tag{14.λ}$$

On the other hand, if F is assumed to be increasing, we have $F(I_{km}) = [F(a_{km}), F(b_{km})]$, whence it follows from (14.κ) that

$$|F(E_m)| \geq \sum_{k=1}^{s(m)} |F(b_{km}) - F(a_{km})| \geq \varepsilon.$$

which contradicts (14.λ). Therefore, we conclude that $F \in AC(I)$. Q.E.D.

Integration by Parts

We now give an Integration by Parts Theorem for Lebesgue integrable functions.

14.16 Theorem. *If $I := [a,b]$ and if $F, G \in AC(I)$, then*

$$FG\Big|_a^b = \int_a^b F'G + \int_a^b FG'. \tag{14.μ}$$

Proof. Since $F, G \in AC(I)$, there exists a null set $Z \subset I$ such that

$$(FG)'(x) = F'(x)G(x) + F(x)G'(x) \qquad \text{for} \quad x \in I - Z.$$

Since $F' \in \mathcal{L}(I)$ and G is continuous (and hence is bounded and measurable), then $F'G \in \mathcal{L}(I)$. Similarly, $FG' \in \mathcal{L}(I)$, so that

$$\int_a^b (F'G + FG') = \int_a^b F'G + \int_a^b FG'.$$

By Theorem 14.5(c), the product $FG \in AC(I)$, whence it follows from Theorem 14.7 that

$$\int_a^b (FG)' = FG\Big|_a^b.$$

Consequently (14.μ) follows from these formulas. Q.E.D.

The reader should compare this result with Theorems 12.2 and 12.3.

Substitution Theorems

We return again to the topic of the substitution theorems. In Theorem 13.5, we required that the substitution function Φ be strictly monotone and have a derivative except on a countable set. Here we will consider the case where Φ is absolutely continuous (and so may fail to have a derivative on a null set). However, we will now require that the functions $f = F'$ or $(f \circ \Phi) \cdot \varphi$ are Lebesgue integrable.

We will state a theorem due to James B. Serrin and Dale E. Varberg. We refer the reader to their paper [S-V] or to [St; p. 325] for a detailed proof.

14.17 Substitution Theorem. *Let $I := [a,b]$ and $J := [c,d]$ and let $\Phi \in AC(I)$, $\Phi(I) \subseteq J$ and $F \in AC(J)$. Further, let φ and f be the derivatives of Φ and F, respectively, when these derivatives exist, and equal to 0 elsewhere.*

Then the following statements are equivalent:

(a) *$F \circ \Phi$ belongs to $AC(I)$.*

(b) *$(f \circ \Phi) \cdot \varphi$ belongs to $\mathcal{L}(I)$ and*

$$\int_{\Phi(s)}^{\Phi(t)} f = \int_s^t (f \circ \Phi) \cdot \varphi \qquad \text{for all} \quad s, t \in I. \tag{14.ν}$$

14.18 Remarks. **(a)** It is quite possible that F and Φ are absolutely continuous, but their composition $F \circ \Phi$ is not absolutely continuous. For example, let $F(x) := \sqrt{x}$ for $x \in [0, 1]$, and $\Phi(x) := (x \sin x^{-1})^2$ for $x \in (0, 1]$ and $\Phi(0) := 0$. Then both F and Φ are absolutely continuous, but $F \circ \Phi$ does not have bounded variation on $[0, 1]$, so it is not absolutely continuous on this interval.

(b) If $\Phi \in AC(I)$ is monotone on I and if $F \in AC(J)$, then it is an exercise to show that $F \circ \Phi$ is also absolutely continuous. Therefore, (14.ν) holds when $f \in \mathcal{L}(J)$ and $\Phi \in AC(I)$ is monotone.

(c) If $\Phi \in AC(I)$ and if F satisfies a Lipschitz condition on I (that is, there exists a constant $K > 0$ such that $|F(x) - F(y)| \le K|x - y|$ for all $x, y \in I$), then it is an exercise to show that $F \circ \Phi$ is absolutely continuous on I.

(d) If $f \in \mathcal{L}(J)$ is bounded on J, then it is an exercise to show that its indefinite integral F satisfies a Lipschitz condition on J. Thus, (14.ν) holds when $f \in \mathcal{L}(J)$ is bounded and $\Phi \in AC(I)$.

Other instances in which (14.ν) holds are given in the next result, which is proved in [St; p. 326].

14.19 Theorem. *Let $\Phi \in AC(I)$ and $f \in \mathcal{L}(J)$. Then equation (14.ν) holds if any one of the following conditions hold:*

(a) *Φ is monotone on I.*

(b) *f is bounded on J.*

(c) *$(f \circ \Phi) \cdot \varphi$ belongs to $\mathcal{L}(I)$.*

It is worth noting that the hypothesis on $(f \circ \Phi) \cdot \varphi$ cannot be dropped in (c). Indeed, if F and Φ are as in Remark 14.18(a), then $f = F'$ belongs to $\mathcal{L}(J)$; however, the function $(f \circ \Phi) \cdot \varphi$ does not belong to $\mathcal{L}(I)$.

More Mapping Properties

We will now show that the indefinite integrals of functions in $\mathcal{R}^*(I)$, although not necessarily in $AC(I)$, have the same mapping property for null sets. This

theorem was known for the indefinite integral of a Denjoy-Perron integrable function, and hence also for a generalized Riemann integrable function. A direct proof was given by Xu Dongfu and Lu Shipan [X-L].

• **14.20 Theorem.** *If F is an indefinite integral of $f \in \mathcal{R}^*(I)$, then F sends null sets to null sets.*

Proof. Let $Z \subset I := [a,b]$ be a null set and let $f_Z(x) := f(x)$ for $x \in I - Z$ and $f_Z(x) := 0$ for $x \in Z$. Since $f \in \mathcal{R}^*(I)$ and Z is a null set, then $f_Z \in \mathcal{R}^*(I)$ and F is also the indefinite integral of f_Z.

Let $\varepsilon > 0$ be given and let δ_ε be a gauge as in Definition 1.7 for f_Z. The family $\mathcal{F} := \{[x-r, x+r] : x \in Z,\ 0 < r \le \delta_\varepsilon(x)\}$ is a Vitali covering of Z. Since F is continuous, the family $\mathcal{F}'$ consisting of the nondegenerate intervals in $\{F(J) : J \in \mathcal{F}\}$ is a Vitali covering of $F(Z)$. Thus, by the Vitali Covering Theorem 5.8, there exist disjoint intervals $F(I_i)$, $i = 1, \cdots, p$, from $\mathcal{F}'$ and closed intervals $\{J_i : i \ge p+1\}$ in $F(I)$ such that

$$(14.\xi) \qquad F(Z) \subseteq \bigcup_{i=1}^{p} F(I_i) \cup \bigcup_{i=p+1}^{\infty} J_i \quad \text{with} \quad \sum_{i=p+1}^{\infty} l(J_i) \le \varepsilon.$$

Since F is continuous, there exist points $a_i, b_i \in I_i$ such that $F(I_i) = [F(a_i), F(b_i)]$. Let x_i be the midpoint of $I_i \in \mathcal{F}$ so that $x_i \in Z$, and choose $c_i \in \{a_i, b_i\}$ so that

$$\left|F(x_i) - F(c_i)\right| \ge \tfrac{1}{2}\left|F(b_i) - F(a_i)\right| = \tfrac{1}{2} l\big(F(I_i)\big).$$

For $i = 1, \cdots, p$, we let J_i be the interval with endpoints x_i and c_i, tagged by x_i. Since the $F(I_i)$ are disjoint, so are the I_i and hence also the J_i. Thus $\dot{\mathcal{P}} := \{(J_i, x_i)\}_{i=1}^{p}$ is a δ_ε-fine subpartition of I. Since $f_Z(x_i) = 0$, it follows from Corollary 5.4 of the Saks-Henstock Lemma applied to f_Z and $\dot{\mathcal{P}}$ that

$$\sum_{i=1}^{p} l\big(F(I_i)\big) \le 2\sum_{i=1}^{p} \left|F(t_i) - F(c_i)\right| \le 4\varepsilon.$$

Using $(14.\xi)$, we deduce that $F(Z)$ is contained in the union of a countable collection of closed intervals with total length $\le 5\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $F(Z)$ is a null set. Q.E.D.

Generalized AC

In the study of the Denjoy and Perron integrals, extensive use is made of classes of functions having bounded variation or absolute continuity in a variety of generalized senses. The standard reference for this material is the

book of Saks [S-2; especially Chapters 7 and 8]. Recently, Gordon [G-3] has given a thorough and modern treatment of this theory, which he relates to the generalized Riemann (= Henstock-Kurzweil) integral that we have been discussing. Gordon's treatment is very lucid, but it is inevitably complicated by the multiplicity of these generalized classes and their interrelations. While we will not go into these matters, we do wish to state another characterization of the indefinite integrals of functions in $\mathcal{R}^*(I)$, where $I := [a,b]$.

The next definition is taken from [G-3; p. 146].

14.21 Definition. **(a)** If $E \subseteq I$, we say that $F : I \to \mathbb{R}$ belongs to the class $AC_\delta(E)$ if for every $\varepsilon > 0$, there exist $\eta_\varepsilon > 0$ and a gauge δ_ε on E such that if $\{([u_i, v_i], t_i)\}_{i=1}^s$ is a (δ_ε, E)-fine subpartition of E such that

$$\sum_{i=1}^{s} |v_i - u_i| \leq \eta_\varepsilon, \qquad \text{then} \qquad \sum_{i=1}^{s} |F(v_i) - F(u_i)| \leq \varepsilon.$$

(b) We say that F belongs to the class $ACG_\delta(I)$ if there exists a sequence $(E_n)_{n=1}^\infty$ of sets in I such that $I = \bigcup_{n=1}^\infty E_n$ and $F \in AC_\delta(E_n)$ for each $n \in \mathbb{N}$.

The reader will note that the definition of $AC_\delta(E)$ contains the ingredients of both of the classes $AC(I)$ and $NV_I(E)$. In this connection, Gordon [G-3; p. 147] proved the following theorem, which is clearly related to our Characterization Theorem 5.12.

14.22 Theorem. *A function f belongs to $\mathcal{R}^*(I)$ if and only if there exists a function $F \in ACG_\delta(I)$ such that $F' = f$ a.e.*

In order to establish that the generalized Riemann integral coincides with the Denjoy and Perron integrals, Gordon shows that the class $ACG_\delta(I)$ coincides with a class $ACG_*(I)$, which affords the simplest treatment of the Denjoy integral. For the record, we will give a definition of this class of functions.

First, we define the **oscillation** of a bounded function F on a set $A \subseteq I$ by

$$\omega_F(A) := \sup\{|F(x) - F(y)| : x, y \in A\}.$$

Next, if $E \subseteq I$, we say that $F \in AC_*(E)$ if for every $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ such that if $\{[u_i, v_i]\}_{i=1}^s$ is a collection of nonoverlapping intervals with endpoints in E and such that $\sum_{i=1}^s |v_i - u_i| \leq \eta_\varepsilon$, then $\sum_{i=1}^s \omega_F([u_i, v_i]) \leq \varepsilon$. Finally, we say that $F \in ACG_*(I)$ if F is continuous on I and there is

a countable collection $(E_n)_{n=1}^{\infty}$ of sets in I with $E = \bigcup_{i=1}^{\infty} E_n$ and $F \in AC_*(E_n)$ for $n \in \mathbb{N}$.

Exercises

14.A A function $F : I \to \mathbb{R}$ is said to satisfy a **Lipschitz condition** on $I := [a, b]$ if there exists a constant $M > 0$ such that $|F(x) - F(y)| \leq M|x - y|$ for all $x, y \in I$. Prove that such an F belongs to $AC(I)$.

14.B If $F : I \to \mathbb{R}$ satisfies $|F'(x)| \leq M$ for all $x \in I$, show that F satisfies a Lipschitz condition on I, and therefore belongs to $AC(I)$.

14.C Let $F : I \to \mathbb{R}$. Show that $F \in AC(I)$ if and only if for every $\varepsilon > 0$ there exists $\zeta_\varepsilon > 0$ such that if $\{[u_j, v_j]\}_{j=1}^{s}$ is any subpartition of I satisfying $\sum_{j=1}^{s}(v_j - u_j) \leq \zeta_\varepsilon$, then $\sum_{j=1}^{s}\big(F(v_j) - F(u_j)\big) \leq \varepsilon$.

14.D Let $F : I \to \mathbb{R}$. Show that $F \in AC(I)$ if and only if for every $\varepsilon > 0$ there exists $\theta_\varepsilon > 0$ such that if $\{[u_j, v_j]\}_{j=1}^{\infty}$ is any sequence of nonoverlapping intervals satisfying $\sum_{j=1}^{\infty} |v_j - u_j| \leq \theta_\varepsilon$, then $\sum_{j=1}^{\infty} |F(v_j) - F(u_j)| \leq \varepsilon$.

14.E (a) Let $S(x) := \sqrt{x}$ on the interval $I_c := [c, b]$ with $c > 0$. Show that S satisfies $|S(x) - S(y)| \leq (1/2\sqrt{c})|x - y|$ for all $x, y \in I_c$, so that $S \in AC(I_c)$.

(b) Show that S does *not* satisfy a Lipschitz condition on $I_0 := [0, b]$.

(c) Prove that $S \in AC(I_0)$. [Hint: Given $\varepsilon > 0$, let $c > 0$ be sufficiently small, and then break $\sum_{j=1}^{s} |S(v_j) - S(u_j)|$ into a sum of intervals in $[0, c]$ plus intervals in $[c, b]$.]

14.F Show that the function $S(x) := \sqrt{x}$ in $AC([0, 1])$ has the property that for any $\eta > 0$ there exists a finite collection of intervals $\{[u_j, v_j]\}_{j=1}^{s}$ satisfying $\sum_{j=1}^{s} |v_j - u_j| \leq \eta$ and $\sum_{j=1}^{s} |S(v_j) - S(u_j)| \geq 1$.

14.G Suppose F is a continuous monotone function on $[a, b]$. If $F \in AC([c, b])$ for every $c \in (a, b]$, prove that $F \in AC([a, b])$.

14.H If $F : I \to \mathbb{R}$ is a continuous function in $BV(I)$ and $F'(x)$ exists except on a countable set, prove that $F \in AC(I)$.

14.I Show that $G(x) := x^r$ for $x \in (0,1]$ and $G(0) := 0$ belongs to $AC([0,1])$ for all $r > 0$.

14.J (a) Let $I := [0,1]$. Give an example of $F : I \to \mathbb{R}$ such that $F \notin BV(I)$ but $|F| \in AC(I)$.
(b) Give an example of $G : I \to \mathbb{R}$ such that $G \in BV(I)$, $|G| \in AC(I)$, but $G \notin AC(I)$.

14.K If $F \in AC(I)$ and if $F(x) > \alpha > 0$ for all $x \in I$, show that $1/F \in AC(I)$.

14.L If $F \in AC(I)$, show that $\mathrm{Var}(F; [a,x]) = \int_a^x |F'|$ for $x \in I$.

14.M Let $s > 0$ and $H(x) := x^r \sin(1/x^s)$ for $x \in (0,1]$ and $H(0) := 0$. Prove that H belongs to $AC([0,1])$ if and only if $s < r$.

14.N Show that $K(x) := x^2|\sin(1/x)|$ for $x \in (0,1]$ and $K(0) := 0$ belongs to $AC([0,1])$ but that $\sqrt{K}$ does not belong to $BV([0,1])$.

14.O Suppose that $F \in AC(I)$, $G \in AC(J)$ and $G(J) \subseteq I$.
(a) Prove that if F satisfies a Lipschitz condition, then $F \circ G \in AC(J)$.
(b) Prove that if G is monotone on J, then $F \circ G \in AC(J)$.

14.P Show that neither of the hypotheses (that F satisfies a Lipschitz condition, or that G is monotone) can be dropped in Exercise 14.O.

14.Q (a) If $F \in AC(I)$, show that $V(x) := \mathrm{Var}(F; [a,x])$ belongs to $AC(I)$.
(b) Show that $F \in AC(I)$ if and only if it is the difference of two increasing functions in $AC(I)$.

14.R Recall the notion of the oscillation of a bounded function $F : I \to \mathbb{R}$ that was given after Theorem 14.22.
(a) Show that if $A \subseteq I$, then $\omega_F(A) = \sup_{x\in A} F(x) - \inf_{x\in A} F(x)$.
(b) Show that if $A \subseteq B \subseteq I$, then $\omega_F(A) \le \omega_F(B) \le 2\sup_{x\in I} |F(x)|$.
(c) If $c \in I$ and $r > 0$, let $B(c;r) := \{x \in I : |x - c| < r\}$. We define the **oscillation of** F **at the point** c to be

$$\omega_F(c) := \inf_{r>0}\big[\omega_F(B(c;r))\big] = \lim_{r\to 0+}\big[\omega_F(B(c;r))\big].$$

Show that the last equality holds.
(d) Show that F is continuous at c if and only if $\omega_F(c) = 0$.

14.S Show that $F \in AC(I)$ if and only if for every $\varepsilon > 0$ there exists $\zeta_\varepsilon > 0$ such that if $\{[u_j, v_j]\}_{j=1}^s$ is any subpartition of I such that

$$\sum_{j=1}^{s} |v_j - u_j| \leq \zeta_\varepsilon, \qquad \text{then} \qquad \sum_{j=1}^{s} \omega_F([u_j, v_j]) \leq \varepsilon.$$

[Note that this is *not* the same as requiring that $F \in AC_*(E)$, when E is a proper subset of I.]

14.T Suppose that $F : I \to \mathbb{R}$ is continuous and has a derivative c.e. on I, and that $Z \subset I$ is a null set. Prove that $F(Z)$ is a null set.

14.U Suppose that (F_n) is a sequence in $AC(I)$ with $F_n(x) \to F(x)$ for all $x \in I$.

(a) Show that F need not belong to $AC(I)$.

(b) Show that F need not belong to $AC(I)$ even when the convergence is uniform on I.

(c) Suppose, in addition, that $\mathrm{Var}(F_m - F_n; I) \to 0$ as $m, n \to \infty$. Show that $F \in AC(I)$.

14.V A sequence (F_n) in $AC(I)$ is said to be **uniformly absolutely continuous** on I if, given $\varepsilon > 0$ there exists a constant η_ε as in Definition 14.4 that can be used for all F_n.

(a) Show that any finite collection of functions in $AC(I)$ is uniformly absolutely continuous.

(b) If $\mathrm{Var}(F_m - F_n; I) \to 0$ as $m, n \to \infty$, show that the sequence (F_n) is uniformly absolutely continuous on I.

(c) If $f_n \in \mathcal{L}(I)$ and $\int_I |f_m - f_n| \to 0$ as $m, n \to \infty$, show that the sequence of indefinite integrals $F_n(x) := \int_a^x f_n$ for $x \in I$ is uniformly absolutely continuous on I.

Part 2

Integration on Infinite Intervals

Introduction to Part 2

In Part 1, we have discussed the integration of real-valued functions defined on compact intervals $[a, b] \subset \mathbb{R}$ and have developed a considerable theory for these functions. However, in applications we often want to integrate real-valued functions defined on unbounded intervals, such as

$$[a, \infty), \qquad (-\infty, b], \quad \text{or} \quad (-\infty, \infty).$$

In elementary calculus courses, the standard approach is to define the integral over $[a, \infty)$ as an "improper integral"; that is, to define it as the limit:

$$\int_a^\infty f := \lim_{\gamma \to \infty} \int_a^\gamma f.$$

Even the Lebesgue integral does not escape this extension procedure, since it is restricted to absolutely integrable functions and can handle nonabsolutely integrable functions only by this limiting process. As we have seen in discussing Hake's Theorem, the generalized Riemann integral does not suffer from this defect. In the remaining sections of this book, we will discuss the generalized Riemann integrable functions defined on infinite intervals, obtaining the Lebesgue integrable ones as a by-product. It will be seen that most of the work in developing this integral has already been done.

In defining the generalized Riemann integral of a function f on $[a, \infty)$ to $\mathbb{R}$, we are immediately confronted with a problem. We note that if

$$(15.\alpha) \qquad \dot{\mathcal{Q}} := \Big\{ \big([x_0, x_1], t_1\big), \cdots, \big([x_{n-1}, x_n], t_n\big), \big([x_n, x_{n+1}), t_{n+1}\big) \Big\}$$

is a tagged partition of $[a, \infty)$, then $x_0 = a$ and $x_{n+1} = \infty$ and so the Riemann sum of f corresponding to $\dot{\mathcal{Q}}$ has the form

$$(15.\beta) \qquad f(t_1)(x_1 - x_0) + \cdots + f(t_n)(x_n - x_{n-1}) + f(t_{n+1})(\infty - x_n).$$

The difficulty is that the final term $f(t_{n+1})(\infty - x_n)$ is not meaningful in $\mathbb{R}$; consequently, we either have to assign a meaning to this term, or we have to suppress it. Since we can think of no useful way of assigning a nonzero meaning to this term, we elect to suppress it. There seem to be two ways of doing this: (i) define the Riemann sum to contain only the first n terms, or (ii) have a procedure that will enable us to deal with the endpoints $\pm\infty$ of the infinite intervals when they occur in caclulations in such a way that we eliminate the final term in (15.β).

In Chapter 10 of the Third Edition of [B-S], we chose to use Method (i) and to avoid the use of extended real numbers, since unsophisticated students often make incorrect uses of the symbols $-\infty$ and ∞.

Method (i)

In [B-S], for integrals over $[a,\infty)$, we dealt with tagged **subpartitions**:

$$\dot{\mathcal{P}} := \Big\{\big([x_0,x_1],t_1\big),\cdots,\big([x_{n-1},x_n],t_n\big)\Big\} \tag{15.γ}$$

that are **full** in the sense that they lack only the final interval $[x_n,\infty)$ to cover $[a,\infty)$; that is, they are such that

$$[a,\infty) = \bigcup_{i=1}^{n}[x_{i-1},x_i] \cup [x_n,\infty). \tag{15.δ}$$

The Riemann sum of f corresponding to this subpartition $\dot{\mathcal{P}}$ is then taken to be

$$S(f;\dot{\mathcal{P}}) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}),$$

all terms of which are meaningful. We defined a **gauge** on $[a,\infty]$ to be an ordered pair (δ, d^*) consisting of a strictly positive function δ on $[a,\infty)$, and a number $d^* > 0$, which can be considered to be $\delta(\infty)$. We define the full subpartition $\dot{\mathcal{P}}$ to be (δ,d^*)**-fine** in case that

$$[x_{i-1},x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \quad \text{for} \quad i = 1,\cdots,n, \tag{15.ε}$$

and that

$$[x_n,\infty) \subseteq [1/d^*,\infty), \tag{15.ζ'}$$

or, equivalently, that

$$1/d^* \le x_n. \tag{15.ζ''}$$

We defined $C \in \mathbb{R}$ to be the generalized Riemann integral of $f : [a, \infty) \to \mathbb{R}$ if: for every $\varepsilon > 0$ there exists a gauge (δ, d^*) on $[a, \infty)$ such that if $\dot{\mathcal{P}}$ is any (δ, d^*)-fine full subpartition of $[a, \infty)$, then $|S(f; \dot{\mathcal{P}}) - C| \le \varepsilon$.

It is easy to modify this procedure for functions defined on other types of infinite intervals. Thus, on $(-\infty, b]$ the tagged subpartition $\dot{\mathcal{P}}$ given in (15.γ) is said to be **full** in case

$$(-\infty, b] = (-\infty, x_0] \cup \bigcup_{i=1}^{n} [x_{i-1}, x_i],$$

and a **gauge** on $(-\infty, b]$ is an ordered pair (d_*, δ) consisting of a number $d_* > 0$ and a strictly positive function δ on $(-\infty, b]$. Here the full subpartition $\dot{\mathcal{P}}$ is (d_*, δ)**-fine** in case (15.ε) holds and

$$(15.\eta') \qquad (-\infty, x_0] \subseteq (-\infty, -1/d_*],$$

or, equivalently,

$$(15.\eta'') \qquad x_0 \le -1/d_*.$$

It will now be clear how to modify this procedure for functions defined on the interval $(-\infty, \infty)$. Here both of the infinite intervals $(-\infty, x_0]$ and $[x_n, \infty)$ are needed to augment a full subpartition to obtain all of $(-\infty, \infty)$, and a gauge consists of two strictly positive numbers d_*, d^* and a strictly positive function δ on $(-\infty, \infty)$ (or, a strictly positive function δ defined on $[-\infty, \infty]$ with $\delta(-\infty) = d_*$ and $\delta(\infty) = d^*$).

Method (ii)

In this book we will use the second procedure. We do this for two reasons; the more pursuasive being that the other expositions of the generalized Riemann integral use the second method and we do not want to make it difficult for the reader to consult these treatments. Another reason is that there is a long-standing practice in measure theory to use the convention

$$(15.\theta) \qquad 0 \cdot (\pm\infty) = 0 = (\pm\infty) \cdot 0.$$

It is also our opinion that, once students have been sheltered from inappropriate use of the symbols $\pm\infty$ in elementary courses, they develop enough sophistication to use them when they are useful.

Thus, we will implement Method (ii) by defining the functions to be integrated to take on the value 0 at the points ∞ or $-\infty$ and we will use

the convention (15.θ). It will be seen that, in fact, this procedure leads to very similar considerations as Method (i) outlined above.

Remark. It will be noticed that the symbols $\pm\infty$ will occur only as points in the domain of the functions; all of our functions will take on only real (or possibly complex) values.

Subsequent Sections

In Section 16, we will introduce the generalized Riemann integral of a real-valued function defined on an interval of the form $[a, \infty), (-\infty, b]$, or $(-\infty, \infty)$, using what we have called Method (ii). It will be seen that the theory is quite parallel to that for compact intervals. In fact, most of the theory extends with little or no modification; however, there are a few changes that need to be considered. The theorem of Hake will provide a corner-stone for this development, for it is this theorem that asserts that there is no such thing as an "improper integral". In this section, we will also give analogues of the classical tests due to Abel, Chartier-Dirichlet, and Du Bois-Reymond for the integrability of the product of two functions.

Since we have elected to develop the integral first for a compact interval, when we turn to infinite intervals, we have three choices: (j) go through the entire process again, making the requisite modifications for the noncompact case and writing out the details, (jj) re-examine the earlier theory and make only the changes that are needed, or (jjj) just say *mutatis mutandis* and be finished. Since the first and third options seem unacceptable, in Section 17, we will go through the (somewhat tedious) process of examining what modifications are needed, either in the statements or the proofs of our earlier results. Items we have marked with a

• need only trivial change in their statement, but occasionally need supplementary arguments in their proofs. Items marked with a

◇ need more substantive changes and require further examination, which is given in Section 17.

It seems to us that this re-examination procedure has some pedagogical value in itself, since it requires that the reader make a careful — though guided — review of Sections 1–14.

To Measure, or not to Measure

The theory of measure is one arena in which there are significant differences between compact and noncompact intervals. Therefore Section 18 is devoted to a full description of this subject. It will be seen that the usual problems associated with the *construction* of Lebesgue measure on certain subsets

of $\mathbb{R}$ evaporate, since the properties of the measurable sets flow from the integral and from the Monotone and Dominated Convergence Theorems for it. Thus the existence and properties of the Lebesgue measurable sets are easy for us to establish. In this section, we will also give a number of results showing how arbitrary Lebesgue measurable sets can be approximated by "simpler" sets.

In the penultimate section, we discuss measurable functions on $\mathbb{R}$ to $\mathbb{R}$. Since the distinction between Borel and Lebesgue measurable functions arises in a natural way, we decided to give a brief exposition of the general notion of measurable spaces and measurable functions on them, and of measures defined on an abstract σ-algebra. It is our opinion that the idea and elementary properties of an abstract measure pose no conceptual difficulties to the student; the only thing that is not clear is whether any nontrivial measures exist. But readers who already have an acquaintance with Lebesgue measure on the line, know that important measures do exist. We touch on integration with respect to an abstract measure (which we regard as being quite simple) only in passing, but do not go into the various decompositions of measures.

Section 20, our final one, revisits the material of Section 11 dealing with the important notions of almost uniform convergence and convergence in measure and the generalization of Egorov's Theorem. It seems to us that these ideas are instances where the abstract notions do help to clarify the situation. In addition, we believe that the discussion here should make it possible for the reader to assimilate readily the abstract theory of integration and its application quite easily when they have the need to do so.

Infinite Intervals

We now consider real-valued functions defined on *infinite* closed intervals and define their integrals. Much of this discussion will be closely parallel to what we have already done; in fact, it would have been possible to handle the general case right from the beginning. However, we feel that it is more advantageous to treat the case of compact intervals first, as we have done. There are a few points where some additional considerations are needed in the infinite interval situation, and we prefer to treat these points separately. In addition, we feel that there is considerable merit in having the reader re-examine what we have already done in the light of a slightly different situation. It will be convenient for us to introduce some additional terminology and notations.

We define the **extended real number system** $\overline{\mathbb{R}}$ to be the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ consisting of the real numbers $\mathbb{R}$ with the two symbols $-\infty$ and $\infty\,(= +\infty)$ adjoined. (The set $\overline{\mathbb{R}}$ can be considered to be the *two-point compactification* of $\mathbb{R}$.) We do *not* consider the symbols $\pm\infty$ to be real numbers; however, it is useful to extend some of the operations in $\mathbb{R}$ to $\overline{\mathbb{R}}$. For example, we will write

$$-\infty < x < \infty \qquad \text{for all} \quad x \in \mathbb{R}.$$

The most important convention concerning the "arithmetic" in $\overline{\mathbb{R}}$ is the rule:

$$(16.\alpha) \qquad 0 \cdot (\pm\infty) = 0 = (\pm\infty) \cdot 0,$$

which will be frequently used. If $x \in \mathbb{R}$, then we also use the rule:

$$x + (\pm\infty) = \pm\infty = (\pm\infty) + x.$$

If $x > 0$, then we have

$$x \cdot (\pm\infty) = \pm\infty = (\pm\infty) \cdot x,$$

while if $y < 0$, then

$$y \cdot (\pm\infty) = \mp\infty = (\pm\infty) \cdot y.$$

We do *not* attach any meaning to notations such as $\infty - \infty$, or $(-\infty)/(+\infty)$, etc. If $a, b \in \mathbb{R}$, we introduce the **closed intervals** in $\mathbb{R}$:

$$[a, \infty) := \{x \in \mathbb{R} : a \leq x\}, \qquad (-\infty, b] := \{y \in \mathbb{R} : y \leq b\},$$
$$(-\infty, \infty) := \mathbb{R},$$

as well as the corresponding **open intervals**:

$$(a, \infty) := \{x \in \mathbb{R} : a < x\}, \qquad (-\infty, b) := \{y \in \mathbb{R} : y < b\}.$$

Sometimes we wish to include the symbols $\pm\infty$ in these intervals. When we do so, we write

$$[a, \infty] := \{x \in \overline{\mathbb{R}} : a \leq x \leq \infty\},$$

and similarly for $(a, \infty]$, $[-\infty, b]$, $[-\infty, b)$ and $[-\infty, \infty] = \overline{\mathbb{R}}$. All of these ten types of intervals are called **infinite intervals**, and there are no other infinite intervals. We define the length $l(I)$ of each of these infinite intervals to be ∞.

With the exception of the length function, all functions in this section will be assumed to have values in $\mathbb{R}$, so that they can be combined by the usual arithmetic rules in $\mathbb{R}$.

Gauges and δ-fineness

We want to define the integral of $f : [a, \infty) \to \mathbb{R}$ along the lines presented in Section 1, but bearing in mind the difficulty mentioned in Section 15. First, we take a tagged partition $\dot{\mathcal{P}}$ of the interval $[a, \infty]$ in $\overline{\mathbb{R}}$:

$$\dot{\mathcal{P}} := \Big\{\big([x_0, x_1], t_1\big), \cdots, \big([x_{n-1}, x_n], t_n\big), \big([x_n, x_{n+1}], t_{n+1}\big)\Big\},$$

so that $x_0 = a$ and $x_{n+1} = \infty$. We define $f(\infty) := 0$ (using the same symbol for the extended function). We will define δ-fineness of $\dot{\mathcal{P}}$ in such a way that the final tag $t_{n+1} = \infty$, so that the final term in the Riemann sum $S(f; \dot{\mathcal{P}})$ is $f(\infty)(\infty - x_n)$, which equals 0, in view of our convention that $0 \cdot \infty = 0$.

We define a **gauge** on $[a, \infty]$ to be a strictly positive real-valued function δ defined on $[a, \infty]$. We will say that the partition $\dot{\mathcal{P}}$ is **δ-fine** if the finite subintervals satisfy

$$(16.3) \qquad [x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \qquad \text{for} \quad i = 1, \cdots, n.$$

and if the infinite subinterval $[x_n, \infty]$ satisfies

$$(16.\gamma') \qquad [x_n, \infty] \subseteq [1/\delta(\infty), \infty],$$

or, equivalently, that

$$(16.\gamma'') \qquad 1/\delta(\infty) \le x_n.$$

Since the final subinterval $[x_n, \infty]$ is the only subinterval in $\dot{\mathcal{P}}$ that contains ∞, the requirement that $\dot{\mathcal{P}}$ be δ-fine forces the tag $t_{n+1} = \infty$, as desired. Since $f(\infty) = 0$, then for any δ-fine partition $\dot{\mathcal{P}}$, the Riemann sum reduces (as we have seen above) to

$$S(f; \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}),$$

and all of these terms are finite real numbers.

It is convenient to have a notation that will express (16.β) and (16.γ') in a uniform way. If $r > 0$ and $t \in \mathbb{R}$ we define $U[t;r] := [t-r, t+r]$, while if $t = \infty$ we define $U[\infty; r] := [1/r, \infty]$, and if $t = -\infty$ we define $U[-\infty; r] := [-\infty, -1/r]$. With this notation, the inclusions (16.β) and (16.γ') can be expressed in the form

$$(16.\delta) \qquad [x_{i-1}, x_i] \subseteq U[t_i; \delta(t_i)] \qquad \text{for} \quad i = 1, \cdots, n+1.$$

The Fineness Theorem

A question arises: Given a gauge δ on the infinite interval $I := [a, \infty]$, does there always exist a δ-fine partition of I? We now show that the answer is: "Yes".

16.1 Fineness (= Cousin's) Theorem. *If $I := [a, \infty]$ and if δ is a gauge on I, then there exist δ-fine partitions of I.*

Proof. Given a gauge δ on I, we take $b \ge \max\{a, 1/\delta(\infty)\}$. The Fineness Theorem 1.4 implies that the interval $[a, b]$ has a δ-fine partition $\{(I_i, t_i)\}_{i=1}^{n}$ with $x_n = b$. It is clear that if we let $I_{n+1} := [b, \infty]$ and $t_{n+1} := \infty$, and adjoin the pair (I_{n+1}, t_{n+1}), then we obtain a δ-fine partition of $[a, \infty]$. Q.E.D.

It is clear that if δ is a gauge on the infinite intervals $[-\infty, b]$ or $[-\infty, \infty]$, then there exist δ-fine partitions.

Note. It is evident that, by defining $f(\infty) = 0$, we accomplish the same result as in Method (i), discussed in Section 15.

Definition of the Integral

After this rather extended prologue, we can now define the integral of a function $f : [a, \infty) \to \mathbb{R}$, which we assume has been extended by setting $f(\infty) := 0$.

16.2 Definition. (a) If $I := [a, \infty]$ and if $f : I \to \mathbb{R}$, then we say that f is **(generalized Riemann) integrable on** $[a, \infty)$, or **on** $[a, \infty]$, if there exists a number $C \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a gauge δ_ε on $[a, \infty]$ such that if $\dot{\mathcal{P}}$ is any δ_ε-fine partition of $[a, \infty]$, then

$$|S(f; \dot{\mathcal{P}}) - C| \leq \varepsilon.$$

The set of all integrable functions on $[a, \infty)$ will be denoted by $\mathcal{R}^*([a, \infty))$ or by $\mathcal{R}^*([a, \infty])$. If f is integrable on $I := [a, \infty]$, we often write

$$C = \int_I f = \int_a^\infty f = \int_a^\infty f(x)\, dx.$$

(b) If both f and $|f|$ belong to $\mathcal{R}^*([a, \infty])$, we say that f is **Lebesgue integrable** (or is **absolutely integrable**) on $[a, \infty]$ and write $f \in \mathcal{L}([a, \infty])$.

It is an easy exercise to show that (as in the Uniqueness Theorem 1.9), there is at most one number C satisfying Definition 16.2.

The intervals $(-\infty, b]$ and $(-\infty, \infty)$

If $g : (-\infty, b] \to \mathbb{R}$ is given, then we extend g to $[-\infty, b]$ by defining

$$g(-\infty) := 0.$$

If δ is a gauge on $[-\infty, b]$, we say that a partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is δ-fine if (16.δ) holds. This involves the inclusion

$$I_1 = [x_0, x_1] \subseteq [-\infty, -1/\delta(-\infty)],$$

which implies that $x_0 = -\infty$ and $x_1 \leq -1/\delta(-\infty)$. The definition of the integral of g on $[-\infty, b]$ is now entirely parallel to Definition 16.2, and the integral will often be denoted by

$$\int_{-\infty}^b g = \int_{-\infty}^b g(x)\, dx.$$

Similarly, if $h : (-\infty, \infty) \to \mathbb{R}$ is given, we extend h to $[-\infty, \infty]$ by defining

$$h(-\infty) := 0 \qquad \text{and} \qquad h(\infty) := 0.$$

If δ is a gauge on $[-\infty, \infty]$, we say that a partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is δ-fine if $I_i = [x_{i-1}, x_i] \subseteq U[t_i; \delta(t_i)]$ holds for $i = 1, \cdots, n$. This involves the inclusions

$$I_1 \subseteq [-\infty, -1/\delta(-\infty)] \qquad \text{and} \qquad I_n \subseteq [1/\delta(\infty), \infty],$$

which imply that $x_0 = -\infty$ and $x_1 \leq -1/\delta(-\infty)$, and that $x_{n-1} \geq 1/\delta(\infty)$ and $x_n = \infty$. This also implies that $t_0 = -\infty$ and $t_n = \infty$, so that the first and last terms in $S(h; \dot{\mathcal{P}})$ vanish. The definition of the integral of h on $[-\infty, \infty]$ is now entirely parallel to Definition 16.2. The integral will usually be denoted by

$$\int_{-\infty}^{\infty} h = \int_{-\infty}^{\infty} h(x)\, dx.$$

Note. In dealing with integrals over $[a, \infty]$, it is sometimes convenient to consider partitions $\dot{\mathcal{P}}$ with $n + 1$ (rather than n) subintervals. Similarly, in dealing with integrals over $[-\infty, \infty]$, we may want to consider partitions with $n + 2$ subintervals.

Two Examples

We will now give two examples of integrable functions on an infinite interval. The first one will be familiar from calculus as an "improper" integral. The second one is very similar to Example 2.7. In both cases there are some tricky details that need to be carefully considered. We will see later in this section that the integrability of these functions (and the value of their integrals) can be established much more easily by using Hake's Theorem.

16.3 Examples. (a) Let $f(x) := 1/x^2$ for $x \in [1, \infty)$ and let $f(\infty) := 0$. We will show directly that f is integrable on $[1, \infty]$ with integral equal to 1. In doing so it will be convenient to use the function $F(x) := -1/x$ for $x \in [1, \infty)$ and $F(\infty) := 0$.

If $\varepsilon > 0$ is given let $\delta_\varepsilon(t) := \varepsilon$ for $t \in [1, \infty]$. Suppose that $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n+1}$ is a δ_ε-fine partition of $[1, \infty]$. Since $f(\infty) l([x_n, \infty]) = 0$, we see that

$$\begin{aligned} S(f; \dot{\mathcal{P}}) - \sum_{i=1}^{n} \big[F(x_i) - F(x_{i-1})\big] \\ = \sum_{i=1}^{n} \Big(\frac{x_i - x_{i-1}}{t_i^2} - \Big(-\frac{1}{x_i} + \frac{1}{x_{i-1}} \Big) \Big) \\ = \sum_{i=1}^{n} \Big(\frac{1}{t_i^2} - \frac{1}{x_{i-1} x_i} \Big) (x_i - x_{i-1}). \end{aligned} \tag{16.ε}$$

We need an algebraic inequality. We note that if $1 \le u \le t \le v$, then we have $0 \le (1/u - 1/t)(1/v + 1/t)$ so that an easy calculation shows that

$$\frac{1}{t^2} - \frac{1}{uv} \le \frac{1}{t} \cdot \left(\frac{1}{u} - \frac{1}{v}\right) \le \frac{1}{u} - \frac{1}{v}.$$

Similarly, the inequality $(1/u + 1/t)(1/v - 1/t) \le 0$ shows that

$$\frac{1}{uv} - \frac{1}{t^2} \le \frac{1}{t} \cdot \left(\frac{1}{u} - \frac{1}{v}\right) \le \frac{1}{u} - \frac{1}{v}.$$

Therefore it follows that if $1 \le u \le t \le v$, then

$$\left|\frac{1}{t^2} - \frac{1}{uv}\right| \le \left(\frac{1}{u} - \frac{1}{v}\right). \tag{16.ζ}$$

Since $1 \le x_{i-1} \le t_i \le x_i$ and $x_i - x_{i-1} \le 2\delta_\varepsilon(t_i) = 2\varepsilon$ for $n = 1, \cdots, n$, we conclude from (16.ζ) that

$$\left|\frac{1}{t_i^2} - \frac{1}{x_{i-1}x_i}\right|(x_i - x_{i-1}) \le \left(\frac{1}{x_{i-1}} - \frac{1}{x_i}\right) \cdot 2\varepsilon. \tag{16.η}$$

We now turn to an appraisal of the terms in (16.ε). To estimate the final sum, we add the inequalities in (16.η) for $i = 1, \cdots, n$, and note the telescoping of the two sums to obtain

$$\begin{aligned} \left|S(f;\dot{\mathcal{P}}) - \left(1 - \frac{1}{x_n}\right)\right| &\le \sum_{i=1}^{n} \left|\frac{1}{t_i^2} - \frac{1}{x_{i-1}x_i}\right|(x_i - x_{i-1}) \\ &\le \sum_{i=1}^{n}\left(\frac{1}{x_{i-1}} - \frac{1}{x_i}\right) \cdot 2\varepsilon = \left(1 - \frac{1}{x_n}\right) \cdot 2\varepsilon < 2\varepsilon. \end{aligned}$$

Therefore, it follows from the δ_ε-fineness of $\dot{\mathcal{P}}$ that $1/x_n \le \varepsilon$, whence

$$|S(f;\dot{\mathcal{P}}) - 1| < 2\varepsilon + 1/x_n \le 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies that $f(x) = 1/x^2$ is (absolutely) integrable on $[1, \infty]$ and that $\int_1^\infty (1/x^2)\,dx = 1$, as asserted.

(b) Let $\sum_{k=1}^\infty a_k$ be a series that converges to A. Define $h : [0, \infty] \to \mathbb{R}$ by $h(x) := a_k$ for $x \in [k-1, k)$, $k \in \mathbb{N}$, and $h(\infty) := 0$. (See Figure 16.1.) We will show that $h \in \mathcal{R}^*([0,\infty])$ and that $\int_0^\infty h = A$. Our arguments will be similar to (but somewhat simpler than) those in Example 2.7.

We let $M \ge \sup\{|a_k| : k \in \mathbb{N}\}$ and $M \ge 1$ and, given $\varepsilon > 0$ with $\varepsilon \le 1$, we let $m(\varepsilon) \in \mathbb{N}$ be such that if $m \ge m(\varepsilon)$ then

$$|a_m| \le \varepsilon \qquad \text{and} \qquad \left|\sum_{k=m}^{\infty} a_k\right| \le \varepsilon.$$

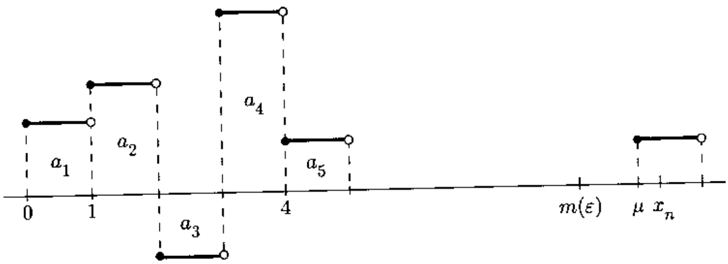

Figure 16.1 The graph of h.

We define the gauge δ_ε on $I := [0, \infty]$ by

$$\delta_\varepsilon(t) := \begin{cases} \frac{1}{2}\operatorname{dist}(t, \mathbb{N}) & \text{if} \quad t \in [0,\infty) - \mathbb{N}, \\ \varepsilon/2^{k+1}M & \text{if} \quad t = k \in \mathbb{N}, \\ 1/m(\varepsilon) & \text{if} \quad t = \infty. \end{cases}$$

Let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n+1}$ be a δ_ε-fine partition of $[0, \infty]$, so that $x_n \geq m(\varepsilon)$. For convenience, let $\mu := \sup\{k \in \mathbb{N} : k \leq x_n\}$ so that $m(\varepsilon) \leq \mu \leq x_n$. It follows from the definition of δ_ε that each integer $k \in \{1, \cdots, \mu\}$ must be a tag for any finite subinterval in $\dot{\mathcal{P}}$ that contains k. We may also assume that each such point $k < x_n$ is a tag for two consecutive subintervals in $\dot{\mathcal{P}}$. We consider the subintervals $[k-1, x_r], \cdots, [x_s, k]$ in $\dot{\mathcal{P}}$ that are contained in $[k-1, k]$. The total contribution T_k to $S(h; \dot{\mathcal{P}})$ from this subinterval is easily seen to be

$$T_k = a_k(x_s - k + 1) + a_{k+1}(k - x_s),$$

whence it follows that $T_k - a_k = (k - x_s)(a_{k+1} - a_k)$, so that

$$|T_k - a_k| \leq 2M|k - x_s|.$$

But, since $|k - x_s| \leq \delta_\varepsilon(k) = \varepsilon/2^{k+1}M$, we conclude that

$$(16.\theta) \qquad |T_k - a_k| \leq \varepsilon/2^k \qquad \text{for} \quad k = 1, \cdots, \mu.$$

There are two cases to consider, depending on whether $x_n \in \mathbb{N}$ or not.

Case 1. $x_n = \mu \in \mathbb{N}$.

Here we have $S(h;\dot{\mathcal{P}}) = \sum_{k=1}^{\mu} T_k$. Since (16.$\theta$) holds, we conclude that

$$\begin{aligned}|S(h;\dot{\mathcal{P}}) - A| &\le \Big|\sum_{k=1}^{\mu}(T_k - a_k)\Big| + \Big|\sum_{k=\mu+1}^{\infty} a_k\Big| \\ &\le \sum_{k=1}^{\mu} |T_k - a_k| + \varepsilon \\ &\le \sum_{k=1}^{\mu} \varepsilon/2^k + \varepsilon \le 2\varepsilon.\end{aligned}$$

Case 2. $\mu < x_n < \mu + 1$.

Here the contribution T^* to $S(h;\dot{\mathcal{P}})$ due to subintervals in $[\mu, x_n]$ is

$$T^* = a_\mu(x_n - \mu).$$

But since $|a_\mu| \le \varepsilon$ and $x_n - \mu < 1$, we have $|T^*| \le \varepsilon$. Therefore we conclude that $|S(h;\dot{\mathcal{P}}) - A| \le 3\varepsilon$ in Case 2.

Therefore, in either case we have $|S(h;\dot{\mathcal{P}}) - A| \le 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary, then $h \in \mathcal{R}^*([0,\infty])$ and

$$(16.\iota) \qquad \int_0^\infty h = A = \sum_{k=1}^{\infty} a_k.$$

16.4 Remark. (a) The absolute value $|h|$ of the function in Example 16.3(b) is given by $|h|(x) := |a_k|$ for $x \in [k-1, k)$, $k \in \mathbb{N}$, and $|h|(\infty) := 0$. If the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then the preceding argument shows that $|h| \in \mathcal{R}^*([0,\infty])$ so that $h \in \mathcal{L}([0,\infty])$ and that

$$\int_0^\infty |h| = \sum_{k=1}^{\infty} |a_k|.$$

It will be seen in Example 16.8(c) that if $\sum_{k=1}^{\infty} a_k$ is not absolutely convergent, then the function $|h| \notin \mathcal{R}^*([0,\infty])$ so that $h \notin \mathcal{L}([0,\infty])$.

Review of Section 3

Most (but not all) of the results in Section 3 can be extended to integrals over infinite intervals with few changes in the proofs that were given there. As indicated in Section 15, we will give the reader a guided review of Section 3. We will treat the case of an integral over an interval of the form $I :=$

$[a, \infty]$, leaving it to the reader to make whatever modifications are needed for intervals of the form $[-\infty, b]$ and $[-\infty, \infty]$.

We recall the meaning of the symbols $\bullet$ and $\diamond$ used in the preceding sections. If *no change* is required in the *statement* (except the replacement of $[a, \infty]$ for $[a, b]$), we have marked that result with the symbol $\bullet$. If only a *minor change* is required, we have marked that statement with the symbol $\diamond$. If the statement is false or requires a major change, we have left that statement unmarked. In some cases a change or supplementary argument is needed in the *proof*. In those cases, it will be indicated how the argument needs to be modified.

(3.1) If δ', δ'' are gauges on $I := [a, \infty]$ and if $\delta(t) := \min\{\delta'(t), \delta''(t)\}$ for all $t \in I$, then it is trivial that δ is a gauge on I. Moreover, it is also true that if $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n+1}$ is δ-fine, then it is both δ'-fine and δ''-fine. This follows since

$$x_n \geq 1/\delta(\infty) = \max\{1/\delta'(\infty), 1/\delta''(\infty)\}.$$

The rest of the proof of 3.1 proceeds as before.

(3.2) and **(3.3)** No change is needed.

(3.4) The analogous result is not true, since nonzero constant functions are not integrable over $[a, \infty]$.

(3.5) No change is needed.

(3.6) — The Cauchy Criterion. ($\Rightarrow$) No change is needed.

($\Leftarrow$) As seen in the remarks concerning (3.1), if $\delta_n(t) \geq \delta_{n+1}(t)$ for all $t \in I$, then a δ_{n+1}-fine partition is also δ_n-fine, so we obtain a Cauchy sequence $(S(f; \dot{\mathcal{P}}_n))$ and proceed as in the earlier proof.

(3.7) ($\Leftarrow$) The third line in the definition of δ_ε needs to apply for $t \in (c, \infty)$, and we need to add the line $\delta_\varepsilon(\infty) := \delta''_\varepsilon(\infty)$. If $\dot{\mathcal{P}}$ is δ_ε-fine, then the point c must be a tag. Since the resulting partition $\dot{\mathcal{P}}_1$ is δ'_ε-fine, and $\dot{\mathcal{P}}_2$ is δ''_ε-fine, the proof is as before.

($\Rightarrow$) No change is needed either for I_1 or I_2.

(3.8) and **(3.9)** No change is needed.

(3.10) We can take β to be ∞.

(3.11) Any of α, β, γ can be taken to be ∞.

(3.12) — The Squeeze Theorem. No change is needed.

(3.13) *We add the requirement that a step function s vanishes outside of some compact interval.*

(3.14) With the modification made in (3.13), a step function on $[a, \infty]$ is integrable.

(3.15) The definition makes sense on $[a, \infty]$.

(3.16) A regulated function on $[a, \infty]$ need not be integrable; for example, $f(x) := 1/x$ on $[1, \infty)$ and $f(\infty) := 0$.

(3.17) — Characterization of regulated functions. If $f : I \to \mathbb{R}$ is regulated, then all of its one-sided limits exist on $[a, \infty)$, since the restriction of f to a compact interval is regulated. The converse is false, for $f(x) := 1$ is not a regulated function on $[a, \infty)$, since the approximating step functions vanish outside compact intervals.

(3.18) A continuous function on $[a, \infty]$ need not be integrable. The example in (3.16) works.

(3.19) A monotone function on $[a, \infty]$ need not be integrable. Same example.

(3.20) No change is needed. If f is regulated on $[a, \infty]$, its restriction to the interval $[a, a + n]$ is regulated for every $n \in \mathbb{N}$.

(3.21) If $0 \le f \in \mathcal{R}^*(I)$ and $\varepsilon > 0$, it follows from the Cauchy Criterion 16.6 (to be given below) that there exists $K(\varepsilon)$ such that if $q > p \ge K(\varepsilon)$, then $\int_p^q f \le \varepsilon$. Since g is a regulated function it is bounded, say by M. Thus $|\int_p^q fg| \le M \int_p^q f \le M\varepsilon$. Therefore $f \cdot g \in \mathcal{R}^*(I)$.

(3.22) Let $I := [a, \infty]$ and $f : I \to \mathbb{R}$. If $r \in \mathbb{R}$, we define $I_r := [a + r, \infty]$ and $f_r(y) := f(y - r)$ for $y \in I_r$. Similarly, if $r > 0$, we define $I_{(r)} := [ar, \infty]$ and $f_{(r)}(z) := f(z/r)$ for $z \in I_{(r)}$. Since we use the conventions $\infty \pm r = \infty$ and $\infty \cdot r = \infty$ for $r > 0$, the arguments in the proof of Theorem 3.22 remain valid when we define $\eta_\varepsilon(s) := \delta_\varepsilon(s - r)$ for $s \in [a + r, \infty)$ and $\eta_\varepsilon(\infty) := \delta_\varepsilon(\infty)$, and make other similar modifications.

Hake's Limit Theorem

We will now establish a result that is extremely useful both in showing that a function is integrable over an interval $[a, \infty]$ and in evaluating the integral. This remarkable result shows that one does *not* obtain an "improper extension" of the (generalized Riemann) integral by taking limits as the limits

become infinite; instead, the integral is already "complete" and cannot be extended by taking limits.

This result was proved in 1921 by the German mathematician Heinrich Hake for the Perron integral. The reader will observe that it is *not* true for the R-integral or the L-integral. This theorem is a close analogue for $[a,\infty]$ of Theorem 12.8, and its proof is very similar to the proof of that result.

16.5 Hake's Theorem on $[a,\infty]$. *Let* $I := [a,\infty]$ *and let* $f : I \to \mathbb{R}$. *Then* $f \in \mathcal{R}^*(I)$ *if and only if* $f \in \mathcal{R}^*([a,c])$ *for every compact interval* $[a,c]$ *with* $c \in [a,\infty)$, *and there exists* $A \in \mathbb{R}$ *such that*

$$\lim_{c\to\infty} \int_a^c f = A. \tag{16.κ}$$

In this case, $\int_a^\infty f = A$.

Proof. $(\Rightarrow)$ Let $A := \int_a^\infty f$ and let $\varepsilon > 0$ be given. Then there exists a gauge η on I such that if $\dot{\mathcal{P}} := \{(I_i,t_i)\}_{i=1}^{n+1}$ is any η-fine partition of I, then $|S(f;\dot{\mathcal{P}}) - A| \le \frac{1}{2}\varepsilon$. Let x_n be the next to the last partition point of $\dot{\mathcal{P}}$ and let c be any number with $c \ge x_n$. By the analogue of Theorem 3.7 for $[a,\infty]$, the function f is integrable on $[a,c]$; hence there exists a gauge η_c on $[a,c]$ such that if $\dot{\mathcal{P}}_c$ is any η_c-fine partition of $[a,c]$, then $|S(f;\dot{\mathcal{P}}_c) - \int_a^c f| \le \frac{1}{2}\varepsilon$. We may also assume that $\eta_c(t) \le \eta(t)$ for all $t \in [a,c]$. Now let $\dot{\mathcal{P}}_c^*$ be obtained from $\dot{\mathcal{P}}_c$ by adjoining the pair $([c,\infty],\infty)$; it is evident that the partition $\dot{\mathcal{P}}_c^*$ is η-fine and that

$$S(f;\dot{\mathcal{P}}_c^*) = S(f;\dot{\mathcal{P}}_c) + f(\infty)l([c,\infty]) = S(f;\dot{\mathcal{P}}_c).$$

Therefore, we have that

$$\Big|\int_a^c f - A\Big| \le \Big|\int_a^c f - S(f;\dot{\mathcal{P}}_c)\Big| + \Big|S(f;\dot{\mathcal{P}}_c^*) - A\Big| \le \varepsilon$$

for any $c \ge x_n$. Since $\varepsilon > 0$ is arbitrary, this implies that (16.κ) holds.

$(\Leftarrow)$ Suppose there exists $A \in \mathbb{R}$ such that for every $c \in (a,\infty)$ the restriction of f belongs to $\mathcal{R}^*([a,c])$ and (16.κ) holds. Now let $(c_k)_{k=0}^\infty$ be a strictly increasing sequence with $a = c_0$ and $\infty = \lim_k c_k$. Given $\varepsilon > 0$, let $r \in \mathbb{N}$ be such that if $b \ge c_r$, then

$$\Big|\int_a^b f - A\Big| \le \varepsilon.$$

If $k \in \mathbb{N}$, let δ_k be a gauge on $I_k := [c_{k-1},c_k]$ such that if $\dot{\mathcal{P}}_k$ is any δ_k-fine partition of I_k, then

$$\Big|S(f;\dot{\mathcal{P}}_k) - \int_{I_k} f\Big| \le \varepsilon/2^k.$$

Without loss of generality, we may assume that

(i) $\delta_1(c_0) \leq \frac{1}{2}(c_1 - c_0)$,

and if $k \geq 1$, that

(ii) $\delta_{k+1}(c_k) \leq \min\{\delta_k(c_k), \frac{1}{2}\operatorname{dist}(c_k, \{c_{k-1}, c_{k+1}\})\}$.

(iii) $\delta_k(t) \leq \frac{1}{2}\operatorname{dist}(t, \{c_{k-1}, c_k\})$ for $t \in (c_{k-1}, c_k)$.

We now define δ on I by:

$$\delta(t) := \begin{cases} \delta_k(t) & \text{if} \quad t \in I_k,\ k \in \mathbb{N}, \\ 1/c_r & \text{if} \quad t = \infty. \end{cases}$$

Thus δ is a gauge on I and we let $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n+1}$ be a δ-fine partition of I, so that the unbounded subinterval $[x_n, \infty]$ in $\dot{\mathcal{P}}$ must have its tag at ∞; moreover, $c_r \leq x_n$.

Now let $s \in \mathbb{N}$ be the smallest positive integer such that $x_n \leq c_s$, so that $r \leq s$. If $k = 1, \cdots, s-1$, then condition (iii) implies that the point c_k must be a tag for any subinterval in $\dot{\mathcal{P}}$ that contains c_k. Using the right-left procedure, we may assume that the points $c_0, \cdots, c_{s-1}$ are also points in $\dot{\mathcal{P}}$. We let

$$\dot{\mathcal{Q}}_1 := \dot{\mathcal{P}} \cap [c_0, c_1], \quad \cdots, \quad \dot{\mathcal{Q}}_{s-1} := \dot{\mathcal{P}} \cap [c_{s-2}, c_{s-1}], \quad \dot{\mathcal{Q}}_s := \dot{\mathcal{P}} \cap [c_{s-1}, x_n].$$

Since each $\dot{\mathcal{Q}}_k$ $(k = 1, \cdots, s-1)$ is a δ_k-fine partition of I_k, we have

$$\left|S(f; \dot{\mathcal{Q}}_k) - \int_{I_k} f\right| \leq \varepsilon/2^k \qquad \text{for} \quad k = 1, \cdots, s-1.$$

Also, since $\dot{\mathcal{Q}}_s$ is a δ_s-fine subpartition of I_s, it follows from the Saks-Henstock Lemma 5.3 that

$$\left|S(f; \dot{\mathcal{Q}}_s) - \int_{c_{s-1}}^{x_n} f\right| \leq \varepsilon/2^s.$$

If $\dot{\mathcal{Q}}_\infty := \{([x_n, \infty], \infty)\}$, then $S(f; \dot{\mathcal{Q}}_\infty) = 0$. Since $\dot{\mathcal{P}} = \dot{\mathcal{Q}}_1 \cup \cdots \cup \dot{\mathcal{Q}}_s \cup \dot{\mathcal{Q}}_\infty$, we have

$$\begin{aligned} |S(f; \dot{\mathcal{P}}) - A| &= \left|\sum_{i=1}^{s} S(f; \dot{\mathcal{Q}}_i) + S(f; \dot{\mathcal{Q}}_\infty) - A\right| \\ &\leq \left|\sum_{i=1}^{s} S(f; \dot{\mathcal{Q}}_i) - \int_a^{x_n} f\right| + |S(f; \dot{\mathcal{Q}}_\infty)| + \left|\int_a^{x_n} f - A\right| \\ &\leq \varepsilon + 0 + \varepsilon = 2\varepsilon. \end{aligned}$$

But since $\varepsilon > 0$ is arbitrary, then $f \in \mathcal{R}^*(I)$ with integral A. Q.E.D.

For future reference, we will give a formal statement of the Cauchy Criterion for $f \in \mathcal{R}^*(I)$ when $I = [a, \infty]$. We leave the details of its proof to the reader as an exercise.

16.6 Cauchy Criterion. *Let $f : I \to \mathbb{R}$ be such that $f \in \mathcal{R}^*([a, c])$ for all $c \geq a$. Then $f \in \mathcal{R}^*(I)$ if and only if for every $\varepsilon > 0$ there exists $K(\varepsilon) \geq a$ such that if $q > p \geq K(\varepsilon)$, then $\left|\int_p^q f\right| \leq \varepsilon$.*

We leave it to the reader to formulate a version of Hake's Theorem for functions on $[-\infty, b]$. However, we will make an explicit statement of the corresponding result for a function on $[-\infty, \infty]$.

16.7 Hake's Theorem for $[-\infty, \infty]$. *Let $h : [-\infty, \infty] \to \mathbb{R}$. Then the following statements are equivalent:*

(i) *$h \in \mathcal{R}^*([-\infty, \infty])$ with integral $A \in \mathbb{R}$.*

(ii) *$h \in \mathcal{R}^*([a, b])$ for every compact interval $[a, b]$ and*

$$\lim_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b h = A.$$

(iii) *For any $c \in \mathbb{R}$, the function h is in $\mathcal{R}^*([-\infty, c])$ and $\mathcal{R}^*([c, \infty])$ and*

$$\int_{-\infty}^{c} h + \int_{c}^{\infty} h = A.$$

Proof. The analogue of Theorem 3.7 is that h is integrable on $[-\infty, \infty]$ if and only if it is integrable on $[-\infty, c]$ and $[c, \infty]$ for any $c \in \mathbb{R}$. We leave the remaining details to the reader. Q.E.D.

Some Applications

The proof of Hake's Theorem 16.5 was delicate, but not much more so than the argument in Example 16.3(b). We now show that Hake's Theorem can be used to establish the results in Example 16.3.

16.8 Examples. **(a)** Let $f(x) := 1/x$ for $x \in [1, \infty)$ and $f(\infty) := 0$ as in Example 16.3(a). If $c \in (1, \infty)$, then it follows from the Fundamental Theorem 4.5 that

$$\int_1^c (1/x^2)\, dx = (-1/x)\Big|_1^c = 1 - 1/c.$$

Since $\lim_{c \to \infty}(1/c) = 0$, Hake's Theorem 16.5 implies that $f \in \mathcal{R}^*([1, \infty])$ with integral equal to 1. Clearly, $f \in \mathcal{L}([1, \infty])$.

(b) Let h be as in Example 16.3(b). If $c \in (0, \infty)$, then the restriction of h to $[0, c]$ is a step function and so is integrable. Since the series $\sum_{k=1}^{\infty} a_k$ is convergent, given $\varepsilon > 0$ there exists $m(\varepsilon) \in \mathbb{N}$ such that if $m \geq m(\varepsilon)$ then $|a_m| \leq \varepsilon$ and $\left|\sum_{k=m}^{\infty} a_k\right| \leq \varepsilon$. If $p, q \in \mathbb{R}$ are such that $m(\varepsilon) \leq p \leq q$ and if $n := \lfloor p \rfloor$ and $m := \lfloor q \rfloor$ are the greatest integers less than or equal to p and q, respectively, then $m(\varepsilon) \leq n + 1 \leq m + 1$, so that

$$\left|\int_p^q h\right| \leq |a_{n+1}| + \left|\sum_{k=n+1}^{m+1} a_k\right| + |a_{m+1}| \leq 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the limit $\lim_{c \to \infty} \int_0^c h$ exists and equals

$$\lim_{n\to\infty} \int_0^n h = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k.$$

(c) If the series $\sum_{k=1}^{\infty} a_k$ is *not* absolutely convergent and if $h : [0, \infty] \to \mathbb{R}$ is defined as in Example 16.3(b), then $H := |h|$ does not belong to $\mathcal{R}^*([0, \infty])$, so that $h \notin \mathcal{L}([0, \infty])$.

Indeed, the restriction of H to every interval $[0, n]$ is integrable. If H is in $\mathcal{R}^*([0, \infty])$, then Hake's Theorem implies that

$$\sum_{k=1}^{n} |a_k| = \int_0^n H \to \int_0^{\infty} H$$

as $n \to \infty$. But this contradicts the assumption that the series is not absolutely convergent.

(d) It is important to realize that Hake's Theorem requires that the limit $\lim_{c\to\infty} \int_a^c f$ must exist, not just over a particular sequence $c_k \to \infty$. For example, let $f(x) := \cos \pi x$ for $x \in [0, \infty)$ and $f(\infty) := 0$. If $c \in (0, \infty)$, then we have

$$\int_0^c \cos \pi x \, dx = \frac{1}{\pi} \sin \pi x \Big|_0^c = \frac{1}{\pi} \sin \pi c.$$

If we take the sequence $(n)_{n=1}^{\infty}$, then the sequence of integrals converges to 0, but if we take $\left(n + \frac{1}{2}\right)_{n=1}^{\infty}$, then the sequence of integrals converges to 1. In fact, the integral $\int_0^{\infty} \cos \pi x \, dx$ does not exist.

(e) Let $g(x) := 1/(1 + x^2)$ for $x \in \mathbb{R}$ and $g(-\infty) := 0 =: g(\infty)$. Then $(\operatorname{Arctan} x)' = g(x)$ for all $x \in \mathbb{R}$. It follows from the Fundamental Theorem 4.7 that if $a, b \in \mathbb{R}$ and $a \leq b$, then

$$\int_a^b \frac{dx}{1 + x^2} = \operatorname{Arctan} x \Big|_a^b = \operatorname{Arctan} b - \operatorname{Arctan} a.$$

The reader will also recall the well-known limits:

$$\lim_{x\to-\infty} \operatorname{Arctan} x = -\tfrac{1}{2}\pi \qquad \text{and} \qquad \lim_{x\to\infty} \operatorname{Arctan} x = \tfrac{1}{2}\pi.$$

Therefore $g \in \mathcal{L}([-\infty, \infty])$ and

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

Three Convergence Tests

We will now present versions of three results with very classical roots that are useful in showing that an integrand belongs to $\mathcal{R}^*(I)$, when $I = [a, \infty]$. The first two can be regarded as extensions of the Multiplier Theorem 10.12 to infinite intervals. For convenience, we will limit our attention to monotone functions φ. Our first result is an integral version of a test for series given in 1826 by the Norwegian genius Niels H. Abel (1802–1829).

16.9 Abel's Test. *Let $f, \varphi : I \to \mathbb{R}$, and suppose that:*

(i) $f \in \mathcal{R}^*(I)$.

(ii) *φ is bounded and monotone on I.*

Then $f\varphi \in \mathcal{R}^*(I)$.

Proof. By hypothesis, there exists $M > 0$ such that $|\varphi(x)| \le M$ for $x \in I$. Since $f \in \mathcal{R}^*(I)$, the Cauchy Criterion 16.6 implies that given $\varepsilon > 0$ there exists $K(\varepsilon) \ge a$ such that if $q > p \ge K(\varepsilon)$, then $|\int_p^q f| \le \varepsilon/2M$. Since φ is monotone, it follows from the Multiplier Theorem 10.12 that $f\varphi \in \mathcal{R}^*([p, q])$ and from the Second Mean Value Theorem 12.5 that there exists $\xi \in [p, q]$ such that

$$\int_p^q f\varphi = \varphi(p) \int_p^\xi f + \varphi(q) \int_\xi^q f.$$

Thus, if $q > p \ge K(\varepsilon)$, then $|\int_p^q f\varphi| \le M \cdot (\varepsilon/2M) + M \cdot (\varepsilon/2M) = \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the Cauchy Criterion implies that $f\varphi \in \mathcal{R}^*(I)$. Q.E.D.

The next two results do not require that $f \in \mathcal{R}^*(I)$, but they impose greater restrictions on φ than Abel's Test does. The first one is often called "Dirichlet's Test", but was formulated for integrals by J. Chartier as early as 1853.

16.10 Chartier-Dirichlet's Test. *Let $f, \varphi : I \to \mathbb{R}$ and suppose that:*

(j) *$f \in \mathcal{R}^*([a, c])$ for all $c \ge a$ and $F(x) := \int_a^x f$ is bounded on $[a, \infty)$.*

(jj) *φ is monotone on I and $\lim_{x\to\infty} \varphi(x) = 0$.*

Then $f\varphi \in \mathcal{R}^*(I)$.

Proof. The Multiplier Theorem 10.12 implies that $f\varphi \in \mathcal{R}^*([a,c])$ for all $c \geq a$. Let $M \geq |F(x)|$ for $x \in [a,\infty)$. Since $\lim_{x\to\infty} \varphi(x) = 0$, given $\varepsilon > 0$ there exists $K(\varepsilon) \geq a$ such that $|\varphi(x)| \leq \varepsilon/4M$ for $x \geq K(\varepsilon)$.

If $q > p \geq K(\varepsilon)$, then the Second Mean Value Theorem 12.5 implies that there exists $\xi \in [p,q]$ such that

$$\begin{aligned}\int_p^q f\varphi &= \varphi(p)\int_p^\xi f + \varphi(q)\int_\xi^q f \\ &= \varphi(p)[F(\xi) - F(p)] + \varphi(q)[F(q) - F(\xi)].\end{aligned}$$

Consequently, if $q > p \geq K(\varepsilon)$, then

$$\left|\int_p^q f\varphi\right| \leq \frac{\varepsilon}{4M}\cdot 2M + \frac{\varepsilon}{4M}\cdot 2M = \varepsilon.$$

The Cauchy Criterion then implies that $f\varphi \in \mathcal{R}^*(I)$. Q.E.D.

Both the Abel and Chartier-Dirichlet Tests deal with monotone φ. The next test, which is essentially due to the German mathematician Paul Du Bois-Reymond (1831–1889), imposes a different type of condition on φ.

16.11 Du Bois-Reymond's Test. *Let $f, \varphi : I \to \mathbb{R}$, and suppose that:*

(j) $f \in \mathcal{R}^*([a,c])$ *for all* $c \geq a$ *and* $F(x) := \int_a^x f$ *is bounded on* I.

(jj′) φ *is differentiable on* $[a,\infty)$ *and* $\varphi' \in \mathcal{L}(I)$.

(jjj′) $F(x)\varphi(x)$ *has a limit as* $x \to \infty$.

Then $f\varphi \in \mathcal{R}^*(I)$.

Proof. From (j) there exists $M > 0$ such that $|F(x)| \leq M$ for $x \in [a,\infty)$. Therefore $|F(x)\varphi'(x)| \leq M|\varphi'(x)|$ for $x \in I$ which implies that $F\varphi' \in \mathcal{L}(I)$ so that $\lim_{x\to\infty}\int_a^x F\varphi'$ exists. If $c \in I$ is given, then $F\varphi' \in \mathcal{L}([a,c])$ and the Integration by Parts Theorem 12.2(b) implies that $f\varphi \in \mathcal{R}^*([a,c])$ and that $\int_a^c f\varphi = F(c)\varphi(c) - \int_a^c F\varphi'$. Therefore, it follows from Hake's Theorem that $f\varphi \in \mathcal{R}^*(I)$. Q.E.D.

We now give some applications of these tests. The reader is invited to supply the details.

16.12 Examples. (a) Consider $\displaystyle\int_2^\infty \frac{\sin x}{\ln x}\,dx$.

If we let $f(x) := \sin x$ and $\varphi(x) := 1/\ln x$, then Abel's Test does not apply since $f \notin \mathcal{R}^*([2,\infty])$. However, both the Chartier-Dirichlet and Du Bois-Reymond Tests apply to give the convergence of the integral.

(b) Consider $\displaystyle\int_1^{\infty} \frac{\sin x}{x + 2\sin x}\,dx.$

If we let $f(x) := \sin x$ and $\varphi(x) := 1/(x + 2\sin x)$, then Abel's Test does not apply since $f \notin \mathcal{R}^*([1,\infty])$ and the Chartier-Dirichlet Test does not apply since φ is not monotone. However Du Bois-Reymond's Test applies.

(c) Consider $\displaystyle\int_0^{\infty} \frac{x}{x+1}\sin(x^2)\,dx.$

Let $f(x) := \sin(x^2)$ and $\varphi(x) := x/(x+1)$. The Chartier-Dirichlet Test does not apply since φ does not converge to 0, but both Abel's and Du Bois-Reymond's Tests apply.

(d) Consider $\displaystyle\int_0^{\infty} \sqrt{x}\sin(x^2)\,dx.$

If we let $f(x) := \sin(x^2)$ and $\varphi(x) := \sqrt{x}$, then none of these three tests applies. (Why?) However, if we substitute $u = x^2$, we are led to the integral $\frac{1}{2}\int_1^{\infty} u^{-1/4}\sin u\,du$ to which the Chartier-Dirichlet Test applies.

(e) Consider $\displaystyle\int_0^{\infty} \frac{\cos x}{\sqrt{x}}\,dx.$

Since $|(\cos x)/\sqrt{x}| \le 1/\sqrt{x}$ on $(0,1]$, the integral exists on $[0,1]$. If we let $f(x) := \cos x$ and $\varphi(x) := 1/\sqrt{x}$ for $x \ge 1$, the Chartier-Dirichlet Test gives the convergence on $[1,\infty]$.

(f) Consider $\displaystyle\int_0^{\infty} \frac{\cos x}{x}\,dx.$

This integral does not exist, since the integrand is essentially $1/x$ when x is near $x = 0$. (Why do the above tests not apply?)

Exercises

Note. It is to be understood that the integrands equal 0 when they are not properly defined by the formulas.

16.A Establish the uniqueness of the value of the integral of $f \in \mathcal{R}^*([a,\infty])$.

16.B Prove the Fineness Theorem for the intervals $[-\infty, b]$ and $[-\infty,\infty]$.

16.C If $p > 0$, define f by $f(x) := 1/x^p$ for $x \in [1, \infty)$ and $f(\infty) := 0$.
(a) If $p > 1$, show that $f \in \mathcal{L}([1,\infty])$ and that $\int_1^\infty (1/x^p)\,dx = 1/(p-1)$.
(b) If $p \le 1$, show that $f \notin \mathcal{R}^*([1,\infty])$.

16.D For what values of $p > 0$ does $\int_0^\infty (1/x^p)\,dx$ exist? What is its value?

16.E If $s \in \mathbb{R}$, let $g(x) := e^{-sx}$ for $x \in [0,\infty)$ and $g(\infty) := 0$.
(a) If $s > 0$, show that $g \in \mathcal{L}([0,\infty])$ and that $\int_0^\infty e^{-sx}\,dx = 1/s$.
(b) If $s \le 0$, show that $g \notin \mathcal{R}^*([0,\infty])$.

16.F For what values of $s \in \mathbb{R}$ does $\int_{-\infty}^{\infty} e^{-sx}\,dx$ exist? What is its value?

16.G Establish the following version of the **Fundamental Theorem**.
Suppose E is a countable set in $[a,\infty)$ and that $f, F : [a,\infty) \to \mathbb{R}$ satisfy:
(a) F is continuous on $[a,\infty)$ and $F(\infty) := \lim_{x\to\infty} F(x)$ exists.
(b) $F'(x) = f(x)$ for all $x \in [a,\infty) - E$.
Then $f \in \mathcal{R}^*([a,\infty])$ and $\int_a^\infty f = F(\infty) - F(a)$.

16.H (a) If $s > 0$, show that $x \mapsto xe^{-sx}$ is in $\mathcal{L}([0,\infty])$ and $\int_0^\infty xe^{-sx}\,dx = 1/s^2$.
(b) If $s > 0$ and $n \in \mathbb{N}$, show that $x \mapsto x^n e^{-sx}$ is in $\mathcal{L}([0,\infty])$ and that $\int_0^\infty x^n e^{-sx}\,dx = n!/s^{n+1}$.

16.I (a) Does the integral $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$ converge? If so, to what?
(b) Show that if $s > 1$, then $\displaystyle\int_1^\infty \frac{\ln x}{x^s}\,dx = \frac{1}{(s-1)^2}$.

16.J Give a proof of the Cauchy Criterion (Theorem 16.6).

16.K Let $a > 0$, $b \in \mathbb{R}$. Integrate by parts twice, regroup, and use Hake's Theorem to evaluate:

(a) $$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2},$$

(b) $$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}.$$

16.L Evaluate the following integrals:

(a) $\displaystyle\int_0^\infty \frac{dx}{\sqrt{e^x}} = 2,$ (b) $\displaystyle\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = 2,$

(c) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2+1}} \approx 0.881,$ (d) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2+1}} \approx 1.571.$

16.M Establish the convergence of the following integrals:

(a) $\displaystyle\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx,$ (b) $\displaystyle\int_0^\infty \sin(x^2)\,dx.$

(c) $\displaystyle\int_0^\infty \frac{\sin x}{\sqrt[4]{x}}\,dx,$ (d) $\displaystyle\int_0^\infty \sqrt{x}\sin(x^2)\,dx.$

[Note that the integrand in (b) does not converge to 0 when $x \to \infty$, and the integrand in (d) becomes unbounded as $x \to \infty$.]

16.N Establish the divergence of the following integrals:

(a) $\displaystyle\int_0^\infty \frac{\sin^2 x}{x}\,dx,$ (b) $\displaystyle\int_0^\infty \frac{1-\cos x}{x}\,dx.$

16.O Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\int_{-c}^{c} f$ exists for all $c > 0$ and $\lim_{c\to\infty} \int_{-c}^{c} f$ exists, but such that $\int_{-\infty}^{\infty} f$ does not exist.

16.P Show that the following integrals converge. Evaluate (a,b,d).

(a) $\displaystyle\int_{-\infty}^\infty e^{-|x|}\,dx,$ (b) $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx,$

(c) $\displaystyle\int_{-\infty}^\infty \frac{2x\,dx}{e^x - e^{-x}},$ (d) $\displaystyle\int_{-\infty}^\infty x^2 e^{-|x|}\,dx.$

16.Q Discuss the convergence or divergence of the following integrals:

(a) $\displaystyle\int_0^\infty \frac{\ln x}{x^2+1}\,dx,$ (b) $\displaystyle\int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx,$

(c) $\displaystyle\int_0^\infty \frac{\operatorname{Arctan} x}{x}\,dx.$ (d) $\displaystyle\int_0^\infty \frac{\operatorname{Arctan} x}{x^2+1}\,dx,$

(e) $\displaystyle\int_0^\infty \frac{\sqrt{x}\cos x}{x+1}\,dx.$ (f) $\displaystyle\int_0^\infty (1/x)\sin(1/x)\,dx.$

16.R Give examples to show that the Chartier-Dirichlet Test 16.10 may fail if either hypothesis that φ is decreasing, or that $\varphi(x) \to 0$ is dropped.

16.S Suppose that φ is continuous and decreases to 0 as $x \to \infty$.

(a) Show that if $\int_a^\infty \varphi$ converges, then the integrals $\int_a^\infty \varphi(x) \sin x \, dx$ and $\int_a^\infty \varphi(x) \cos x \, dx$ are absolutely convergent.

(b) Show that if $\int_a^\infty \varphi$ diverges, then the integrals $\int_a^\infty \varphi(x) \sin x \, dx$ and $\int_a^\infty \varphi(x) \cos x \, dx$ are conditionally convergent.

16.T Which of the following integrals are convergent? Which are absolutely convergent?

(a) $\displaystyle\int_0^\infty \frac{\cos 2x}{\sqrt{1+x^2}}\, dx,$ (b) $\displaystyle\int_2^\infty \frac{x \sin x}{x^2 - x - 1}\, dx,$

(c) $\displaystyle\int_0^\infty e^{-x/2} \sin x \, dx,$ (d) $\displaystyle\int_2^\infty \frac{\sin x \cos x}{\ln x}\, dx.$

16.U For what values of $s \in \mathbb{R}$ are the following integrals convergent? For which values are they absolutely convergent?

(a) $\displaystyle\int_1^\infty \frac{\cos x}{x^s},$ (b) $\displaystyle\int_0^\infty \frac{\cos x}{x^s},$

(c) $\displaystyle\int_0^\infty \frac{\sin x \, dx}{(x+1)x^s},$ (d) $\displaystyle\int_0^\infty \frac{x^s \cos x}{1+x^2},$

(e) $\displaystyle\int_1^\infty \frac{\sin^2 x \, dx}{x^s},$ (f) $\displaystyle\int_0^\infty \frac{\sin^2 x \, dx}{x^s}.$

16.V Establish the convergence of the following integrals:

(a) $\displaystyle\int_0^\infty \frac{\operatorname{Arctan} x \cos x}{\sqrt{x}}\, dx,$

(b) $\displaystyle\int_1^\infty \frac{(\tanh x + \coth x) \cos x}{x}\, dx,$

(c) $\displaystyle\int_0^\infty \frac{x}{x+1} \sin(x^2)\, dx,$

(d) $\displaystyle\int_0^\infty \frac{x}{\sqrt{x+1}} \sin(x^2)\, dx.$

Further Re-examination

In this section we conduct a guided review of the results in Sections 4–14, discussing the validity of these results for infinite intervals. It is likely that the reader will consult this section primarily when a particular result that requires some change is needed. Results marked by a • can be confidently used for infinite intervals.

As in Section 16, we will comment specifically about intervals of the form $[a, \infty]$, since the case of an interval $[-\infty, b]$ is treated in exactly the same way, and since the case of the interval $[-\infty, \infty]$ is a composite of these two cases. *Throughout this section, we denote* $I := [a, \infty]$ *and* $I_0 := [a, \infty)$. We also recall that if $a \in \mathbb{R}$ and $r > 0$, then $U[a; r] := [a - r, a + r]$, while $U[\infty; r] := [1/r, \infty]$ and $U[-\infty; r] := [-\infty, -1/r]$.

The sets $\{\infty\}, \{-\infty\}$, and $\{-\infty, \infty\}$ are considered to be null sets.

We recall that, in order for a function $f : I \to \mathbb{R}$ to be *integrable*, we always define $f(\pm\infty) := 0$. However, we do not automatically make this extension of *every* function; for example, the primitive F of a integrable function f will also be understood to be defined at ∞, but it is not required to vanish there (and usually does not).

Continuity and Primitives

Before we proceed, we wish to extend to infinite intervals certain terms that were introduced in Section 4 for finite intervals.

17.1 Definition. A function $F : I \to \mathbb{R}$ is said to be **continuous at** ∞ if

$$\lim_{x \to \infty} F(x) = F(\infty) \in \mathbb{R}.$$

Equivalently, this means that given any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $x \in U[\infty; \delta_\varepsilon]$, then $|F(x) - F(\infty)| \leq \varepsilon$.

17.2 Definition. Let f, F be defined on $I \to \mathbb{R}$.

(a) We say that F is a **primitive of** f **on** I if F is continuous on I and $F'(x) = f(x)$ for all $x \in I_0$.

(b) We say that F is an **a-primitive** [respectively, **c-primitive, f-primitive**] **of** f **on** I if F is continuous on I and there exists a null [respectively, countable, finite] set $E \subset I$ such that $F'(x) = f(x)$ for all $x \in I_0 - E$. The set E is called the **exceptional set**.

Remark. Note that we require the primitive (or an a-, c-, or f-primitive) to be continuous at ∞, but *we impose no condition concerning the existence of the derivative of F at ∞*. (Indeed, we never use the phrase "differentiable at ∞".)

We will now be explicit about what we mean by an "indefinite integral" of $f \in \mathcal{R}^*(I)$, where $I := [a, \infty]$.

17.3 Definition. (a) If $u \in I_0$, then the function defined by

$$F_u(x) := \int_u^x f \qquad \text{for} \quad x \in I,$$

is called the **indefinite integral of f with base point** u.

(b) The function defined by

$$F_\infty(x) := \int_\infty^x f = -\int_x^\infty F \qquad \text{for} \quad x \in I,$$

is called the **indefinite integral of f with base point** ∞.

(c) Any function on I that differs from F_a by a constant is called an **indefinite integral of** f.

17.4 Lemma. **(a)** *If F_u and F_v are indefinite integrals of $f \in \mathcal{R}^*(I)$ with base points $u, v \in I$, then*

$$(17.\alpha) \qquad F_v(x) = F_u(x) + \int_v^u f \qquad \textit{for all} \quad x \in I.$$

(b) *If G is any indefinite integral of $f \in \mathcal{R}^*(I)$ and $u \in I$, then we have $G(x) = G(u) + F_u(x)$ for all $x \in I$.*

Proof. (a) By the extension of Theorem 3.11 to $\overline{\mathbb{R}}$, we have

$$F_v(x) = \int_v^x f = \int_v^u f + \int_u^x f = F_u(x) + \int_v^u f.$$

(b) Since F_a and F_u differ by a constant, then $G(x) = K + F_u(x)$ for $x \in I$. Since $F_u(u) = 0$, we have $G(u) = K$, whence $G(x) = G(u) + F_u(x)$. Q.E.D.

Review of Section 4

We now examine the results of Section 4 individually.

(4.5) and **(4.7)** — Fundamental Theorems, I. We change the statement to read: *If $f : I \to \mathbb{R}$ has a primitive (or c-primitive) F on $I := [a, \infty]$, then $f \in \mathcal{R}^*(I)$ and $\int_a^\infty f = F(\infty) - F(a)$.*

Indeed, if f has a c-primitive F on I, then the restriction of F to any compact interval $[a, c]$ is a c-primitive of f, so that $\int_a^c f = F(c) - F(a)$. Since F is continuous at ∞, then

$$\int_a^c f = F(c) - F(a) \to F(\infty) - F(a) \qquad \text{as} \quad c \to \infty.$$

Hence, by Hake's Theorem 16.5, $f \in \mathcal{R}^*(I)$ with integral $F(\infty) - F(a)$.

(4.8) — Fundamental Theorem, II. This theorem is concerned with finding a derivative of an indefinite integral at a finite point $c \in I_0$. Consequently, no change is needed when we deal with a finite base point. It is seen from (17.α) with $v = \infty$ and $u = a$ that $F'_\infty(c + 0) = F'_a(c + 0)$ if $c \in I_0$, so the same result also holds for the indefinite integral with base point ∞.

(4.9) and **(4.10)** These need no change since we differentiate only at finite points. We can also take $u = \infty$.

(4.11) — Fundamental Theorem, II*. No change is needed in the statement.

Indeed, if $f \in \mathcal{R}^*(I)$ then, by Hake's Theorem, f is integrable on every compact interval $[a, c]$ and $F_a(c) \to \int_a^\infty f$ as $c \to \infty$. It follows from Theorem 4.11 that F_a is continuous on I_0. Since $F_a(c) \to \int_a^\infty f$ as $c \to \infty$, then F_a is also continuous at ∞ if we define $F_a(\infty) := \int_a^\infty f$. The continuity of an arbitrary indefinite integral follows readily from Lemma 17.4(b).

Turning to the differentiability of F_a, we note that Theorem 4.11 implies that on each interval $I_n := [a + n - 1, a + n]$ $(n \in \mathbb{N})$ there exists a null set

$Z_n \subset I_n$ such that if $x \notin Z_n$, then F_a is differentiable at x and $F_a'(x) = f(x)$. Since $Z := \bigcup_{n=1}^{\infty} Z_n$ is a null set (by 2.5(d)), the indefinite integral F_a is differentiable outside of the null set Z and $F_a'(x) = f(x)$ for $x \in I_0 - Z$. The differentiability of an arbitrary indefinite integral of f follows from Theorem 17.4(b). Consequently, any indefinite integral of f is an a-primitive of f.

(4.12) Change the hypothesis to: If $f \in \mathcal{R}^*(I)$ is a regulated function $\cdots$.

For, $f(x) := 1/x$ for $x \in [1, \infty)$ is a regulated function, but it is not in $\mathcal{R}^*([1,\infty))$ since the indefinite integral $F(x) := \ln x$ does not have a limit *that belongs to* $\mathbb{R}$ at ∞. If $f \in \mathcal{R}^*(I)$, then the indefinite integrals F_u are finite-valued, continuous on I, and have derivatives outside a countable set.

17.5 Remark. If $f \in \mathcal{R}^*(I)$ is continuous on I, then it follows from the extensions of 4.9 and 4.11 that f has a primitive on I. Indeed, 4.11 assures that the indefinite integral $F_a(x) := \int_a^x f$ is continuous on I and 4.9 guarantees that $F_a'(x) = f(x)$ for all $x \in I_0$; hence F_a is a primitive of f on I.

Review of Section 5

(5.1) No changes are needed, except that $J_j \subseteq U[t_j; \delta(t_j)]$ in (c) and (d).

(5.3) — Saks-Henstock Lemma. No changes are needed.

Note that the pair (J_s, ∞), where $J_s = [x_{s-1}, \infty]$, makes the contribution $f(\infty)l(J_s) = 0$ to $S(f; \dot{\mathcal{P}})$ and that $\int_{J_s} f = \int_{x_{s-1}}^{\infty} f$, which is small when x_s is large, by Hake's Theorem.

(5.4) and **(5.5)** These results follow from 5.3, so no changes are needed.

(5.6) See the comments concerning 4.11.

(5.7) and **(5.8)** The discussion here is for bounded sets.

(5.9) — Differentiation Theorem. The theorem implies that on every interval $I_n := [a+n-1, a+n]$ in I there is a null set Z_n such that $F'(x) = f(x)$ for $x \in I_n - Z_n$. Now let $Z := \bigcup_n Z_n$.

(5.10) — Characterization of null functions. No change is needed in the statement.

Note that ψ and $|\psi|$ are null functions on I if and only if their restrictions to every interval I_n are null functions. In (d), if $c = \infty$, recall that $(17.\alpha)$ with $v = \infty$ and $u = a$ implies that $\Psi_\infty(x) = \Psi_a(x) - \int_a^\infty \psi$.

(5.11) The definition makes sense for I as stated.

Note that if the tag $t_s = \infty$, then we must also have $v_s = \infty$ and $|F(v_s) - F(u_s)| = |F(\infty) - F(u_s)|$. In particular, this implies that if $\infty \in E$ and $F \in NV_I(E)$, then F is continuous at ∞.

(5.12) — Characterization Theorem. Note that the null set Z contains ∞.

The proof of $(\Rightarrow)$ needs no change.

For $(\Leftarrow)$, we will use the Arctangent Lemma, given in Appendix C. Suppose that Z is a null set and $G : I \to \mathbb{R}$ is such that $G'(x) =: f(x)$ for all $x \in I - Z$ and $G \in NV_I(Z)$; then G is continuous on I. Let $f(x) := 0$ for $x \in Z$ and let δ_ε be defined on Z as in Definition 5.11 and (using the Straddle Lemma 4.11) such that if $t \in I - Z$ and $t \in [u, v] \subseteq [t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)]$ then $|G(v) - G(u) - f(t)(v - u)| \le \varepsilon(v - u)/[2\pi(1 + t^2)]$. It follows that if $\dot{\mathcal{P}} := \{([u_i, v_i], t_i)\}_{i=1}^n$ is a δ_ε-fine partition of I, then since $f(t_i) = 0$ for $t_i \in Z$, we have (see Corollary C.4)

$$\begin{aligned} &\left|G(\infty) - G(a) - S(f;\dot{\mathcal{P}})\right| \\ &\le \sum_{t_i \in Z} |G(v_i) - G(u_i)| + \sum_{t_i \in I - Z} |G(v_i) - G(u_i) - f(t_i)(v_i - u_i)| \\ &\le \varepsilon + \sum_{t_i \in I - Z} \frac{\varepsilon(v_i - u_i)}{2\pi(1 + t_i^2)} \le \varepsilon + \varepsilon \sum_{t_i \in I - Z} \vartheta([u_i, v_i]) \le 2\varepsilon. \end{aligned}$$

Therefore $f \in \mathcal{R}^*(I)$ and $\int_a^\infty f = G(\infty) - G(a)$.

Review of Section 6

It will be useful to expand on the earlier discussion somewhat.

17.6 Definition. A **step function** on $I_0 := [a, \infty)$ is a function $s : I_0 \to \mathbb{R}$ such that there exists a partition $\{[c_{i-1}, c_i]\}_{i=1}^{n+1}$ of $[a, \infty]$ and real numbers $\{\alpha_i\}_{i=1}^n$ such that

$$s(x) = \begin{cases} \alpha_i & \text{for } x \in (c_{i-1}, c_i),\ i = 1, \cdots, n; \\ 0 & \text{for } x \in (c_n, \infty). \end{cases}$$

Remarks. (a) The step function s also has values at the points c_i, which may differ from the values α_i, but these values are usually not important. Frequently we consider step functions defined on $[a, \infty]$, in which case we define $s(\infty) := 0$.

(b) A step function is said to be **complex-valued** in case $\alpha_i \in \mathbb{C}$.

17.7 Theorem. *A function $f : I_0 \to \mathbb{R}$ has the property that every restriction $f|[\alpha, \beta]$ is measurable in the sense of Definition 6.1 if and only if there exists a sequence (s_k) of step functions on I_0 that converges to f a.e. on I_0.*

Proof. ($\Rightarrow$) Let $J^n := [a+n-1, a+n)$ for $n \in \mathbb{N}$ and let $(s_k^n)_{k=1}^\infty$ be a sequence of step functions on J^n such that $\lim_{k\to\infty} s_k^n(x) = f(x)$ a.e. on J^n. Now let t_k be defined by

$$t_k(x) := \begin{cases} s_k^i(x) & \text{for } x \in J^i,\ i = 1, \cdots, k, \\ 0 & \text{for } x \geq a+k, \end{cases}$$

so that t_k is a simple function on I_0 and $t_k(x) \to f(x)$ a.e. on I_0.

($\Leftarrow$) Let (t_k) be a sequence of step functions on I_0 that converges a.e. to f on I_0 and let $J := [\alpha, \beta]$. Then the sequence of restrictions $(t_k|J)_{k=1}^\infty$ is a sequence of step functions on J that converges a.e. on J to $f|J$. Q.E.D.

17.8 Definition. A function $f : I_0 \to \mathbb{R}$ is said to be **measurable** if it satisfies the properties in Theorem 17.7. The collection of all measurable functions on I_0 will be denoted by $\mathcal{M}(I_0)$. If $f \in \mathcal{M}(I_0)$ has been extended to I by $f(\infty) := 0$ we write $f \in \mathcal{M}(I)$.

(6.2) Every step [respectively, continuous, monotone, null] function on I_0 belongs to $\mathcal{M}(I_0)$ and its extension to I also belongs to $\mathcal{M}(I)$. However, the extension of a continuous (or monotone) function to I by defining $f(\pm\infty) := 0$ need not lead to a continuous function (and almost never leads to a monotone one).

(6.3) Except for statement (b), the validity of these results for I is clear. In (b), we understand that if $h \in \mathcal{M}(I)$ is such that $h(x) \neq 0$ on I_0, then $1/h$ is defined by $(1/h)(x) := 1/h(x)$ for $x \in I_0$ and $(1/h)(\infty) := 0$.

(6.4) – (6.6) These statements deal with behavior of the functions at individual points. Hence they extend to measurable functions on I.

(6.7) ($\Leftarrow$) If there exists a sequence (h_k) of continuous functions that converges a.e. on I to f, then the restriction to each interval $[\alpha, \beta]$ has the same property.

($\Rightarrow$) This direction is also true, but its proof requires a form of Luzin's theorem; it will be given in 19.19

(6.8) No change is needed in the statement.

If $J := [\alpha, \beta] \subset I_0$, then by Corollary 3.8, the restriction $f|J$ is measurable on J. Thus $f \in \mathcal{M}(I)$.

(6.9) No change is needed.

(6.10) This result does not hold for I. For, let $f(x) := 1$ for $x \in I_0$.

(6.11) No change is needed.

(6.12) — Multiplier Theorem. No change is needed.

Note that φ is assumed to be monotone on $I = [a, \infty]$; this means that φ is bounded on I_0. If φ is only assumed to be monotone on I_0, we need to impose the additional hypothesis that φ is bounded. See also the remarks concerning (10.12) below. Also see the tests 16.9–16.11.

(6.14) We will discuss measurable and integrable sets in $\mathbb{R}$ at length in later sections. Integrable sets in $\mathbb{R}$ are defined in 18.1 and measurable sets in 18.4.

(6.16) It was seen in Theorem 10.8 that one can define the integral over an arbitrary measurable set in I when $f \in \mathcal{L}(I)$. Further discussion of indefinite integrals is given in 19.20–19.23.

Review of Section 7

(7.1) The terminology needs no change. We denote the collection of all absolutely integrable functions on I by $\mathcal{L}(I)$ or $\mathcal{L}([a, \infty])$.

(7.3) If $\varphi : I \to \mathbb{R}$, we can define $\mathrm{Var}(\varphi; I)$ exactly as in (7.α). (*Here we do not require that* $\varphi(\infty) = 0$.)

Note that this definition implies that φ is bounded on I. Moreover, the restriction of φ to every interval $[a, c] \subset I_0$ belongs to $BV([a, c])$ and $\mathrm{Var}(\varphi; [a, c]) \le \mathrm{Var}(\varphi; I)$. Since the map $c \mapsto \mathrm{Var}(\varphi; [a, c])$ is increasing and bounded, then

$$\lim_{c \to \infty} \mathrm{Var}(\varphi; [a, c]) \le \mathrm{Var}(\varphi; I). \tag{17.β}$$

The example $\varphi(x) := 0$ for $x \in I_0$ and $\varphi(\infty) := 1$, shows that we can have strict inequality in (17.β). See also the remarks made below in connection with Section 14.

(7.5) — Characterization of Absolute Integrability. No change in the statement is needed (read $b = \infty$).

($\Rightarrow$) No change in proof.

($\Leftarrow$) The only change in the proof is to define $\delta^*(t)$ as in (7.δ) for $t \in I_0$ and $\delta^*(\infty) := \delta_\varepsilon(\infty)$. Alternatively, we could argue that since $F \in BV(I)$, then $F \in BV([a, c])$ for every $c \in I_0$. Hence, by the theorem, $|f|$ is integrable on every $[a, c]$ with $\mathrm{Var}(F; [a, c]) = \int_a^c |f|$. Since $c \mapsto \mathrm{Var}(F; [a, c])$ is increasing and bounded as is seen in the inequality (17.β) given above, it is convergent as $c \to \infty$. Therefore, Hake's Theorem implies that $\int_a^\infty |f|$ exists and equals $\lim_c \mathrm{Var}(F; [a, c])$. The equality (7.$\beta$) follows from the observation just made, inequality (17.β) and the inequality at the end of the first part of the proof.

(7.7) — Comparison Test. No change is needed.

(7.8) and **(7.9)** No change is needed.

(7.10) This result is not true for I, since a nonzero constant function is not integrable on $[a, \infty]$.

(7.11) – **(7.13)** No change is needed.

Review of Section 8

(8.2) The definition makes sense on I whether we require the functions to vanish at ∞ or not.

(8.3) — Uniform Convergence Theorem. This theorem may fail on I.

For example, let $I := [0, \infty]$ and let $f_k(x) := 1/k$ for $x \in [0, k]$ and $f_k(x) := 0$ for $x \in (k, \infty]$. Then $(f_k) \subset \mathcal{R}^*(I)$ and converges uniformly on I to the function $f(x) := 0$ for all $x \in I$. However, $\int_0^\infty f_k = 1$ for all k, while $\int_0^\infty f = 0$.

(8.5) — Monotone Convergence Theorem. No change is needed in the statement (read $b = \infty$).

We consider the case when the sequence (f_k) is increasing. We note that it is no loss of generality to suppose that $f_k(x) \geq 0$ for all $x \in I$, since otherwise we can replace f_k by $f_k - f_1$.

($\Rightarrow$) No change is needed.

($\Leftarrow$) Because of the importance of this result, we will give *two* methods for extending this result to $I = [a, \infty]$.

Method 1. Let $A := \sup\{\int_a^\infty f_k : k \in \mathbb{N}\}$ so that the monotone sequence $(\int_a^\infty f_k)_k$ converges to A. If $c > a$, then since $f_k \geq 0$, the sequence $(\int_a^c f_k)$ is increasing and is bounded by A. The Monotone Convergence Theorem 8.5 applied to $I_c := [a, c]$ implies that f is integrable on I_c and that

$$\int_a^c f = \lim_{k\to\infty} \int_a^c f_k. \tag{17.γ}$$

Further, for each $k \in \mathbb{N}$, since $f_k \geq 0$, the indefinite integral of f_k is increasing on I and, since f_k is integrable on I, Hake's Theorem 16.5 implies that

$$\int_a^c f_k \leq \lim_{c\to\infty} \int_a^c f_k = \int_a^\infty f_k \leq A.$$

Therefore, using (17.γ), we conclude that if $c \geq a$, then

$$\int_a^c f = \lim_{k\to\infty} \int_a^c f_k \leq \lim_{k\to\infty} \int_a^\infty f_k \leq A.$$

Since $f \geq 0$, the function $c \mapsto \int_a^c f$ is increasing on $[a, \infty)$ and, since it is bounded, we conclude that it is convergent as $c \to \infty$. Therefore it follows from Hake's Theorem that f is integrable on $I = [a, \infty]$ and that

$$(17.\delta) \qquad \int_a^\infty f = \lim_{c\to\infty} \int_a^c f \leq A.$$

We now show that equality holds in $(17.\delta)$. Let $\varepsilon > 0$ be given; since

$$A = \lim_{k\to\infty} \int_a^\infty f_k = \lim_{k\to\infty} \lim_{c\to\infty} \int_a^c f_k,$$

there exists $p \in \mathbb{N}$ such that

$$A - \tfrac{1}{2}\varepsilon \leq \lim_{c\to\infty} \int_a^c f_p \leq A.$$

Therefore, there exists c_ε such that if $c \geq c_\varepsilon$, then

$$A - \varepsilon \leq \int_a^c f_p \leq A.$$

But, since $\int_a^c f_p \leq \int_a^c f$, the relation $(17.\delta)$ implies that

$$A - \varepsilon \leq \int_a^c f \leq A$$

for any $c \geq c_\varepsilon$. Therefore, we conclude that $A - \varepsilon \leq \int_a^\infty f \leq A$. Since $\varepsilon > 0$ is arbitrary, this implies that $\int_a^\infty f = A$.

Method 2. We now use the Arctangent Lemma (proved in Appendix C) to prove the Monotone Convergence Theorem for $I = [a, \infty]$, rather than to obtain the unbounded case from the compact case, as just done.

The difficulty in the proof of 8.5 is that the inequality $(8.\beta)$ leads to the upper bound of the first term in $(8.\gamma)$ being $(b - a)\varepsilon$, which depends on the length of the interval. The Arctangent Lemma will enable us to replace the (unbounded) length function by the (bounded) ϑ-length function. One needs to make the following changes in the proof of 8.5.

Replace the sentence ending with the formula $(8.\beta)$ by:

Also, since $f(x) = \lim f_k(x)$, then for each $x \in I \cap \mathbb{R}$ there exists an integer $k(x) \geq r$ such that

$$(8.\beta') \qquad 0 \leq f(x) - f_{k(x)}(x) < \frac{\varepsilon}{2\pi(1 + x^2)},$$

and we take $k(x) := r$ if $x = \infty$. Let γ be the function in Corollary C.4 for $x \in \mathbb{R}$ and let $\gamma(\infty) := 1$. Define $\delta_\varepsilon(t) := \min\{\delta_{k(t)}(t), \gamma(t)\}$ for $t \in I$, so that δ_ε is a gauge on I. Now let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ be a δ_ε-fine partition of I. We want to show that $|S(f;\dot{\mathcal{P}}) - A| \le 3\varepsilon$. By the Triangle Inequality we have

Formula (8.γ)

(i) If the tag $t_i \in \mathbb{R}$, then by (8.β') we have

$$\left|f(t_i) - f_{k(t_i)}(t_i)\right| < \frac{\varepsilon}{2\pi(1+t_i^2)}.$$

Since $I_i \subseteq [t_i - \gamma(t_i), t_i + \gamma(t_i)]$ for $t_i \in \mathbb{R}$, it follows from Corollary C.4 that

$$(8.\beta'') \qquad \left|f(t_i)l(I_i) - f_{k(t_i)}(t_i)l(I_i)\right| \le \frac{\varepsilon l(I_i)}{2\pi(1+t_i^2)} \le \varepsilon\vartheta(I_i).$$

Further, if the tag $t_i = \infty$, then all of the functions f_k, f vanish at t_i and so the left side of (8.β'') also vanishes and therefore (8.β'') remains true for all i. Consequently we conclude, using Lemma C.3, that

$$\left|\sum_{i=1}^n f(t_i)l(I_i) - \sum_{i=1}^n f_{k(t_i)}(t_i)l(I_i)\right| \le \sum_{i=1}^n \varepsilon\vartheta(I_i) = \varepsilon\vartheta(I) \le \varepsilon.$$

Therefore, the first term on the right in (8.γ) is dominated by ε.

Now resume the rest of the argument in the proof, beginning with (ii).

(8.6) No change is neeeded.

(8.7) — Fatou's Lemma. No change is needed.

(8.8) — Dominated Convergence Theorem. No change is needed.

(8.9) — Mean Convergence Theorem. No change is needed.

(8.10) The definition makes sense when $b = \infty$.

(8.11) — Equi-integrability Theorem. No change in the statement.

(8.12) — Gordon's Theorem. No change in the statement.

Again the upper bounds (for example, in (8.o) and (8.ρ)) involve the length of the interval, so we need to use the ϑ-length instead. Our argument will be much like the proof of 8.11, but we will present all of the details.

($\Leftarrow$) We will first show that the sequence $(\int_I f_k)$ is a Cauchy sequence. To do so, let γ be as in Corollary C.4 and, given $\varepsilon > 0$, let δ_ε be a gauge as

in the stated condition in the theorem. With no loss of generality, we may suppose that $\delta_\varepsilon(x) \le \gamma(x)$ for all $x \in I_0$.

Now let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ be a fixed δ_ε-fine partition of I. Since $\dot{\mathcal{P}}$ has only a finite number of tags $t_1, \cdots, t_{n-1}$ in I_0, there exists an integer $K_\varepsilon \ge K_{\dot{\mathcal{P}}}$ such that if $h \ge k \ge K_\varepsilon$, then

$$\left|f_k(t_i) - f_h(t_i)\right| \le \frac{\varepsilon}{2\pi(1+t_i^2)}.$$

Since $t_i \in I_i \subseteq [t_i - \gamma(t_i), t_i + \gamma(t_i)]$, it follows from Corollary C.4 that if $h \ge k \ge K_\varepsilon$ and $i = 1, \cdots, n-1$, then

$$\text{(17.}\varepsilon\text{)} \qquad \left|f_k(t_i) - f_h(t_i)\right| l(I_i) \le \varepsilon \vartheta(I_i).$$

Since all of the functions f_k vanish at the tag $t_n = \infty$, if we add the terms in (17.ε), we conclude that if $h \ge k \ge K_\varepsilon$, then

$$\text{(8.}o'\text{)} \qquad \left|S(f_k; \dot{\mathcal{P}}) - S(f_h; \dot{\mathcal{P}})\right| \le \sum_{i=1}^n \left|f_k(t_i) - f_h(t_i)\right| l(I_i) \le \varepsilon \vartheta(I) \le \varepsilon.$$

If we let $h \to \infty$ in this inequality, we obtain

$$\text{(8.}\pi'\text{)} \qquad \left|S(f_k; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{P}})\right| \le \varepsilon \qquad \text{for} \quad k \ge K_\varepsilon.$$

It follows from the condition in the theorem that $|\int_I f_k - S(f_k; \dot{\mathcal{P}})| \le \varepsilon$ for all $k \in \mathbb{N}$ such that $k \ge K_\varepsilon \ge K_{\dot{\mathcal{P}}}$. Thus, if $h \ge k \ge K_\varepsilon$, then the condition in the theorem and (8.o') give

$$\begin{aligned} \left|\int_I f_k - \int_I f_h\right| &\le \left|\int_I f_k - S(f_k; \dot{\mathcal{P}})\right| + \left|S(f_k; \dot{\mathcal{P}}) - S(f_h; \dot{\mathcal{P}})\right| \\ &\quad + \left|S(f_h; \dot{\mathcal{P}}) - \int_I f_h\right| \le 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then $(\int_I f_k)$ is a Cauchy sequence, and so converges to some number $A \in \mathbb{R}$. If we let $h \to \infty$ in the above inequality, we obtain

$$\text{(8.}\rho'\text{)} \qquad \left|\int_I f_k - A\right| \le 3\varepsilon \qquad \text{for} \quad k \ge K_\varepsilon.$$

To show that $f \in \mathcal{R}^*(I)$ with integral A, given $\varepsilon > 0$, we see that if $\dot{\mathcal{P}} \ll \delta_\varepsilon$ and $k \ge K_\varepsilon$, then

$$\begin{aligned} \left|S(f; \dot{\mathcal{P}}) - A\right| &\le \left|S(f; \dot{\mathcal{P}}) - S(f_k; \dot{\mathcal{P}})\right| + \left|S(f_k; \dot{\mathcal{P}}) - \int_I f_k\right| + \left|\int_I f_k - A\right| \\ &\le \varepsilon + \varepsilon + 3\varepsilon = 5\varepsilon. \end{aligned}$$

where we have used $(8.\pi')$ to estimate the first term, $(8.\xi)$ to estimate the second term, and $(8.\rho')$ to estimate the third term. Since $\varepsilon > 0$ is arbitrary, then f is integrable on I with integral A.

$(\Rightarrow)$ If $(8.*)$ holds and $\varepsilon > 0$, there exists $M_\varepsilon \in \mathbb{N}$ such that if $k \geq M_\varepsilon$ then $(8.\sigma)$ holds. Since $f \in \mathcal{R}^*(I)$, there exists a gauge δ_ε such that if $\dot{\mathcal{P}} \ll \delta_\varepsilon$, then $(8.\tau)$ holds. Now choose $K_{\dot{\mathcal{P}}} \geq M_\varepsilon$ such that if $k \geq K_{\dot{\mathcal{P}}}$, then $|f_k(t_i) - f(t_i)| \leq \varepsilon/[3 \cdot 2\pi(1+t_i^2)]$ for $i = 1, \cdots, n$, whence

$$(8.\upsilon') \qquad \begin{aligned} \left|S(f_k;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})\right| &\leq \sum_{i=1}^{n} |f_k(t_i) - f(t_i)| l(I_i) \\ &\leq (\varepsilon/3)\sum_{i=1}^{n} \vartheta(I_i) \leq \varepsilon/3. \end{aligned}$$

Consequently, if $k \geq K_{\dot{\mathcal{P}}} \geq M_\varepsilon$, then

$$\begin{aligned} \left|S(f_k;\dot{\mathcal{P}}) - \int_I f_k\right| &\leq \left|S(f_k;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})\right| + \left|S(f;\dot{\mathcal{P}}) - \int_I f\right| \\ &\quad + \left|\int_I f - \int_I f_k\right| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

where we have used $(8.\upsilon'), (8,\tau)$, and $(8.\sigma)$. Since $\varepsilon > 0$ is arbitrary, it follows that $(8.\xi)$ holds for $k \geq K_{\dot{\mathcal{P}}}$.

(8.13) No change is needed.

(8.15) The argument given in the discussion related to (5.12) shows that the uniformity hypotheses given here imply that the sequence (f_k) is equi-integrable on $I = [a, \infty]$.

Review of Section 9

(9.1) — Integrability Theorem. Since (as seen in Theorem 17.7) a measurable function on I_0 is the a.e. limit of a sequence of step functions on I_0, no change is needed.

The two Remarks do not hold for an unbounded interval, as is seen by taking the characteristic function of I in (a), or the interval I in (b).

(9.2) — Measurable Limit Theorem. It follows that $f|[\alpha,\beta]$ is measurable for every compact interval $[\alpha,\beta]$. Thus, by Theorem 17.7, f is measurable on I.

(9.3) — Increasing Sequence Theorem. No change is needed.

(9.4) — Beppo Levi's Theorem. No change is needed.

(9.5) This definition has nothing to do with an interval.

(9.6) The definition makes sense when I is an unbounded interval.

(9.7) – (9.9) No change is needed.

(9.11) The definition makes sense.

(9.12) — Completeness Theorem. No change is needed.

(9.13) No change is needed.

(9.14) — Truncation Theorem. No change is needed.

(9.15) — Density Theorem. The proof of (a) needs no change.

For (b), note that there exists a step function s_k with $\|f^{[n]} - s_k\| \leq \varepsilon/4$. Since s_k vanishes outside a compact interval, we can use the construction in the proof of Theorem 5.7 to construct a continuous function h_k vanishing oustide of a compact interval such that $\|s_k - h_k\| \leq \varepsilon/4$. Thus $h_k \in \mathcal{L}(I)$ and $\|f - h_k\| \leq \varepsilon$.

(9.16) No change is needed.

(9.17) — Riemann-Lebesgue Theorem. No change is needed.

Review of Section 10

(10.1) – (10.3) Integrable sets in $\mathbb{R}$ are the analogue of measurable (= integrable) sets in $I := [a, b]$. Results paralleling those in 10.1 are given in 18.2 and 18.3 for integrable sets. Gereral Lebesgue measurable sets in $\mathbb{R}$ can have infinite measure, which poses certain problems; see 18.4–18.8.

In Sections 19 and 20, we will discuss abstract measurable spaces and measures on them. In Exercises 20.A and 20.B we give the basic properties of measures.

(10.4) – (10.6) See Theorems 19.2 and 19.15.

(10.7) In Theorem 19.14 this result is extended to measurable functions on a general measure space.

(10.8) No change is needed.

(10.9) These ideas are discussed in 19.20–19.23.

(10.10) The general notion of absolute continuity for measures is taken up in 19.27. The analogue of 10.10 is presented in 20.16.

(10.11) No change is needed.

The Riemann-Stieltjes integral is ordinarily treated only for compact intervals $[a, b]$, and we have limited ourselves to this case in Appendix H. Thus some changes are needed for the statements in the final results in Section 10.

We have noted in our review of Section 7 that if $I := [a, \infty]$, then the definition of $\varphi \in BV(I)$ is just as in Definition 7.3. Recall also that such a function is bounded on I, and note that φ can be given as the difference of two bounded increasing functions on I.

(10.12) — Multiplier Theorem. Change the statement to:

If $f \in \mathcal{R}^*(I)$ and $\varphi \in BV(I)$, then $f \cdot \varphi \in \mathcal{R}^*(I)$ and

$$(10.\xi') \qquad \int_a^\infty f\varphi = \lim_{b\to\infty} \int_a^b \varphi\, dF = \lim_{b\to\infty} \Big[F(b)\varphi(b) - \int_a^b F\, d\varphi\Big],$$

where $F(x) := \int_a^x f$ for $x \in I$ and the integrals on the right are Riemann-Stieltjes integrals.

We will consider the case that φ is increasing on I. Then 10.12 implies that $f\varphi \in \mathcal{R}^*([a, b])$ for every $b \in I_0$. Since $f \in \mathcal{R}^*(I)$, Hake's Theorem 16.5 and the Cauchy Criterion 16.6 imply that given $\varepsilon > 0$, there exists $K(\varepsilon) \in I_0$ such that if $K(\varepsilon) \le p < q < \infty$, then $|\int_p^q f| \le \varepsilon$. If M is an upper bound for $|\varphi(x)|$ on I, the Second Mean Value Theorem 12.5 implies that there exists $\xi \in [p, q]$ such that

$$\int_p^q f\varphi = \varphi(p) \int_p^\xi f + \varphi(q) \int_\xi^q f,$$

whence it follows that $|\int_p^q f\varphi| \le 2M\varepsilon$. By the Cauchy Criterion the limit $\int_a^b f\varphi$ exists as $b \to \infty$, so Hake's Theorem implies that $\int_I f\varphi$ exists. The validity of $(10.\xi)$ for all $b \in I_0$ now implies $(10.\xi')$.

It might be asked if the Multiplier Theorem remains true when φ has *locally* bounded variation in the sense to be discussed later in this section under our review of Section 14. The answer is "No", as is seen by taking $\varphi(x) := x$ for $x \in [1, \infty)$, $\varphi(\infty) := 0$, and $f(x) := 1/x^2$ for $x \in [1, \infty)$.

(10.13) Change the statement to read:

If $f \in \mathcal{R}^*(I)$ has a c-primitive F on I and if $\varphi \in BV(I)$, then $f \cdot \varphi$ has a c-primitive on I that is given by (10.o) for $x \in I_0$ and by

$$(10.o') \qquad \Pi(\infty) := \lim_{x\to\infty} \int_a^x \varphi\, dF = \lim_{x\to\infty} \Big[\varphi(x)F(x) - \int_a^x F\, d\varphi\Big].$$

The given argument shows that Π is continuous and has a derivative equal to $f(x) \cdot \varphi(x)$ for x outside of a countable set, so that $\int_a^x f \cdot \varphi = \Pi(x)$ for $x \in I_0$. By 10.12 and Hake's Theorem we know that $\lim_{x\to\infty} \int_a^x f \cdot \varphi$ exists, and the existence of the other limits then follows from (10.ξ).

(10.14) Change the final assertion to: given by (10.o) and (10.o′).

Review of Section 11

The extensions of the results in this section to $\mathbb{R}$ (and general measure spaces) are discussed primarily in Section 20.

(11.1) – (11.2) See Lemma 20.1 and the material before it, where the discussion is for an abstract measure space.

(11.3) As seen in Example 20.2, Egorov's Theorem 11.3 is not true for infinite measure spaces. However, extensions of the theorem to such spaces are given in some detail in 20.3–20.12.

(11.4) An extension of Luzin's Theorem is given in Theorem 19.18.

(11.5) See the discussion in Section 20 for general measure spaces.

(11.6) No change is needed.

(11.7) No change is needed in (a). Since we have not seriously discussed the integral in a general space, we cannot prove (b) in that context. But it is noted before 20.14 that (b) is true for $(\mathbb{R}, \mathbb{M}, \lambda)$.

(11.9) The theorem extends, but the proof that $g_k \to f$ a.u. used Egorov's Theorem and needs to be replaced; see 20.13.

(11.10) No change is needed; see 20.14.

(11.11) No change is needed; see 20.15.

(Diagram 11.1) Since Egorov's Theorem fails, this diagram does not remain true for $\mathbb{R}$. Two diagrams are given after 20.15; the first diagram is for

an infinite interval, and no additional conditions on the functions. The second diagram is for sequences that are dominated by an absolutely integrable function.

(11.13) – (11.14) It is shown in the Remarks before 20.16 that neither of these Vitali Theorems is valid without some change. However, very similar results are given in 20.19 and 20.20, using the notion of equifiniteness.

Review of Section 12

In the statements read $[a, \infty]$ for $[a, b]$.

(12.1) — Integration by Parts. No change is needed.

If f, g have c-primitives F, G on I, then 12.1 implies that for every $c \in (a, \infty)$, the function $Fg + fG$ has a c-primitive FG on $[a, c]$ and that

$$\int_a^c (Fg + fG) = F(c)G(c) - F(a)G(a).$$

Since F, G are continuous at ∞, Hake's Theorem 16.5 implies that $Fg+fG \in \mathcal{R}^*(I)$ and that (12.α) holds with $b = \infty$. If $Fg \in \mathcal{R}^*(I)$, then Theorem 3.1 implies that $fG \in \mathcal{R}^*(I)$ and that (12.β) holds.

(12.2) — Integration by Parts*. No change is needed. (Here $b = \infty$.)

We take the base point to be a.

(a) If $b \in I_0$, the argument given shows that $Fg + fG$ belongs to $\mathcal{R}^*([a, b])$ and that (12.α) holds. Since $F(b) \to F(\infty)$ and $G(b) \to G(\infty)$, Hake's Theorem implies that $Fg + fG \in \mathcal{R}^*(I)$ and that (12.α) holds with $b = \infty$.

(b) As before.

(12.4) — First Mean Value Theorem. No change is needed. (Here $b = \infty$.)

The Bolzano Intermediate Value Theorem holds for a continuous function on $I = [a, \infty]$.

(12.5) — Second Mean Value Theorem. Add the hypothesis that g is continuous on $[a, \infty]$.

For $n \in \mathbb{N}$, there exists $\xi_n \in [a, a + n]$ such that (12.λ) holds with $b = a + n$ and $\xi = \xi_n$. If $\xi \in I$ is the limit of a subsequence of (ξ_n), then the continuity of g and F on I imply that

$$(12.\lambda') \qquad \int_a^\infty fg = g(a) \int_a^\xi f + g(\infty) \int_\xi^\infty f,$$

which is (12.λ) with $b = \infty$.

(12.6) — Bonnet's Mean Value Theorem. The result holds when g is continuous on I and $g(a) = 0$, but then this follows from the Second Mean Value Theorem.

(12.8) — Hake's Theorem. The case $b = \infty$ is Theorem 16.5.

(12.10) The hypothesis makes sense when $b = \infty$. It is understood that $f(\infty, t) = 0$ for all $t \in T$.

(12.11) — Limit Theorem. No change is needed.

(12.12) — Continuity Theorem. No change is needed.

(12.13) — Differentiation Theorem. No change is needed.

(12.14) — Leibniz's Theorem. Here we are dealing with the integral of f over compact intervals $[u(t), v(t)] \subset \mathbb{R}$.

(12.15) — Integration Theorem. No change is needed.

Review of Section 13

This is one of the few places in this book where we allow functions to have values in $\overline{\mathbb{R}}$; here we do so only for the substitution function Φ (and its inverse Ψ, when this inverse exists). Even these functions are allowed to take $\pm\infty$ only at the endpoints of their intervals of definition. Note that Φ appears only in (i) $f \circ \Phi$ or $F \circ \Phi$, or (ii) in the limits of integration. Since we do not define the derivative at $\pm\infty$, the functions φ, ψ are understood to vanish at these points.

In this review, we will limit our attention to an interval $J := [c, d]$ on which f and F are defined and permit $d = \infty$, and to an interval $I := [a, b]$ on which φ and Φ are defined and permit $b = \infty$ or $\Phi(b) = \infty$. We let $J_0 := [c, d)$, and $I_0 := [a, b)$ if either $b = \infty$ or $\Phi(b) = \infty$. (The other cases are handled similarly.)

In Definition 17.2(b) we defined $F : J \to \mathbb{R}$ to be a c-primitive of f on J if F is continuous on J and $F'(x) = f(x)$ for $x \in J - C_f$, where C_f is countable. We will also say that $\Phi : I \to \overline{\mathbb{R}}$ is a c-primitive of φ on I if Φ is continuous on I (which requires $\lim_{x \to b} \Phi(x) = \Phi(b) \in \overline{\mathbb{R}}$) and $\Phi'(x) = \varphi(x)$ for $x \in I - C_\varphi$, where C_φ is countable.

(13.1) — First Substitution Theorem, I. With the above terminology and the condition that $\Phi(I_0) \subseteq J_0$, the statement stands.

Let $d \in C_f$, $b \in C_\varphi$. As in the argument, $F \circ \Phi : I \to \mathbb{R}$ is continuous on I and differentiable to $(f \circ \Phi) \cdot \varphi$ on $I - C$. Thus the extension of 4.7 implies that $(f \circ \Phi) \cdot \varphi$ belongs to $\mathcal{R}^*(I)$ and that equation (13.α) holds.

17.9 Example. **(a)** Consider $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$, where the integrand equals 0 at $x = 1$.

Let $u = \Phi(x) := \sqrt{x^2-1}$ for $x \in [1, \infty)$ and $\Phi(\infty) := \infty$, so that Φ is a c-primitive of $\varphi(x) := x/\sqrt{x^2-1}$ for $x \in [1, \infty] - \{1, \infty\}$. The function $f(u) := 1/(u^2+1)$ for $u \in [0, \infty)$ and $f(\infty) := 0$ has a primitive $F(u) = \operatorname{Arctan} u$ (here $F(\infty) := \pi/2$) on $J := [0, \infty]$. Evidently $\Phi([1, \infty)) \subseteq [0, \infty)$ and Φ is one-to-one. Note that $(f \circ \Phi)(x) = 1/x^2$ for $x \in [1, \infty)$ so that $(f \circ \Phi) \cdot \varphi$ is the given integrand on $(1, \infty)$. Hence we have

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^\infty (f \circ \Phi) \cdot \varphi = \int_{\Phi(1)}^{\Phi(\infty)} f = F(\infty) - F(0) = \tfrac{1}{2}\pi.$$

Remark. In (a) a primitive of f was known. We will now treat this integral using other substitutions where a c-primitive is not known explicitly.

(b) We consider the same integral as in (a), but this time we use the substitution $x = \Omega(v) := \sec v$ for $v \in [0, \pi/2)$ and $\Omega(\pi/2) := \infty$. Thus Ω is continuous on $[0, \pi/2]$ and $\omega(v) := \Omega'(v) = \sec v \cdot \tan v$ for $v \in [0, \pi/2)$. The function $\tilde{f}(x) := 1/x\sqrt{x^2-1}$ is known to be in $\mathcal{R}^*([1, \infty])$ (why?), but we do not know a c-primitive of it. However, since $\tilde{f}$ is continuous on $(1, \infty)$ it has an indefinite integral (with base point 2, say) which (by Corollary 4.10) is a c-primitive on every interval $[1, \beta] \subset [1, \infty)$. The theorem enables us to replace the given integral by one that we can evaluate. Indeed, if $B := \sec^{-1} \beta$, then we have

$$\begin{aligned}\int_1^\beta \frac{dx}{x\sqrt{x^2-1}} &= \int_0^B (\tilde{f} \circ \Omega) \cdot \omega = \int_0^B \frac{\sec v \cdot \tan v}{\sec v \cdot \tan v}\, dv = \int_0^B dv \\ &= B = \sec^{-1} \beta \to \tfrac{1}{2}\pi\end{aligned}$$

as $\beta \to \infty$. The use of Hake's Theorem is justified since $\tilde{f} \in \mathcal{R}^*([1, \infty])$.

(c) We consider the same integral as in (a), but this time we use the substitution $x = \Omega_1(t) := \cosh t$ for $t \in [0, \infty)$ and $\Omega_1(\infty) := \infty$; thus Ω_1 is continuous on $[0, \infty]$ and $\omega_1(t) := \Omega_1'(t) = \sinh t$ for $t \in [0, \infty)$. As in (b), the function $\tilde{f}(x) := 1/x\sqrt{x^2-1}$ has an (unknown) c-primitive on every

interval $[1,\beta] \subset [1,\infty)$. If $B := \cosh^{-1}\beta$, then we have

$$\int_1^\beta \frac{dx}{x\sqrt{x^2-1}} = \int_0^B \frac{\sinh t\,dt}{\cosh t \cdot \sinh t} = \int_0^B \operatorname{sech} t\,dt$$
$$= \sin^{-1}(\tanh t)\Big|_0^B \rightarrow \tfrac{1}{2}\pi$$

as $\beta \to \infty$. The use of Hake's Theorem is justified since $\tilde{f} \in \mathcal{R}^*([1,\infty])$.

(13.3) — Second Substitution Theorem, I. We allow the endpoints a, b, c, d and $\Phi(a), \Phi(b)$ to be $\pm\infty$. We require $\Phi\big((a,b)\big) \subseteq (c,d)$. Note that hypothesis (j′) implies that Φ is strictly monotone.

The proof requires no change.

(13.5) — First Substitution Theorem, II. We allow the endpoints a, b, c, d and $\Phi(a), \Phi(b)$ to be $\pm\infty$. We require $\Phi\big((a,b)\big) \subseteq (c,d)$.

We will treat the case of a strictly increasing Φ where $a, \Phi(a) \in \mathbb{R}$.

If $\beta \in (a,b)$, let $I_\beta := [a,\beta]$, so that the hypotheses of 13.5 hold on I_β. Therefore $f \in \mathcal{R}^*(\Phi(I_\beta))$ if and only if $(f \circ \Phi) \cdot \varphi \in \mathcal{R}^*(I_\beta)$, in which case

$$\int_a^\beta (f \circ \Phi) \cdot \varphi = \int_{\Phi(a)}^{\Phi(\beta)} f. \tag{17.ζ}$$

We first suppose that $(f \circ \Phi) \cdot \varphi \in \mathcal{R}^*(I)$. By Hake's Theorem 16.5, for every $\beta \in (a,b)$, we have $(f \circ \Phi) \cdot \varphi \in \mathcal{R}^*(I_\beta)$ and

$$\int_a^b (f \circ \Phi) \cdot \varphi = \lim_{\beta \to b} \int_a^\beta (f \circ \Phi) \cdot \varphi.$$

But, since $\Phi(\beta) \to \Phi(b)$ as $\beta \to b$, it follows from (17.ζ) that

$$\lim_{\gamma \to \Phi(b)} \int_{\Phi(a)}^\gamma f = \lim_{\beta \to b} \int_a^\beta (f \circ \Phi) \cdot \varphi = \int_a^b (f \circ \Phi) \cdot \varphi.$$

Thus, Hake's Theorem implies that $f \in \mathcal{R}^*(\Phi(I))$ and that (17.ζ) holds with β replaced by b.

Now suppose that $f \in \mathcal{R}^*\big(\Phi(I)\big) = \mathcal{R}^*\big([\Phi(a), \Phi(b)]\big)$. Hake's Theorem 16.5 implies that for every $\gamma \in (\Phi(a), \Phi(b))$, we have $f \in \mathcal{R}^*([\Phi(a), \gamma])$ and that

$$\int_{\Phi(a)}^{\Phi(b)} f = \lim_{\gamma \to \Phi(b)} \int_{\Phi(a)}^\gamma f.$$

Since $\gamma = \Phi(\beta)$ for some $\beta \in (a, b)$, it follows from (17.ζ) that

$$\lim_{\beta \to b} \int_a^\beta (f \circ \Phi) \cdot \varphi = \lim_{\beta \to b} \int_{\Phi(a)}^{\Phi(\beta)} f = \lim_{\gamma \to \Phi(b)} \int_{\Phi(a)}^{\gamma} f = \int_{\Phi(a)}^{\Phi(b)} f.$$

Thus Hake's Theorem implies that $(f \circ \Phi) \cdot \varphi \in \mathcal{R}^*(I)$ and that (17.ζ) holds with β replaced by b.

(13.7) — Second Substitution Theorem, II. We require that $\Phi\big((a,b)\big) \subseteq (c,d)$.

The proof needs no changes since we can take $\{a, b\} \subseteq C$.

(13.8) — First Substitution Theorem, III. We require that $\Phi\big((a,b)\big) \subseteq (c,d)$.

The proof needs no change, since only I_m is infinite and we can apply the extended 13.5 to it.

Review of Section 14

If $I := [a, b]$ with $b \le \infty$, we recall that in (7.3) we defined the variation of $F : I \to \mathbb{R}$ to be

$$\operatorname{Var}(F; I) := \sup\Big\{\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|\Big\},$$

where the supremum is taken over all partitions of I. If $\operatorname{Var}(F; I) < \infty$, we say that F has **bounded variation on** I and write $F \in BV(I)$. It is easy to see that if $F \in BV(I)$, then F is bounded by $|F(a)| + \operatorname{Var}(F; I)$ on I.

It is clear that if J is any compact interval contained in I and if $F \in BV(I)$, then $\operatorname{Var}(F; J) \le \operatorname{Var}(F; I)$ so that $F \in BV(J)$. The converse is not true, however, for if $F_1(x) := x$ for $x \in [0, \infty)$ and $F_1(\infty) := 0$, then $F_1 \in BV(J)$ for every compact interval $J \subset I$, but $F_1 \notin BV(I)$. We will say that a function F has **locally bounded variation on** I and write $F \in LBV(I)$, if $F \in BV(J)$ for every compact interval $J \subset I$. The reader may show that $F \in LBV(I)$ if and only if F is the difference of two increasing functions, and that $F \in BV(I)$ if and only if F is the difference of two *bounded* increasing functions on I. (A good reference for these notions is the book of Asplund and Bungart [A-B], where one will also find a proof of Lebesgue's Differentiation based on the Riesz "Rising Sun Lemma".)

(14.1) — Lebesgue's Differentiation Theorem. The statement requires no change. Since differentiation is a local process it is enough to assume that $F \in LBV(I)$.

Suppose that $I := [0, \infty]$ and let $I_n := [n-1, n]$ for $n \in \mathbb{N}$. If $F \in BV(I_n)$, then there exists a null set $Z_n \subset I_n$ such that $F'(x)$ exists for $x \in I_n - Z_n$. Now let $Z := \mathbb{N} \cup \{\infty\} \cup \bigcup_{n=1}^{\infty} Z_n$ so that $F'(x)$ exists for $x \notin Z$.

(14.2) No change is needed.

If F is increasing on I, then $F' \geq 0$ a.e. It follows from 14.2 that if $c \in I_0$, then $F' \in \mathcal{L}([a,c])$ and $\int_a^y F' \leq F(y) - F(a)$ for $y \leq c$. Since F is bounded on I, the indefinite integral $y \mapsto \int_a^y F'$ is bounded and increasing on I. Therefore it has a limit as $y \to \infty$, so Hake's Theorem implies that $F' \in \mathcal{L}(I)$. If $x \in I_0$ is given, let $c > x$ and use 14.2 applied to the interval $[a, c]$ to obtain (14.α).

(14.4) There are two natural ways to define absolute continuity for a function F on an infinite interval $I := [a, \infty]$. Method (1) is to use Definition 14.4 exactly as stated, which implies that only compact intervals $[u_j, v_j]$ are used in (14.γ). Method (2) is to require that the restriction of F to each compact interval $J \subset I$ belongs to $AC(J)$. Neither method imposes a condition at ∞.

We will use Method (1) and denote the collection of functions satisfying this condition by $AC(I)$. We say that functions satisfying the condition in Method (2) are **locally absolutely continuous on** I and denote this collection by $LAC(I)$. It is clear that $AC(I) \subseteq LAC(I)$, and the example $F_2(x) := x^2$ for $x \in [0, \infty)$ and $F_2(\infty) := 0$ shows that this inclusion is proper. The function $F_1(x) := x$, $F_1(\infty) := 0$, mentioned earlier shows that functions in $AC(I)$ and $LAC(I)$ may be unbounded; therefore, neither of these collections is contained in $BV(I)$. Also, functions in $AC(I)$ and $LAC(I)$ need not be continuous at ∞.

(14.5) Replace **(b)** by **(b′)**: *If* $F \in AC(I)$ *then* $F \in LBV(I)$.

In **(c)**, add the hypothesis that F, G are bounded to conclude that the product $F \cdot G \in AC(I)$.

(b′) If $F \in AC(I)$, then $F \in LAC(I)$ so that $F \in LBV(I)$. The function $F_3(x) := \sin x$ for $x \in [0, \infty)$ and $F_3(\infty) := 0$ is a bounded function in $AC([0, \infty])$, but does not belong to $BV([0, \infty])$; also note that F_3 is not continuous at ∞. The function $F_4(x) := (1/x)\sin(\pi x/2)$ on $[1, \infty)$, $F_4(\infty) := 0$ is a bounded function in $AC([1, \infty])$ and is continuous at ∞; however, it does not belong to $BV([1, \infty])$.

(c′) The boundedness of F and G was used in the proof that $F \cdot G \in AC(I)$.

The example $F_5(x) := x \sin x$ for $x \in [0, \infty)$ and $F_5(\infty) := 0$, shows that it is not enough for just *one* of F and G to be bounded.

(14.6) We either require that $Z \subset I_0$, or that F is continuous at ∞.

If $Z \subset I_0$, the construction of δ_ε is as given. If $\infty \in Z$ and F is continuous at ∞, then given $\varepsilon > 0$ we let $\delta_\varepsilon(\infty) > 0$ be such that if $x \geq 1/\delta_\varepsilon(\infty)$ then $|F(x) - F(\infty)| \leq \varepsilon$. Now define $\delta_\varepsilon(t)$ for $t \in Z \cap \mathbb{R}$ as before. Then, if $\dot{\mathcal{P}}_0$ is a (δ_ε, Z)-fine subpartition of I, we consider the terms corresponding to finite tags and to $t_i = \infty$ to get $\sum |F(v_j) - F(u_j)| \leq 2\varepsilon$.

(14.7) — Characterization Theorem. Replace statements **(b)** and **(c)** by:

(b′) *$F \in AC(I)$ and F is continuous at ∞.*

(c′) *$F \in LBV(I)$ and if ...*

(a) $\Rightarrow$ (b′) The proof that $F \in AC(I)$ is as before. Hake's Theorem implies that F is continuous at ∞.

(b′) $\Rightarrow$ (c′) If $F \in AC(I)$, we have seen from the modification of Theorem 14.5(b′) that $F \in LBV(I)$. Here ∞ belongs to Z and, since F is continuous at ∞, the modification of 14.6 shows that $F \in NV_I(Z)$.

(c′) $\Rightarrow$ (a) The same argument applies.

(14.8) Change the statement to: *Let $I := [a, \infty]$ and let $F : I \to \mathbb{R}$. Then $F \in AC(I)$ and is continuous at ∞ if and only if $F \in LBV(I)$ and $F \in NV_I(Z)$ for every null set $Z \subset I$.*

($\Rightarrow$) If $F \in AC(I)$, then $F \in LBV(I)$. The modification of 14.6 shows that if Z is any null set, then $F \in NV_I(Z)$.

($\Leftarrow$) If $F \in LBV(I)$, then F' exists a.e. on I_0, so the set Z where the derivative does not exist is a null set containing ∞ and so $F \in NV_I(Z)$. If we apply the modification of 14.7 we conclude that $F \in AC(I)$ and is continuous at ∞.

Corollary 14.8 is false for I as stated, since $F_1(x) := x$ for $x \in [0, \infty)$, and $F_1(\infty) := 0$, is in $AC([0, \infty])$ but not $BV([0, \infty])$.

(14.9) Change the statement to: *Let $F : I \to \mathbb{R}$ be increasing on $[a, \infty]$. Then $F \in AC(I)$ and F is continuous at ∞ if and only if*

$$\int_a^\infty F' = F(\infty) - F(a). \tag{14.ε'}$$

($\Rightarrow$) The modification of 14.7 implies that F is an indefinite integral of a function $f \in \mathcal{L}(I)$, whence $F(x) = F(a) + \int_a^x f$ for all $x \in I$. Further, $F'(x) = f(x)$ a.e. Take $x = \infty$.

($\Leftarrow$) The earlier argument shows that F is an indefinite integral of $F' \in \mathcal{L}(I)$. Thus, the modification of 14.7 shows that $F \in AC(I)$ and F is continuous at ∞.

(14.10) The definition makes sense.

(14.11) Change the statement to: *If $F \in AC(I)$ is continuous at ∞ and is singular on $I := [a, \infty]$, then F is a constant function.*

The theorem implies that F is constant on I_0; since F is continuous at ∞, it is constant on I.

The function $F_6(x) := 0$ for $x \in [0, \infty)$, $F_6(\infty) := 1$ shows that the continuity condition is needed.

(14.12) — Lebesgue Decomposition Theorem. Change the statement to: *Let $I := [a, \infty]$. If $F \in LBV(I)$, then F can be written as the sum (14.η), where $F_a \in AC(I)$ and is continuous at ∞, and $F_s \in LBV(I)$ is singular on I. Moreover, this representation is unique up to a constant.*

Only minor changes are needed in the given argument.

(14.13) No change is needed.

For each $n \in \mathbb{N}$, let $Z_n := Z \cap [-n, n]$, so that Z_n is a null set. The theorem implies that $F(Z_n)$ is a null set and hence $F(Z) = F\left(\bigcup_{n=1}^{\infty} Z_n \cup \{\infty\}\right) \subseteq \bigcup_{n=1}^{\infty} F(Z_n) \cup \{F(\infty)\}$ is a null set.

(14.14) – (14.15) We have not yet discussed measurable sets in $\overline{\mathbb{R}}$.

(14.16) Change the statement to: *If $I := [a, \infty]$ and if $F, G \in AC(I)$ are continuous at ∞, then (14.μ) holds with $b = \infty$.*

Both F and G are bounded and continuous on I, so $F'G$ and FG' are in $\mathcal{L}(I)$. The product FG is continuous at ∞. Let $b \to \infty$ in (14.μ) and use Hake's Theorem.

(14.20) No change is needed. Argue as in (14.13) above.

(14.21) These classes are usually discussed only for compact I.

It does not seem to be necessary to offer a list of exercises for this section.

Measurable Sets

In Section 6 we introduced the notions of a measurable subset of a *compact* interval, and of its measure. These ideas were studied in Section 10, and it was asserted that many of these results extend with little or no change to subsets of $\mathbb{R} := (-\infty, \infty)$.

In this section we will give a more systematic discussion of these notions. For the convenience of the reader, we will give a largely independent discussion based on the generalized Riemann integral on $\overline{\mathbb{R}}$. Consequently, there will be a slight bit of repetition of previous material, but this should do no serious harm. After we develop the properties of the measurable sets, we will show that all of the familiar collections of sets in $\mathbb{R}$ consist of measurable sets. We then give a number of relatively deep results that show that the most general measurable sets can be approximated in various ways by sets having topological interest.

Finally we establish the invariance of the measurable sets under translations, and use this property to construct a set in $\mathbb{R}$ that is *not measurable.*

Integrable Sets in $\mathbb{R}$

For subsets of a *compact* interval, there is no distinction between a "measurable set" and an "integrable set". However, for subsets of $\mathbb{R}$, it is important to distinguish between these closely related types of sets.

We recall that the **characteristic function** (or **indicator function**) of a set $A \subseteq \mathbb{R}$ is the function $\mathbf{1}_A$ defined on $\mathbb{R}$ by

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Often we will also define $\mathbf{1}_A(\pm\infty) := 0$.

18.1 Definition. (a) A subset A of $\mathbb{R}$ is said to be **integrable** if its characteristic function $\mathbf{1}_A$ is integrable on $\overline{\mathbb{R}}$. In this case the number $\int_{-\infty}^{\infty} \mathbf{1}_A$ is called **the Lebesgue measure**, or simply **the measure**, of A. We will denote the measure of A by $|A|$ or by $\lambda(A)$.

(b) The collection of all integrable subsets of $\mathbb{R}$ is denoted by $\mathbb{I}(\mathbb{R})$, or simply by $\mathbb{I}$.

Remarks. (a) It is important to note that the measure of each subset of $\mathbb{I}(\mathbb{R})$ is a *finite* real number.

(b) Many of the proofs given below make use of extension of the results in earlier sections to $\mathbb{R}$. If these results were marked by $\bullet$, the reader may use these results with confidence. If these results were marked by $\diamond$, the reader may consult the relevant entry in Section 17 to determine the extent to which the result applies.

In the next two theorems we will give the basic properties of the collection $\mathbb{I}(\mathbb{R})$ of all integrable subsets of $\mathbb{R}$.

18.2 Theorem. *The collection* $\mathbb{I}(\mathbb{R})$ *has the properties:*

(0) *The empty set* $\emptyset$ *belongs to* $\mathbb{I}(\mathbb{R})$.

(a) Every compact interval I belongs to $\mathbb{I}(\mathbb{R})$.

(b) *If* $A, B \in \mathbb{I}(\mathbb{R})$, *then* $A \cup B, A \cap B$ *and* $A - B$ *belong to* $\mathbb{I}(\mathbb{R})$.

(c) *If* $(A_n)_{n=1}^{\infty}$ *is a sequence in* $\mathbb{I}(\mathbb{R})$, *then the intersection* $A_\infty := \bigcap_{n=1}^{\infty} A_n$ *also belongs to* $\mathbb{I}(\mathbb{R})$.

Proof. (0) Since $\mathbf{1}_\emptyset$ is identically equal to 0, it belongs to $\mathcal{R}^*(\mathbb{R})$ so $\emptyset \in \mathbb{I}$.

(a) If $I = [a, b]$, then we have seen in Example 2.1(a) that $\mathbf{1}_I \in \mathcal{R}^*(I)$. It follows from (the extension of) Corollary 3.8 that $\mathbf{1}_I \in \mathcal{R}^*(\mathbb{R})$ so that $I \in \mathbb{I}$.

(b) It is easily seen that $\mathbf{1}_{A\cup B} = \mathbf{1}_A \vee \mathbf{1}_B$ and $\mathbf{1}_{A\cap B} = \mathbf{1}_A \wedge \mathbf{1}_B$. Thus if $A, B \in \mathbb{I}$, then it follows from Theorem 7.12 that $\mathbf{1}_{A\cup B}$ and $\mathbf{1}_{A\cap B}$ also belong to $\mathcal{R}^*(\mathbb{R})$, so that $A \cup B$ and $A \cap B$ belong to $\mathbb{I}$.

Since $\mathbf{1}_{A-B} = \mathbf{1}_A - \mathbf{1}_{A\cap B}$, it follows from Theorem 3.1 that $\mathbf{1}_{A-B} \in \mathcal{R}^*(\mathbb{R})$, so that $A - B \in \mathbb{I}$.

(c) We let $B_n := \bigcap_{i=1}^{n} A_i$ so that $B_n \in \mathbb{I}$ for each $n \in \mathbb{N}$, and it is easily seen that $A_\infty = \bigcap_{n=1}^{\infty} B_n$. Since the sequence $(\mathbf{1}_{B_n})$ is a monotone decreasing sequence in $\mathcal{R}^*(\mathbb{R})$ that is bounded below by the zero function and converges on $\mathbb{R}$ to $\mathbf{1}_{A_\infty}$, the Monotone Convergence Theorem 8.5 implies that $\mathbf{1}_{A_\infty}$ belongs to $\mathcal{R}^*(\mathbb{R})$, so that $A_\infty \in \mathbb{I}$. Q.E.D.

Our next result gives information about the measures of various combinations of sets in $\mathbb{I}(\mathbb{R})$. It should be compared with Theorem 10.2; but see the Remarks after Theorem 18.8.

18.3 Theorem. *The measure function on* $\mathbb{I}(\mathbb{R})$ *satisfies:*

(a) $|\emptyset| = 0$.

(b) *If* $I := [a, b]$ *is a compact interval, then* $|I| = b - a$.

(c) *If* $A, B \in \mathbb{I}(\mathbb{R})$, *then* $|A \cup B| + |A \cap B| = |A| + |B|$.

(d) *If* $A, B \in \mathbb{I}(\mathbb{R})$ *and if* $A \subseteq B$, *then* $|B - A| = |B| - |A|$.

(e) *If* $A, B \in \mathbb{I}(\mathbb{R})$ *and if* $A \subseteq B$, *then* $|A| \leq |B|$.

(f) *If* $A_1, \cdots, A_n$ *belong to* $\mathbb{I}(\mathbb{R})$, *then* $\left|\bigcup_{i=1}^n A_i\right| \leq \sum_{i=1}^n |A_i|$.

(g) *If* $A_1, \cdots, A_n$ *belong to* $\mathbb{I}(\mathbb{R})$ *and if* $|A_i \cap A_j| = 0$ *for* $i \neq j$, *then*

$$\left|\bigcup_{i=1}^n A_i\right| = \sum_{i=1}^n |A_i|.$$

(h) *If* $B_1 \supseteq \cdots \supseteq B_n \supseteq \cdots$ *belong to* $\mathbb{I}(\mathbb{R})$, *then* $B_\infty := \bigcap_{n=1}^\infty B_n$ *belongs to* $\mathbb{I}(\mathbb{R})$ *and*

$$|B_\infty| = \left|\bigcap_{n=1}^\infty B_n\right| = \lim_{n\to\infty} |B_n|.$$

(i) *If* $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ *belong to* $\mathbb{I}(\mathbb{R})$, *then* $A^\infty := \bigcup_{n=1}^\infty A_n$ *belongs to* $\mathbb{I}(\mathbb{R})$ *if and only if the sequence* $(|A_n|)_{n=1}^\infty$ *is bounded in* $\mathbb{R}$. *In this case,* A^∞ *belongs to* $\mathbb{I}(\mathbb{R})$ *and*

$$|A^\infty| = \left|\bigcup_{n=1}^\infty A_n\right| = \lim_{n\to\infty} |A_n|.$$

(j) *If* $(C_n)_{n=1}^\infty$ *is a sequence in* $\mathbb{I}(\mathbb{R})$ *with* $|C_i \cap C_j| = 0$ *for* $i \neq j$, *then the set* $C^\infty := \bigcup_{n=1}^\infty C_n$ *belongs to* $\mathbb{I}(\mathbb{R})$ *if and only if the series* $\sum_{n=1}^\infty |C_n|$ *is convergent in* $\mathbb{R}$. *In this case,*

$$|C^\infty| = \left|\bigcup_{n=1}^\infty C_n\right| = \sum_{n=1}^\infty |C_n|.$$

(k) *If* $(D_n)_{n=1}^\infty$ *is a sequence in* $\mathbb{I}(\mathbb{R})$ *and if the series* $\sum_{n=1}^\infty |D_n|$ *is convergent in* $\mathbb{R}$, *then* $D^\infty := \bigcup_{n=1}^\infty D_n$ *belongs to* $\mathbb{I}(\mathbb{R})$ *and*

$$|D^\infty| = \left|\bigcup_{n=1}^\infty D_n\right| \leq \sum_{n=1}^\infty |D_n|.$$

Proof. (a) Since $\mathbf{1}_\emptyset = 0$, then $|\emptyset| = \int_{-\infty}^{\infty} 0\,dx = 0$.

(b) By the extension of Corollary 3.8 the integral of $\mathbf{1}_I$ over $\overline{\mathbb{R}}$ equals the integral of 1 over I, which is $b - a$.

(c) Since $\mathbf{1}_{A\cup B} + \mathbf{1}_{A\cap B} = \mathbf{1}_A + \mathbf{1}_B$, the assertion follows from Theorem 3.1.

(d) Since $\mathbf{1}_{B-A} = \mathbf{1}_B - \mathbf{1}_A$, this also follows from Theorem 3.1.

(e) Since $0 \le |B - A|$, this is immediate from (d).

(f) If $n = 2$ this follows from (c) since $|A \cap B| \ge 0$. Now use induction.

(g) If $n = 2$ this follows from (c) since $|A \cap B| = 0$. Now use induction. Note that if the sets A_i are pairwise disjoint, the hypothesis is satisfied.

(h) We apply the Monotone Convergence Theorem 8.5 to the decreasing sequence $(\mathbf{1}_{B_n})$.

(i) We apply the Monotone Convergence Theorem 8.5 to the increasing sequence $(\mathbf{1}_{A_n})$.

(j) Let $A_n := \bigcup_{i=1}^n C_i$, so that (A_n) is an increasing sequence in $\mathbb{I}(\mathbb{R})$, and $|A_n| = \sum_{i=1}^n |C_i|$ from part (g). Now apply (i).

(k) Let $A_n := D_1 \cup \cdots \cup D_n$ so that $(A_n)_{n=1}^\infty$ is an increasing sequence in $\mathbb{I}(\mathbb{R})$, and let $d := \sum_{i=1}^\infty |D_i|$. It follows from (f) that $|A_n| \le \sum_{i=1}^n |D_i| \le d$. If we let $A^\infty := \bigcup_{n=1}^\infty A_n$, then (i) implies that $A^\infty \in \mathbb{I}(\mathbb{R})$ and that $|A^\infty| \le d$. Since $D^\infty = A^\infty$, the conclusion results. Q.E.D.

Remarks. **(i)** The most important consequence of 18.3(c) is the **additivity property**: if $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.

(ii) Property 18.3(d) is sometimes called the **subtractive property** of the measure. (We will see in an exercise that it needs to be modified somewhat when the measure is allowed to take on the value ∞.)

(iii) Property 18.3(e) is called the **monotone property** of the measure.

(iv) Property 18.3(f) is called the **finite subadditivity property** of the measure.

(v) Property 18.3(g) is called the **finite additivity property** of the measure.

(vi) Properties 18.3(h) and (i) give the behavior of the measure function on decreasing and on increasing sequences of integrable sets. Note that these properties are *not exactly* similar. That is because a decreasing sequence of nonnegative real numbers is always convergent in $\mathbb{R}$, while an increasing sequence of nonnegative numbers is convergent if and only if it is bounded.

(vii) Property 18.3(j) is a version of property 18.3(i) for series instead of sequences. It is called the **countable additivity property** of the measure on $\mathbb{I}(\mathbb{R})$.

(viii) Property 18.3(k) is called the **countable subadditivity property** of the measure on $\mathbb{I}(\mathbb{R})$.

Measurable Sets

We now introduce the second collection of sets in $\mathbb{R}$; in a sense, we are merely adjoining certain sets of infinite measure to the collection $\mathbb{I}(\mathbb{R})$ of integrable sets. More precisely, we adjoin those sets with the property that their intersection with *every* compact interval is integrable. Many interesting sets have this character; for example, an unbounded interval in $\mathbb{R}$ is not an integrable set, but its intersection with any compact interval is an integrable set.

18.4 Definition. A set $A \subseteq \mathbb{R}$ is said to be **Lebesgue measurable** (or simply **measurable**), if $A \cap I$ is integrable for every compact interval $I \subset \mathbb{R}$. The collection of all measurable sets in $\mathbb{R}$ is denoted by $\mathbb{M}(\mathbb{R})$, or simply by $\mathbb{M}$.

We now state the basic properties of the set of measurable sets in $\mathbb{R}$. For convenience, we denote the **complement** of $A \subseteq \mathbb{R}$ by A^c instead of $\mathbb{R} - A$.

18.5 Theorem. *The collection* $\mathbb{M}(\mathbb{R})$ *has the properties:*

(0) $\mathbb{I}(\mathbb{R}) \subset \mathbb{M}(\mathbb{R})$.

(a) *The sets* $\emptyset$ *and* $\mathbb{R}$ *belong to* $\mathbb{M}(\mathbb{R})$.

(b) *If* $A, B \in \mathbb{M}(\mathbb{R})$, *then* $A \cup B$ *and* A^c *belong to* $\mathbb{M}(\mathbb{R})$.

(c) *If* $(B_n)_{n=1}^{\infty}$ *is any sequence of sets in* $\mathbb{M}(\mathbb{R})$, *then the countable union* $B^\infty := \bigcup_{n=1}^{\infty} B_n$ *also belongs to* $\mathbb{M}(\mathbb{R})$.

Proof. (0) If $A \in \mathbb{I}$ and I is any compact interval, then Theorem 18.2(a) and (b) imply that $A \cap I \in \mathbb{I}$ so that $A \in \mathbb{M}$. Thus $\mathbb{I} \subseteq \mathbb{M}$.

(a) We note that $\emptyset \in \mathbb{I} \subset \mathbb{M}$. If I is any compact interval, then $\mathbb{R} \cap I = I$, so that $\mathbb{R} \in \mathbb{M}$. Since $\mathbb{R} \notin \mathbb{I}$, then $\mathbb{I} \subset \mathbb{M}$.

(b) If $A, B \in \mathbb{M}$ and I is any compact interval, then $A \cap I$ and $B \cap I$ belong to $\mathbb{I}$. Since $(A \cup B) \cap I = (A \cap I) \cup (B \cap I)$, Theorem 18.2(b) implies that $(A \cup B) \cap I \in \mathbb{I}$. Since this holds for all I, then $A \cup B \in \mathbb{M}$.

If $A \in \mathbb{M}$ and I is any compact interval, then since $A^c \cap I = (\mathbb{R} - A) \cap I = I - A \cap I \in \mathbb{I}$, it follows that $A^c \in \mathbb{M}$.

(c) By induction, (b) implies that if $B_1, \cdots, B_n$ belong to $\mathbb{M}$, then $E_n := B_1 \cup \cdots \cup B_n \in \mathbb{M}$. If I is any compact interval, since the sequence $(E_n)_{n=1}^{\infty}$ is increasing, then the sequence $(E_n \cap I)_{n=1}^{\infty}$ in $\mathbb{I}$ is also increasing. But since $|E_n \cap I| \leq |I|$, Theorem 18.3(i) implies that $\bigcup_{n=1}^{\infty}(E_n \cap I)$ belongs to $\mathbb{I}$. Now since $B^{\infty} \cap I = (\bigcup_{n=1}^{\infty} B_n) \cap I = \bigcup_{n=1}^{\infty}(E_n \cap I)$, we conclude that B^{∞} belongs to $\mathbb{M}$, as asserted. Q.E.D.

The next corollary is an immediate consequence of what we have done, but it is worth stating formally.

18.6 Corollary. **(a)** *If $A, B \in \mathbb{M}(\mathbb{R})$, then $A \cap B$ and $A - B$ are in $\mathbb{M}(\mathbb{R})$.*

(b) *If $(B_n)_{n=1}^{\infty}$ is any sequence in $\mathbb{M}(\mathbb{R})$, then the countable intersection $B_{\infty} := \bigcap_{n=1}^{\infty} B_n$ also belongs to $\mathbb{M}(\mathbb{R})$.*

Proof. (a) By DeMorgan's Laws, $A \cap B = (A^c \cup B^c)^c$, so property 18.5(b) implies that $A \cap B \in \mathbb{M}$. Since $A - B = A \cap B^c$, it follows that $A - B \in \mathbb{M}$.

(b) By DeMorgan's Laws, $B_{\infty} = \bigcap_{n=1}^{\infty} B_n = \left(\bigcup_{n=1}^{\infty}(B_n)^c\right)^c$ and so properties 18.5(b) and (c) apply. Q.E.D.

The Lebesgue Measure

We will now extend the definition of the Lebesgue measure to arbitrary sets in $\mathbb{M}(\mathbb{R})$; we do this by declaring that the measure of a measurable set is equal to the extended real number ∞ in case the set is not integrable.

18.7 Definition. We define the **Lebesgue measure**, or simply the **measure**, to be the function $\lambda : \mathbb{M}(\mathbb{R}) \to [0, \infty]$ given by $\lambda(E) := |E|$ if E is an integrable set in $\mathbb{R}$, and $\lambda(E) := \infty$ if E is a measurable but not integrable subset of $\mathbb{R}$.

Note: It is an immediate consequence of the definition that if $E \in \mathbb{M}(\mathbb{R})$ and $\lambda(E) < \infty$, then $E \in \mathbb{I}(\mathbb{R})$ and $|E| = \lambda(E)$.

The properties of Lebesgue measure on $\mathbb{M}(\mathbb{R})$ follow from its properties on $\mathbb{I}(\mathbb{R})$ that were stated in Theorem 18.3. We will state formally only the most important ones in the next result.

18.8 Theorem. *The Lebesgue measure λ as function on $\mathbb{M}(\mathbb{R})$ satisfies:*

(a) $\lambda(\emptyset) = 0$.

(b) $0 \leq \lambda(E) \leq \infty$ *for all $E \in \mathbb{M}(\mathbb{R})$.*

(c) *If $(E_n)_{n=1}^{\infty}$ is a pairwise disjoint sequence in $\mathbb{M}(\mathbb{R})$, then*

$$\lambda\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \lambda(E_n), \tag{18.α}$$

where it is understood that either the series on the right is (absolutely) convergent, or that it diverges to ∞.

(d) *If $(A_n)_{n=1}^{\infty}$ is any sequence of sets in $\mathbb{M}(\mathbb{R})$, then*

$$\lambda\Big(\bigcup_{n=1}^{\infty} A_n\Big) \le \sum_{n=1}^{\infty} \lambda(A_n). \tag{18.β}$$

Proof. Statements (a) and (b) are trivial.

(c) We note that the terms in the series on the right are either nonnegative numbers or are ∞. Therefore, this series is either absolutely convergent or it diverges to ∞. If any of the terms $\lambda(E_n) = \infty$, then the series is divergent; in this case the union $E := \bigcup_{n=1}^{\infty} E_n$ has infinite measure. (Explain why.) If all of the terms in the series are finite, then all of the sets E_n are integrable, and we can apply Theorem 18.3(j).

(d) We note that if the right side of (18.β) diverges, then the inequality is trivial. If the right side is convergent, we apply Theorem 18.3(k). Q.E.D.

The property 18.8(c) is often called the **countable additivity property** of the Lebesgue measure λ; it is a key property of the measure. Property 18.8(d) is called the **countable subadditivity property** of λ.

Remarks. Since λ is allowed to take the value ∞, we need to exercise certain precautions and add an occasional condition.

(a) Some of the terms in the analogue of 18.3(c, e, f, g) for λ may be infinite, so the equality must be understood as being in $\overline{\mathbb{R}}$.

(b) In the analogue for 18.3(b), if $A \subseteq B$, then $\lambda(B-A) = \lambda(B) - \lambda(A)$ provided $\lambda(A) < \infty$, but is undefined otherwise.

(c) The analogue of 18.3(h) is true if there exists B_k with $\lambda(B_k) < \infty$. Otherwise it may fail, as is seen by taking $B_n := [n, \infty)$, $n \in \mathbb{N}$.

(d) The analogue of 18.3(i) holds in the sense that $\lambda(A^*)$ and $\lim_n \lambda(A_n)$ are either both infinite, or both finite and equal.

(e) If a single term $\lambda(C_n)$ in the analogue of 18.3(j) is infinite, then both $\lambda(C^\infty)$ and $\sum_k \lambda(C_k)$ are infinite. But if one of these latter terms is finite, the other one equals it.

(f) If $\sum \lambda(D_n) = \infty$, then the analogue of 18.3(k) is trivially true. If this series is convergent in $\mathbb{R}$, then $\lambda(\bigcup_n D_n) < \infty$ and the inequality holds.

(g) If $(E_n) \subset \mathbb{M}$, we always have $\lambda(\liminf_n E_n) \leq \liminf_n \lambda(E_n)$. Also, if $\lambda(\bigcup_n E_n) < \infty$, then we have $\limsup_n \lambda(E_n) \leq \lambda(\limsup_n E_n)$. Otherwise this latter inequality may fail, as is seen by taking $B_n := [n, \infty)$, $n \in \mathbb{N}$.

(h) If $(E_n) \subset \mathbb{M}$ and $\sum_n \lambda(E_n) < \infty$, then $\lambda(\bigcup_n E_n) < \infty$ and so we can conclude that $\lambda(\limsup_n E_n) = 0$. This is the **Borel-Cantelli Lemma**.

Examples of Measurable Sets

We will now compile a list of sets in $\mathbb{R}$ that are integrable or measurable. It will be seen that every set the reader is familiar with is measurable; indeed, it is not a trivial matter to produce a *single set* that is not measurable.

It has been noted in Theorem 18.2(0) that the empty set $\emptyset$ is an integrable, and hence a measurable set with $|\emptyset| = \lambda(\emptyset) = 0$. Since $\mathbb{R} = \emptyset^c$, the entire real line is a measurable (though not an integrable) set with $\lambda(\mathbb{R}) = \infty$.

Points and Intervals

Every compact interval $I := [a, b]$ belongs to $\mathbb{I}(\mathbb{R})$ and hence also to $\mathbb{M}(\mathbb{R})$ with $|I| = \lambda(I) = l(I) = b - a$. Taking $a = b$, we conclude that the set $\{a\}$ consisting of the single point $a \in \mathbb{R}$ is also in $\mathbb{I}(\mathbb{R}) \subset \mathbb{M}(\mathbb{R})$. The noncompact bounded intervals with endpoints $a < b$ are:

$$[a, b), \qquad (a, b] \qquad \text{and} \qquad (a, b).$$

Since $[a, b) = [a, b] - \{a\}$, it follows that this closed-open interval is in $\mathbb{I}(\mathbb{R})$, and similarly for the other two types of bounded intervals. The measure of all of these three types of intervals agrees with their length and is $b - a$.

All of the unbounded intervals:

$$(-\infty, b), \qquad (-\infty, b], \qquad \mathbb{R} = (-\infty, \infty), \qquad (a, \infty), \qquad [a, \infty),$$

are measurable with measure equal to ∞. None of these unbounded intervals is integrable. We summarize these statements in a formal statement.

18.9 Theorem. *Every bounded interval in $\mathbb{R}$ belongs to $\mathbb{I}(\mathbb{R}) \subset \mathbb{M}(\mathbb{R})$. Every unbounded interval in $\mathbb{R}$ is in $\mathbb{M}(\mathbb{R})$.*

Null Sets

The reader will recall from Definition 2.4 that a set $Z \subset \mathbb{R}$ is said to be a **null set** if, given any $\varepsilon > 0$ there is a countable collection $\{J_k\}_{k=1}^{\infty}$ of open intervals such that

$$Z \subseteq \bigcup_{k=1}^{\infty} J_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} |J_k| \leq \varepsilon.$$

It is an exercise to show that the content of the definition does not change

if the intervals J_k are required to be closed (or of the form $(a, b]$, etc.), or with length $l(I_n) = |I_n| < d$ for pre-assigned $d > 0$.

18.10 Theorem. *Every null set $Z \subset \mathbb{R}$ is in $\mathbb{I}(\mathbb{R})$ and $|Z| = 0$.*

Proof. Let $Z_n := Z \cap [-n, n]$ for $n \in \mathbb{N}$. It was seen in Example 2.5(a) that Z_n is a null set and in Example 2.6(a) that $|Z_n| = \int_{-\infty}^{\infty} \mathbf{1}_{Z_n} = 0$. If we apply Theorem 18.3(i), the assertion follows. Q.E.D.

Later in this section we will establish the converse of this theorem: *Every set $W \in \mathbb{M}(\mathbb{R})$ with $\lambda(W) = 0$ is a null set.* This result is not obvious, since it is not clear how to find a sequence of intervals with arbitrarily small total length whose union contains W.

Open Sets

A set $G \subseteq \mathbb{R}$ is said to be **open** in $\mathbb{R}$ if, for every $x \in G$ there exists $r_x > 0$ such that the open interval $(x - r_x, x + r_x) \subseteq G$.

Since the empty set $\emptyset$ contains no points, the empty set is open by default. Similarly, it is trivial that the entire set $\mathbb{R}$ is open, since we can take $r_x = 1$ for all $x \in \mathbb{R}$. It is reassuring that an open interval $I := (a, b)$ is an open set, since if $x \in I$, then we can take $r_x := \min\{x - a, b - x\}$. Similarly, one can show that the intervals

$$(-\infty, b) \qquad \text{and} \qquad (a, \infty)$$

are open sets in $\mathbb{R}$.

It is not difficult to prove that: *The union of* ***any*** *collection of open sets in $\mathbb{R}$ is open, and the intersection of any* ***finite*** *collection of open sets in $\mathbb{R}$ is open.*

Indeed, if $x \in G := \bigcup_\alpha G_\alpha$, then there exists a set $G_{\alpha(x)}$ such that $x \in G_{\alpha(x)}$; hence there exists $r_x > 0$ such that $(x - r_x,\, x + r_x) \subseteq G_{\alpha(x)} \subseteq G$. Similarly, if $x \in \bigcap_{i=1}^n G_i$, then $x \in G_i$ for all $i = 1, \cdots, n$. Since the sets G_i are open and contain x, there exist $r_{i,x} > 0$ such that the intervals $(x - r_{i,x},\, x + r_{i,x}) \subseteq G_i$. If we let $r_x := \min\{r_{1,x}, \cdots, r_{n,x}\}$, then it is clear that

$$(x - r_x,\, x + r_x) \subseteq \bigcap_{i=1}^{n} (x - r_{i,x},\, x + r_{i,x}) \subseteq \bigcap_{i=1}^{n} G_i.$$

Therefore, a finite intersection of open sets is an open set.

It is possible to give a simple characterization of open sets in $\mathbb{R}$: *A subset of $\mathbb{R}$ is open if and only if it is the union of a* ***countable*** *collection of pairwise disjoint open intervals.* Indeed, it follows from the fact just

established that the countable union of open intervals is an open set. The converse is somewhat more involved and is proved in [B-S; p. 315].

We will prove here the weaker statement: *An open set is a* ***countable*** *union of open intervals.* In fact, if G is a nonempty open set in $\mathbb{R}$, we let $\{q_1, q_2, \cdots\}$ be an enumeration of the rational numbers in G. Since G is open, for each i there exists the smallest integer n_i such that the open interval $I(q_i) := (q_i - 1/n_i,\, q_i + 1/n_i)$ is contained in G. It is clear that we have

$$G_0 := \bigcup_{i=1}^{\infty} I(q_i) \subseteq G.$$

We will show that $G_0 = G$. Indeed, let $y \in G$; since G is open, let n_y be the smallest integer such that $J_y := (y - 1/n_y,\, y + 1/n_y) \subseteq G$. Now look at the nonempty open interval $K_y := (y - 1/2n_y, y + 1/2n_y)$. Since the set of rational numbers is dense in $\mathbb{R}$, there exists a rational number in $G \cap K_y$; let q_y be the first such rational number in the enumeration given above. It is an exercise to show that $y \in (q_y - 1/2n_y,\, q_y + 1/2n_y) \subseteq I(q_y) \subseteq G_0$. Since $y \in G$ is arbitrary, we conclude that $G \subseteq G_0$, and therefore $G = G_0$.

18.11 Theorem. *Every open set in $\mathbb{R}$ belongs to $\mathbb{M}(\mathbb{R})$.*

Proof. We have seen that every open set is a countable union of open intervals, so we can apply Theorems 18.9 and 18.5(c). Q.E.D.

Closed and Compact Sets

A set $F \subseteq \mathbb{R}$ is said to be **closed** in $\mathbb{R}$ if its complement $F^c = \mathbb{R} - F$ is an open set. Consequently, the empty set $\emptyset$ and $\mathbb{R}$ are closed sets (since $\mathbb{R} = \emptyset^c$ and $\emptyset = \mathbb{R}^c$ are open sets). Similarly, since

$$[a, b]^c = (-\infty, a) \cup (b, \infty)$$

and the intervals $(-\infty, a)$ and (b, ∞) are open sets, it follows that the interval $[a, b]$ is a closed set. As an exercise, the reader can show that the intervals

$$(-\infty, b] \qquad \text{and} \qquad [a, \infty)$$

are also closed sets in $\mathbb{R}$.

From the properties of open sets mentioned above and DeMorgan's Laws, it follows that: *The intersection of* ***any*** *collection of closed sets in $\mathbb{R}$ is closed, and the union of any* ***finite*** *collection of closed sets is a closed set.*

It is *not true* that every closed set can be expressed as the union (or as the intersection) of a countable collection of closed intervals. However, since $\mathbb{M}(\mathbb{R})$ contains the complement of each of its sets, the following theorem follows from Theorem 18.11.

18.12 Theorem. *Every closed set in $\mathbb{R}$ belongs to* $\mathbb{M}(\mathbb{R})$.

We recall that a set $K \subset \mathbb{R}$ is said to be **compact** if, whenever it is contained in the union of a collection of open sets, then it is contained in a *finite* number of sets in this collection; see [B-S; pp. 319–322] for a discussion of compact sets in $\mathbb{R}$.

The most important theorem about compact sets in $\mathbb{R}$ is the Heine-Borel Theorem: *A set in $\mathbb{R}$ is compact if and only it is both closed and bounded.*

18.13 Theorem. *Every compact set in $\mathbb{R}$ belongs to* $\mathbb{I}(\mathbb{R}) \subset \mathbb{M}(\mathbb{R})$.

Proof. A compact set is closed, so it belongs to $\mathbb{M}(\mathbb{R})$. Since it is also bounded, it is in $\mathbb{I}(\mathbb{R})$. Q.E.D.

The Cantor Set

We recall (see 4.15) that the **Cantor set** Γ is constructed from the unit interval $[0,1]$ by the successive removal of "middle third sets". At the nth stage of this construction we have a set Γ_n that is the union of 2^n disjoint compact intervals each of which has measure (= length) $1/3^n$, so that $|\Gamma_n| = (2/3)^n$. Since $\Gamma = \bigcap_{n=1}^{\infty} \Gamma_n$, Theorem 18.2(c) implies that $\Gamma \in \mathbb{I}$ and Theorem 18.3(h) implies that $|\Gamma| = 0$.

There are two other properties of the Cantor set that are worth mentioning formally. One is that: *The set Γ does not contain any nonempty open intervals.* Another (see Theorem 4.16) is that: *The Cantor set Γ has an uncountable number of points.* This can also be shown by noting that every point $x \in \Gamma$ has a ternary (= base 3) expansion:

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n},$$

where a_n is either 0 or 2. The same type of "diagonal argument" used to show that the interval $[0,1]$ contains uncountably many numbers can also be used to show that Γ contains uncountably many numbers.

A "Cantor-like" Set

Let $H_0 := [0,1]$, and let H_1 be obtained from H_0 by deleting an open interval with length $1/10$ from the middle of H_0; let H_2 be obtained from H_1 by deleting two intervals each of length $1/10^2$ from the middle of the two intervals in H_1; let H_3 be obtained from H_2 by deleting 2^2 intervals each of length $1/10^3$ from the middle of the parts of H_2, etc. Now let H be the intersection of these closed sets H_n, $n \in \mathbb{N}$. It is clear that H has some of

the same properties of the Cantor set (for example, H does not contain any nonempty open intervals). However, it is also different in some ways from Γ; for example, we leave it to the reader to examine the total measure of the sets that were deleted to show that $\lambda(H) = \frac{7}{8}$.

A set constructed in this way — by removing open middle intervals of length $r_1 \geq r_2 \geq \cdots$ from $[0,1]$ — is often called a "Cantor-like" set. In connection with such sets, a natural question arises: How large can the measure of a Cantor-like set be?

F_σ- and G_δ-Sets

Closed sets are often denoted by the letter F (from the French word "fermé" meaning closed) and open sets are often denoted by the letter G (from the German word "Gebiet" meaning region). The intersection of any sequence (or countable family) of open sets is often called a **G_δ-set**. It is not difficult to show that such a set is not necessarily an open set (give an example!). However, by Theorem 18.11 and Corollary 18.6(b), it is a measurable set.

Similarly, the union of a sequence (or countable family) of closed sets is often called an **F_σ-set**. Such a set is not necessarily a closed set (example please!), but it is a measurable set, by Theorems 18.12 and 18.5(c).

Further, a set that is the union of a sequence of G_δ-sets, is often called a **$G_{\delta\sigma}$-set**; such a set is also a measurable set. Further, a set that is the intersection of a sequence of F_σ-sets, is often called an **$F_{\sigma\delta}$-set**; such a set is also a measurable set. If we continue in this way, we obtain the classes:

$$G_{\delta\sigma\delta}\,,\ F_{\sigma\delta\sigma}\,;\quad G_{\delta\sigma\delta\sigma}\,,\ F_{\sigma\delta\sigma\delta}\,;\quad \cdots.$$

All of the sets obtained in this way are measurable sets.

To explain the notation: the subscript σ stands for a countable union or sum (in German, "Summe"), and the subscript δ stands for a countable intersection (in German, "Durchschnitt").

These classes of sets, and their relation to the Borel sets discussed below are thoroughly explored in the book by J. Foran [Fo-1; Chapter 4].

Borel Sets

A collection $\mathcal{A}$ of subsets of an abstract set X is said to be a **σ-algebra** if:

(a) $\emptyset$ and X belong to $\mathcal{A}$;

(b) if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$ and $A^c \in \mathcal{A}$; and

(c) if $(B_n)_{n=1}^\infty$ is any sequence of sets in $\mathcal{A}$, then $\bigcup_{n=1}^\infty B_n$ belongs to $\mathcal{A}$.

The content of Theorem 18.5 is that $\mathbb{M}(\mathbb{R})$ is a σ-algebra of subsets of $\mathbb{R}$ containing the collection $\mathbb{I}(\mathbb{R})$ of integrable sets. However, in addition to the σ-algebra $\mathbb{M}(\mathbb{R})$ of *all* Lebesgue measurable sets, it is sometimes useful to work with a somewhat smaller σ-algebra of subsets of $\mathbb{R}$.

18.14 Definition. The smallest σ-algebra of subsets of $\mathbb{R}$ containing all open sets is called the **Borel σ-algebra**, and is denoted by $\mathbb{B}(\mathbb{R})$, or simply by $\mathbb{B}$. Any set in $\mathbb{B}(\mathbb{R})$ is called a **Borel set**.

We need to justify this definition and show that a smallest such σ-algebra actually exists; this will be done in the exercises. It is also an exercise to show that $\mathbb{B}(\mathbb{R})$ contains all closed sets in $\mathbb{R}$, all G_δ-sets, all F_σ-sets, all $G_{\delta\sigma}$-sets, all $F_{\sigma\delta}$-sets, et cetera. (See also [Fo-1].)

A question arises: Are there any sets in $\mathbb{M}(\mathbb{R})$ that are not in $\mathbb{B}(\mathbb{R})$? The answer is: "Yes". One way of showing this is to "construct" a Lebesgue measurable set that is not a Borel set; see [B-1; pp. 171–174]. Another way is to show (by using some cardinality arguments) that there are "more" Lebesgue measurable sets than Borel sets. In fact, if c denotes the cardinality of the set $\mathbb{R}$, then it can be shown that the cardinality of $\mathbb{B}$ is c and the cardinality of $\mathbb{M}$ is 2^c. Since $c < 2^c$, there are many (!) more Lebesgue measurable sets than Borel sets.

Approximation of Measurable Sets

We will now present a number of results showing that measurable sets can be approximated (in various senses) by intervals, by open and closed sets, and by G_δ- or F_σ-sets. In order to do so, we will first establish a lemma proving that an arbitrary set $E \subseteq \mathbb{R}$ can be covered by a sequence of nonoverlapping compact intervals that are controlled by a pre-assigned gauge. These intervals will be used in Theorem 18.16 to construct a δ-fine subpartition that covers an integrable set E arbitrarily closely.

18.15 Dyadic Covering Lemma. *Let δ be a gauge on $\mathbb{R}$ and let $E \subseteq \mathbb{R}$. Then there exists a countable collection $\{(J_i, t_i)\}$, where*

$$(18.\gamma) \qquad t_i \in E \cap J_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \qquad \text{for all} \quad i,$$

and where the J_i are nonoverlapping compact intervals with $E \subseteq \bigcup_i J_i$.

Proof. First consider the case where $E \subseteq [0, 1]$. Let (I_n) be the sequence of dyadic intervals:

$$[0, \tfrac{1}{2}], [\tfrac{1}{2}, 1], \quad [0, \tfrac{1}{4}], [\tfrac{1}{4}, \tfrac{1}{2}], [\tfrac{1}{2}, \tfrac{3}{4}], [\tfrac{3}{4}, 1], \quad [0, \tfrac{1}{8}], \cdots.$$

For each $t \in E$, since $\delta(t) > 0$, there is a smallest natural number $n(t)$ such that $t \in I_{n(t)} \subseteq [t - \delta(t), t + \delta(t)]$. We let (I_i') be the (possibly terminating)

subsequence of (I_n) obtained by deleting those intervals that are not of the form $I_{n(t)}$ for some $t \in E$. We then let (I''_i) be the (possibly terminating) subsequence of (I'_i) obtained by deleting those intervals I'_j such that $I'_j \subset I'_i$ for some $i < j$. Finally, we let $J_i := I''_i$, and note that this construction assures that the intervals (J_i) are nonoverlapping.

If $t \in E$, then the interval $I_{n(t)}$ belongs to the sequence (I'_i) and is later deleted only if it is already contained in some interval that has been retained. Therefore, $E \subseteq \bigcup_i J_i$. Further, each interval $J_i = I_{n(t)}$ for some point $t \in J_i \cap E$, and we choose such a point as a tag for J_i. Thus property $(18.\gamma)$ is satisfied. This proves the lemma when $E \subseteq [0,1]$.

In the general case, we apply this argument to each set $E \cap [n, n+1]$, $n \in \mathbb{Z}$, that is not empty. Q.E.D.

18.16 Approximation Theorem. *If $E \in \mathbb{I}(\mathbb{R})$ and $\varepsilon > 0$, there exists a countable collection $\{J_n\}_{n=1}^{\infty}$ of nonoverlapping compact intervals such that*

$$(18.\delta) \qquad E \subseteq \bigcup_{n=1}^{\infty} J_n \qquad \textit{and} \qquad |E| \leq \sum_{n=1}^{\infty} |J_n| \leq |E| + \varepsilon.$$

Proof. By hypothesis, the function $\mathbf{1}_E$ is integrable on $\overline{\mathbb{R}}$. Hence, given $\varepsilon > 0$ there exists a gauge δ_ε on $\overline{\mathbb{R}}$ such that if $\dot{\mathcal{P}}$ is a δ_ε-fine partition of $\overline{\mathbb{R}}$, then $\left|S(\mathbf{1}_E; \dot{\mathcal{P}}) - |E|\right| < \varepsilon$. Now let δ be the restriction of δ_ε to $\mathbb{R}$. It follows from the Dyadic Covering Lemma 18.15 that there exists a countable collection $\{(J_i, t_i)\}$ such that $E \subseteq \bigcup_i J_i$ and $t_i \in E \cap J_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$.

To establish the second assertion in $(18.\delta)$, fix $n \in \mathbb{N}$ and look at the collection $\dot{\mathcal{Q}}_n := \{(J_i, t_i)\}_{i=1}^{n}$, so that $\dot{\mathcal{Q}}_n$ is a δ_ε-fine subpartition of $\overline{\mathbb{R}}$. Therefore, the Saks-Henstock Lemma 5.3 (for $\overline{\mathbb{R}}$) implies that

$$(18.\varepsilon) \qquad \left|\sum_{i=1}^{n} l(J_i) - \sum_{i=1}^{n} |E \cap J_i|\right| = \left|S(\mathbf{1}_E; \dot{\mathcal{Q}}_n) - \sum_{i=1}^{n} \int_{J_i} \mathbf{1}_E\right| \leq \varepsilon.$$

Since $\bigcup_{i=1}^{n} E \cap J_i \subseteq E$ and the intervals $J_1, \cdots, J_n$ are nonoverlapping, then

$$(18.\zeta) \qquad \sum_{i=1}^{n} |E \cap J_i| \leq |E|.$$

Since $l(J_i) = |J_i|$, it follows from $(18.\varepsilon)$ and $(18.\zeta)$ that $\sum_{i=1}^{n} |J_i| \leq |E| + \varepsilon$. Now let $n \to \infty$ to obtain the second part of $(18.\delta)$. Q.E.D.

As a by-product of the Approximation Theorem 18.16, we obtain the characterization of null sets that was promised earlier.

18.17 Null Set Characterization Theorem. *A set $Z \subset \mathbb{R}$ is a null set if and only if $Z \in \mathbb{M}(\mathbb{R})$ and $\lambda(Z) = 0$.*

Proof. ($\Rightarrow$) This was proved in Theorem 18.10.

($\Leftarrow$) If $\lambda(Z) = 0 < \infty$, then $Z \in \mathbb{I}(\mathbb{R})$. The Approximation Theorem 18.16 implies that, given $\varepsilon > 0$ there exists a countable collection of compact intervals $\{J_n\}_{n=1}^{\infty}$ such that $Z \subseteq \bigcup_{n=1}^{\infty} J_n$ and $\sum_{n=1}^{\infty} |J_n| \leq \varepsilon$. As noted before, this implies that Z is a null set. Q.E.D.

We are now prepared to characterize measurable sets as those sets that can be approximated arbitrarily closely "from above" by open sets. Moreover, they are those sets that are the difference of a G_δ-set and a null set.

18.18 Measurable-Open Set Theorem. *If $E \subseteq \mathbb{R}$, then the following statements are equivalent:*

(a) $E \in \mathbb{M}(\mathbb{R})$.

(b) *Given $\varepsilon > 0$ there exists an open set G with $E \subseteq G$ and $|G - E| \leq \varepsilon$.*

(c) *There exist a G_δ-set H and a null set Z such that $E = H - Z$.*

Proof. (a) $\Rightarrow$ (b) Let $E_n := E \cap [-n, n]$ for $n \in \mathbb{N}$. Then $E_n \in \mathbb{I}(\mathbb{R})$ and, for each $n \in \mathbb{N}$, the Approximation Theorem 18.16 implies that there exists a sequence $(J_k^n)_{k=1}^{\infty}$ of compact intervals such that

$$E_n \subseteq \bigcup_{k=1}^{\infty} J_k^n \qquad \text{and} \qquad \sum_{k=1}^{\infty} |J_k^n| \leq |E_n| + \varepsilon/2^{n+1}.$$

By enclosing each compact interval J_k^n in a slightly larger open interval with the same center, we may suppose that the union of these open intervals is an open set $G_n \in \mathbb{I}(\mathbb{R})$ with $E_n \subseteq G_n$ and $|G_n| \leq |E_n| + \varepsilon/2^n$. By Theorem 18.3(d), we conclude that $|G_n - E_n| = |G_n| - |E_n| \leq \varepsilon/2^n$.

We now set $G := \bigcup_{n=1}^{\infty} G_n$ so that G is an open set and

$$E = \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} G_n = G.$$

It is also readily seen that $G - E \subseteq \bigcup_{n=1}^{\infty}(G_n - E_n)$. Since $\sum_{n=1}^{\infty} |G_n - E_n| \leq \varepsilon$, it follows from Theorem 18.3(k) that $\bigcup_{n=1}^{\infty}(G_n - E_n)$ belongs to $\mathbb{I}(\mathbb{R})$ and has measure $\leq \varepsilon$, whence the measurable set $G - E$ also belongs to $\mathbb{I}(\mathbb{R})$ and $|G - E| \leq \varepsilon$.

(b) $\Rightarrow$ (c) For each $n \in \mathbb{N}$, let G_n be an open set such that $E \subseteq G_n$ and $|G_n - E| \leq 1/n$, which implies that $G_n - E \in \mathbb{I}(\mathbb{R})$. We let $H_n := \bigcap_{i=1}^{n} G_i$

and $H := \bigcap_{i=1}^{\infty} G_i$. Each set H_n is open and $E \subseteq H_n$ and since

$$H_n - E = \bigcap_{i=1}^{n} (G_i - E) \subseteq G_n - E,$$

we conclude that $H_n - E \in \mathbb{I}(\mathbb{R})$ and $|H_n - E| \leq 1/n$. But the sequence $(H_n - E)_{n=1}^{\infty}$ is a decreasing sequence in $\mathbb{I}(\mathbb{R})$ and $|H_n - E| \leq 1/n$, so Theorem 18.3(h) implies that the intersection $H - E$ belongs to $\mathbb{I}(\mathbb{R})$ and $|H - E| = 0$. The Null Set Characterization Theorem 18.17 implies that $Z := H - E$ is a null set. Finally, we note that H is a G_δ-set and that $E = H - Z$.

(c) $\Rightarrow$ (a) Since G_δ-sets and null sets are in $\mathbb{M}(\mathbb{R})$, the conclusion follows from Corollary 18.6(a). Q.E.D.

The next result is dual to the preceding theorem. Here we approximate "from below" by closed sets or F_σ-sets.

18.19 Measurable-Closed Set Theorem. *If $E \subseteq \mathbb{R}$, then the following statements are equivalent:*

(a) $E \in \mathbb{M}(\mathbb{R})$.

(b) *Given $\varepsilon > 0$ there exists a closed set F with $F \subseteq E$ and $|E - F| \leq \varepsilon$.*

(c) *There exist an F_σ-set K and a null set Z such that $E = K \cup Z$.*

Proof. (a) $\Rightarrow$ (b) If $E \in \mathbb{M}(\mathbb{R})$, then $E^c \in \mathbb{M}(\mathbb{R})$, so given $\varepsilon > 0$, there exists an open set G such that $E^c \subseteq G$ and $|G - E^c| \leq \varepsilon$, so that $G - E^c \in \mathbb{I}(\mathbb{R})$. Then $F := G^c$ is a closed set with $F \subseteq E$. Since

$$E - F = E \cap F^c = G \cap E = G - E^c,$$

it follows that $E - F \in \mathbb{I}(\mathbb{R})$ and $|E - F| = |G - E^c| \leq \varepsilon$.

(b) $\Rightarrow$ (c) For each $n \in \mathbb{N}$, let F_n be a closed set such that $F_n \subseteq E$ and $|E - F_n| \leq 1/n$, which implies that $E - F_n \in \mathbb{I}(\mathbb{R})$. We let $K_n := \bigcup_{i=1}^{n} F_i$ and $K := \bigcup_{i=1}^{\infty} F_i$. Each set K_n is closed and $K_n \subseteq E$ and since

$$E - K_n = \bigcap_{i=1}^{n} (E - F_i) \subseteq E - F_n,$$

we conclude that $E - K_n \in \mathbb{I}(\mathbb{R})$ and $|E - K_n| \leq 1/n$. But the sequence $(E - K_n)_{n=1}^{\infty}$ is a decreasing sequence in $\mathbb{I}(\mathbb{R})$ and $|E - K_n| \leq 1/n$, so Theorem 18.3(h) implies that the intersection $E - K$ belongs to $\mathbb{I}(\mathbb{R})$ and $|E - K| = 0$. Thus the set $Z := E - K$ is a null set, K is an F_σ-set, and $E = K \cup Z$.

(c) $\Rightarrow$ (a) Since F_σ-sets and null sets are in $\mathbb{M}(\mathbb{R})$, the conclusion follows from Theorem 18.5(b). Q.E.D.

We conclude these approximation theorems by giving a result in which an integrable set E is approximated by a particularly simple type of set, but where the approximating set is neither inside nor outside of the set E. We recall that the "symmetric difference" of two sets A, B is the set

$$(18.\eta) \qquad A \triangle B := (A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

18.20 Approximation by Compact Sets. *Let $E \in \mathbb{I}(\mathbb{R})$ and let $\varepsilon > 0$. Then there exists a set K that is a finite union of pairwise disjoint compact intervals, such that*

$$(18.\theta) \qquad |E \triangle K| = \int_{-\infty}^{\infty} |\mathbf{1}_E - \mathbf{1}_K| \le \varepsilon.$$

Proof. By the Approximation Theorem 18.16 (with ε replaced by $\varepsilon/2$), there exists a countable collection $\{J_n\}_{n=1}^{\infty}$ of nonoverlapping compact intervals such that $E \subseteq H := \bigcup_{n=1}^{\infty} J_n$ and such that $|H - E| = |H| - |E| \le \varepsilon/2$. Let N be such that $|\bigcup_{N+1}^{\infty} J_n| \le \varepsilon/2$ and let $K := J_1 \cup \cdots \cup J_N$. Therefore $E - K \subseteq H - K$, so that $|E - K| \le |H - K| \le \varepsilon/2$. Also $K - E \subseteq H - E$, so that $|K - E| \le |H - E| \le \varepsilon/2$. Therefore

$$|E \triangle K| \le |E - K| + |K - E| \le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Since $\mathbf{1}_{E\triangle K} = |\mathbf{1}_E - \mathbf{1}_K|$, the equality in $(18.\theta)$ follows.

We note that the set K, being the finite union of compact intervals, is also a compact set. In addition, the intervals $J_1, \cdots, J_N$ are nonoverlapping. Therefore, by combining intervals that have endpoints in common, we obtain K as a finite union of pairwise disjoint compact intervals. Q.E.D.

Translation Invariance

If $A \subseteq \mathbb{R}$ and if $r \in \mathbb{R}$, then (as in 3.22) the set $A_r := \{r + y : y \in A\}$ is called the r**-translate** of A. Further, if $f : \mathbb{R} \to \mathbb{R}$ is given, then its r**-translate** is the function $f_r(y) := f(y - r)$ for all $y \in A_r$. For example, if $I := [a, b]$, then $I_r = [r + a, r + b]$, and it is clear that the length $l(I_r) = l(I)$. It is an important property of Lebesgue measure that this *translation invariance property* holds for arbitrary integrable and measurable sets.

18.21 Invariance Theorem. **(a)** *If $A \in \mathbb{I}(\mathbb{R})$ and $r \in \mathbb{R}$, then $A_r \in \mathbb{I}(\mathbb{R})$ and $|A_r| = |A|$.*

(b) *If $A \in \mathbb{M}(\mathbb{R})$ and $r \in \mathbb{R}$, then $A_r \in \mathbb{M}(\mathbb{R})$ and $\lambda(A_r) = \lambda(A)$.*

Proof. (a) Suppose that $A \in \mathbb{I}(\mathbb{R})$ and $r \in \mathbb{R}$. We first suppose that A is bounded and let I be a compact interval with $A \subseteq I$. If $\mathbf{1}_A : \mathbb{R} \to \mathbb{R}$ is the characteristic function of A, it is easy to see that its translate $(\mathbf{1}_A)_r$ is the characteristic function of A_r. It we apply Theorem 3.22, we obtain

$$\begin{aligned} |A| &= \int_{-\infty}^{\infty} \mathbf{1}_A = \int_I \mathbf{1}_A = \int_{I_r} (\mathbf{1}_A)_r \\ &= \int_{I_r} \mathbf{1}_{A_r} = \int_{-\infty}^{\infty} \mathbf{1}_{A_r} = |A_r|. \end{aligned}$$

If A is not bounded, let $A_n := A \cap [-n, n]$ and use Theorem 18.3(i).

(b) If $A \in \mathbb{M}(\mathbb{R})$ is not in $\mathbb{I}(\mathbb{R})$, then $A \cap I \in \mathbb{I}(\mathbb{R})$ for every compact interval I. Let $J := I_{-r}$ so that J is a compact interval and $I = J_r$. Since it is easily seen that $A \cap I = A_r \cap J_r = (A \cap J)_r$, and since $(A \cap J)_r$ belongs to $\mathbb{I}(\mathbb{R})$, we deduce that A_r also belongs to $\mathbb{M}(\mathbb{R})$. If $\lambda(A) = \infty$, then we must have $\lambda(A_r) = \infty$; otherwise, $A_r \in \mathbb{I}(\mathbb{R})$ which implies that $A = (A_r)_{-r}$ belongs to $\mathbb{I}(\mathbb{R})$, which is a contradiction. Q.E.D.

A Nonmeasurable Set

We close this section by giving an example, due to G. Vitali, of a subset of $\mathbb{R}$ that is *not* Lebesgue measurable. In fact, it follows from the construction that: *Every Lebesgue measurable set E with $\lambda(E) > 0$ contains a subset that is not Lebesgue measurable.* (See Exercise 18.W.)

18.22 Theorem. *There exists a set $V \subset (0, 1)$ that is not Lebesgue measurable.*

Proof. We define a relation on $\mathbb{R}$ by: $x \sim y$ means that $x - y \in \mathbb{Q}$. It is readily seen that $\sim$ is an equivalence relation; that is, it satisfies (i) $x \sim x$, (ii) $x \sim y \Rightarrow y \sim x$, and (iii) $x \sim y$ and $y \sim z \Rightarrow x \sim z$. Therefore this relation divides $\mathbb{R}$ into a collection of disjoint equivalence classes. It is evident that each equivalence class has the form $\mathbb{Q}_x = \{x + q : q \in \mathbb{Q}\}$ for some $x \in \mathbb{R}$. Thus, each equivalence class has nonvoid intersection with each nonempty open set. By the Axiom of Choice, there is a set $V \subset (0, 1)$ that intersects each distinct equivalence class in exactly one point. We will show that the supposition that V is measurable leads to a contradiction to the Invariance Theorem 18.21.

For, if V is measurable, then it is integrable and so is its translate V_r for any $r \in \mathbb{Q}$; moreover, $|V_r| = |V|$. We note that if $r, s \in \mathbb{Q}$ and $r \neq s$, then the intersection $V_r \cap V_s = \emptyset$. For, if this intersection is not empty, there exist elements $v_1, v_2 \in V$ such that $r + v_1 = s + v_2$, whence $v_1 - v_2 = s - r \in \mathbb{Q}$. But this implies that $v_1 \sim v_2$ and $v_1 \neq v_2$, which contradicts the fact that V

contains only one element from each equivalence class. Thus the collection $\{V_r : r \in \mathbb{Q}\}$ of integrable sets is pairwise disjoint.

Now let $C := \mathbb{Q} \cap (0,2)$, so that C is a countable set. We will show that

$$(1,2) \subseteq \bigcup_{r \in C} V_r \subseteq (0,3). \tag{18.ι}$$

Indeed, if $x \in \mathbb{R}$, then there exists a unique element $v_x \in V$ such that $x \sim v_x$ and we let $r := x - v_x \in \mathbb{Q}$. If $x \in (1,2)$, then since $v_x \in (0,1)$, it follows that $r \in (0,2)$. Therefore $x \in V_r$ for some $r \in C$. On the other hand, since $V \subseteq (0,1)$, we also have $V_r \subseteq (0,3)$ for each $r \in (0,2)$ and the second inclusion also holds.

In view of the pairwise disjointness of the sets $\{V_r : r \in C\}$ and the countable additivity of Lebesgue measure, it follows from (18.ι) that

$$1 = |(1,2)| \leq \sum_{r \in C} |V_r| \leq |(0,3)| = 3.$$

We recall that $|V_r| = |V|$. Hence all of the terms in the sum are equal to $|V|$. If $|V| > 0$, then this sum diverges, contradicting that its sum is ≤ 3. On the other hand, if $|V| = 0$, then this sum equals 0, contradicting that its sum is ≥ 1.

We conclude that the set V is not Lebesgue measurable. Q.E.D.

For other results concerning nonmeasurable sets, see [B-1; Chapter 17].

Exercises

18.A Let $\Omega \neq \emptyset$. A nonempty collection $\mathcal{D}$ of subsets of Ω is said to be a **ring of sets** in Ω if $A \cup B$ and $A - B$ belong to $\mathcal{D}$ whenever $A, B \in \mathcal{D}$.

(a) If $\mathcal{D}$ is a ring of sets in Ω, show that the empty set $\emptyset$, the intersection $A \cap B$, and the symmetric difference $A \Delta B := (A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ belong to $\mathcal{D}$ whenever $A, B \in \mathcal{D}$.

(b) If $A_1, \cdots, A_n$ belong to $\mathcal{D}$, show that the sets $A_1 \cup \cdots \cup A_n$ and $A_1 \cap \cdots \cap A_n$ also belong to $\mathcal{D}$.

18.B A ring $\mathcal{D}$ of sets in Ω is said to be a δ-**ring** if for every sequence $(A_n)_{n=1}^{\infty}$ of sets in $\mathcal{D}$, the intersection $\bigcap_{n=1}^{\infty} A_n$ also belongs to $\mathcal{D}$. Now let Ω be an *uncountable* set.

(a) Show that the collection $\mathcal{D}_1$ of *all* subsets of Ω is a ring of sets. Show that it is a δ-ring.

(b) Show that the collection $\mathcal{D}_2$ of all *finite* subsets of Ω is a ring of sets. Show that it is a δ-ring.

(c) Show that the collection $\mathcal{D}_3$ of all *countable* subsets of Ω is a ring of sets. Show that it is a δ-ring.

18.C A nonempty collection $\mathcal{A}$ of sets in Ω is said to be an **algebra of sets** in Ω in case it is a ring of sets and contains the entire set Ω.

(a) Show that a nonempty collection $\mathcal{A}$ of subsets in Ω is an algebra if and only if the union $A \cup B$ and the complement $A^c := \Omega - A$ belong to $\mathcal{A}$ whenever $A, B \in \mathcal{A}$.

(b) Show that the collection $\mathcal{A}_1$ of *all* subsets of a nonempty set Ω is an algebra of sets.

(c) Show that the collection $\mathcal{A}_2$ of all *finite* subsets of an infinite set Ω is not an algebra of sets in Ω.

(d) Show that the collection $\mathcal{A}_3$ of all *countable* subsets of a set Ω is an algebra of sets if Ω is countable, but is not an algebra if Ω is uncountable.

(e) If Ω is an uncountable set, show that the collection $\mathcal{A}_4$ of all sets in Ω that are either countable or have countable complements is an algebra of sets in Ω.

(f) Which of the collections $\mathcal{A}_1, \cdots, \mathcal{A}_4$ of (b)–(e) are σ-algebras in the sense defined immediately before Definition 18.14 ?

18.D (a) If $\mathcal{D}_1$ and $\mathcal{D}_2$ are rings of sets in a set $\Omega \neq \emptyset$, show that the intersection $\mathcal{D}_1 \cap \mathcal{D}_2 := \{E \subseteq \Omega : E \in \mathcal{D}_1 \text{ and } E \in \mathcal{D}_2\}$ is a ring of sets in Ω.

(b) Show that the union $\mathcal{D}_1 \cup \mathcal{D}_2 := \{E \subseteq \Omega : E \in \mathcal{D}_1 \text{ or } E \in \mathcal{D}_2\}$ may not be a ring of sets in Ω.

(c) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-algebras in Ω, show that $\mathcal{A}_1 \cap \mathcal{A}_2$ is a σ-algebra of sets in Ω.

18.E Let $\mathcal{F}$ be a nonempty collection of sets in a nonempty set Ω.

(a) If $\mathcal{D}_1$ and $\mathcal{D}_2$ are rings of sets in Ω containing the sets in $\mathcal{F}$, show that $\mathcal{D}_1 \cap \mathcal{D}_2$ is a ring of sets containing $\mathcal{F}$.

(b) If $\{\mathcal{D}_\alpha\}$ is the collection of all rings of sets in Ω containing $\mathcal{F}$, show that $\mathcal{D} := \bigcap_\alpha \mathcal{D}_\alpha$ is a ring of sets containing $\mathcal{F}$. Moreover, $\mathcal{D}$ is

the smallest such ring in the sense that if $\mathcal{E}$ is any ring of sets in Ω containing $\mathcal{F}$, then $\mathcal{D} \subseteq \mathcal{E}$.

(c) If $\{\mathcal{A}_\alpha\}$ is the collection of all σ-algebras of sets in Ω containing $\mathcal{F}$, show that $\mathcal{A} := \bigcap_\alpha \mathcal{A}_\alpha$ is a σ-algebra of sets in Ω containing $\mathcal{F}$, and is the smallest such σ-algebra.

18.F Show that a set $Z \subset \mathbb{R}$ is a null set if and only if for every $\varepsilon > 0$ there exists a sequence $(K_i)_{i=1}^\infty$ of compact intervals such that $Z \subseteq \bigcup_{i=1}^\infty K_i$ and $\sum_{i=1}^\infty |K_i| \leq \varepsilon$.

18.G Show that every countable set in $\mathbb{R}$ is a null set. Use this to show that the set $\mathbb{R}$ is not countable.

18.H Show that every null set is a subset of a Borel null set.

18.I (a) Show that the set $\mathbb{Q}$ of all rational numbers is an F_σ-set. Show that $\mathbb{Q}$ is not open in $\mathbb{R}$, and that $\mathbb{Q}$ is not closed in $\mathbb{R}$.

(b) Given $\varepsilon > 0$ show that there exists an open set G_ε such that $\mathbb{Q} \subseteq G_\varepsilon$ and $|G_\varepsilon| \leq \varepsilon$.

(c) Does part (b) imply that $\mathbb{Q}$ is a G_δ-set?

18.J (a) Give an example of a set $U \in \mathbb{I}(\mathbb{R})$ with $|U| \leq 1$ and such that $U \cap [n, n+1] \neq \emptyset$ for all $n \in \mathbb{N}$.

(b) Give an example of an open set U with the property in (a).

18.K Show that a set $F \subseteq \mathbb{R}$ is closed if and only if, for every convergent sequence of points $(x_n)_{n=1}^\infty$ in F, the limit $x_0 := \lim(x_n)$ belongs to F.

18.L Let F be a closed set in $\mathbb{R}$.

(a) Show that the set $G_1 := \{x \in \mathbb{R} : \mathrm{dist}(x, F) < 1\}$ is an open set.

(b) Show that there exists a sequence $(G_n)_{n=1}^\infty$ of open sets such that $F = \bigcap_{n=1}^\infty G_n$.

(c) Show that there exists a sequence $(K_n)_{n=1}^\infty$ of compact sets such that $F = \bigcup_{n=1}^\infty K_n$.

18.M A point z is said to be an **interior point** of a set $A \subseteq \mathbb{R}$ if there exists $r > 0$ such that $(z - r, z + r) \subseteq A$.

(a) Show that a set A is open if and only if every point in A is an interior point of A.

(b) Show that if G is any nonempty open set in $\mathbb{R}$, then $\lambda(G) > 0$.

18.N We have seen that the sets $\emptyset$ and $\mathbb{R}$ are *both* open and closed in $\mathbb{R}$. The purpose of this exercise is to show that no other subset of $\mathbb{R}$ is both open and closed. For suppose that H with $\emptyset \subset H \subset \mathbb{R}$ is both open and closed and let $x \in H$, $y \in H^c$. In case we have $x < y$, let $z := \sup\{u \in H : u \leq y\}$. Show that either of the hypotheses $z \in H$ or $z \in H^c$ leads to a contradiction.

18.O We say that a closed set $F \subseteq \mathbb{R}$ is **nowhere dense** if it does not contain any interior points of F (in the sense of Exercise 18.M).

(a) Give an example of a nowhere dense closed set with $|F| > 0$.

(b) If F is closed and nowhere dense and $y \notin F$, show that there exists a nondegenerate closed interval J with center y such that $F \cap J = \emptyset$.

18.P We now show that $\mathbb{R}$ is not the union of a sequence $(F_n)_{n=1}^{\infty}$ of closed nowhere dense sets. [This is a form of the **Baire Category Theorem** proved, in 1899, by René Baire; it had been proved earlier by W. F. Osgood.]

(a) Suppose that $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$, where each set F_n is closed and nowhere dense. Use the argument in Exercise 18.O(b) to find a nondegenerate closed interval J_1 such that $F_1 \cap J_1 = \emptyset$. Then find a nondegenerate closed interval $J_2 \subseteq J_1$ such that $F_2 \cap J_2 = \emptyset$ and the length of J_2 is $\leq 1/2$.

(b) Construct a sequence $(J_n)_{n=1}^{\infty}$ of nondegenerate closed intervals such that $F_n \cap J_n = \emptyset$ and $J_{n+1} \subseteq J_n$ for $n \in \mathbb{N}$ and the length of J_n is $\leq 1/n$.

(c) Use the Nested Intervals Theorem (see [B-S; p. 46]) to obtain a point $\xi \in \bigcap_{n=1}^{\infty} J_n$ that does not belong to $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$.

18.Q (a) Use the Baire Category Theorem to show that the set $\mathbb{Q}$ is not a G_δ-set.

(b) Give an example of a G_δ-set that is not closed, not open, and not an F_σ-set.

18.R Let $C := [0,1] - \Gamma$, where Γ is the Cantor set.

(a) Show that C is an open set in $\mathbb{R}$.

(b) Show that C is the countable union of pairwise disjoint open intervals.

(c) What is the measure of the set C ?

18.S Use the argument in the Dyadic Covering Lemma 18.15 to show that if G is a nonempty open set in $\mathbb{R}$, then there exists a nonoverlapping sequence $(J_n)_{n=1}^{\infty}$ of closed intervals such that $G = \bigcup_{n=1}^{\infty} J_n$.

18.T (a) Show that a set $E \subseteq \mathbb{R}$ is measurable if and only if for every $\varepsilon > 0$ there are an open set $G \supseteq E$ and a closed set $F \subseteq E$ such that $|G - F| \leq \varepsilon$.

(b) Show that a set $E \subset \mathbb{R}$ is integrable if and only if for every $\varepsilon > 0$ there is a compact set $K \subseteq E$ such that $|E - K| \leq \varepsilon$.

(c) Show that the conclusion of (b) may fail if E is measurable.

18.U If $A \subset \mathbb{R}$, define the **difference set** $\Delta(A) := \{x - y : x, y \in A\}$.

(a) If $A \subseteq B \subset \mathbb{R}$, show that $\Delta(A) \subseteq \Delta(B)$.

(b) If V is a Vitali set, constructed in the proof of Theorem 18.22, show that $\mathbb{R}$ is the union of the pairwise disjoint sets $\{V_r : r \in \mathbb{Q}\}$.

(c) Show that $\Delta(V)$ does not contain any nonzero rational numbers.

18.V Let $K \subset \mathbb{R}$ be a compact set with $|K| > 0$.

(a) Show that there exists a bounded open set G such that $K \subset G$ and $|G| < 2|K|$.

(b) Let $\delta := \inf\{|k - x| : k \in K,\ x \in G^c\}$. Show that $\delta > 0$ and that if $|y| < \delta$, then the translate $K_y \subseteq G$.

(c) Show that if $|y| < \delta$, then $K_y \cap K \neq \emptyset$.

(d) Show that the difference set $\Delta(K)$ contains the open interval $(-\delta, \delta)$.

(e) If $E \in \mathbb{M}(\mathbb{R})$ is any set with $\lambda(E) > 0$, then there exists $\delta > 0$ such that $(-\delta, \delta) \subseteq \Delta(E)$. (This result is called **Steinhaus's Theorem**.)

18.W Let $H \in \mathbb{M}(\mathbb{R})$ with $\lambda(H) > 0$.

(a) Show that there exists a set $E \subseteq H$ with $E \in \mathbb{I}(\mathbb{R})$ and $|E| > 0$.

(b) Let V be as in the proof of Theorem 18.22 and let $E(r) := E \cap V_r$ for $r \in \mathbb{Q}$. Show that at least one of the sets $E(r)$ does not belong to $\mathbb{M}(\mathbb{R})$.

(c) Show that there exist nonmeasurable sets $B, C \subseteq E$ such that $E = B \cup C$ and $B \cap C = \emptyset$.

Section 19

Measurable Functions

We will now discuss measurable functions defined on $\mathbb{R}$ with values in $\mathbb{R}$. Among other things, we will obtain results that are very similar to those in Section 10 for functions defined on a compact interval $[a, b]$.

We define a function $s : \mathbb{R} \to \mathbb{R}$ to be a **step function** if it is a finite linear combination of characteristic functions of bounded intervals in $\mathbb{R}$. (Although this definition may appear to be slightly different from Definition 17.6 for functions on $[a, \infty)$, it is easily seen to be equivalent to it.) Note that a step function equals 0 outside some bounded interval. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is **Lebesgue measurable**, or simply **measurable**, if there exists a sequence $(s_n)_{n=1}^{\infty}$ of step functions on $\mathbb{R}$ such that

$$f(x) = \lim_{n\to\infty} s_n(x) \qquad \text{almost everywhere on } \mathbb{R}.$$

It is seen as in Theorem 17.7 that: *A function $f : \mathbb{R} \to \mathbb{R}$ is measurable if and only if its restriction to every compact interval $I \subset \mathbb{R}$ is measurable in the sense of Definition 6.1.* The collection of all measurable functions on $\mathbb{R}$ will be denoted by $\mathcal{M}(\mathbb{R})$.

Although our main interest is in Lebesgue measurable functions, we will consider briefly what is meant by a measurable space $(X, \mathcal{A})$ and by the $\mathcal{A}$-measurability of a function $f : X \to \mathbb{R}$. This will enable us to put the notion of a Borel measurable function, etc., into proper perspective. We also give a strengthened version of Luzin's Theorem, encountered in Theorem 11.4. Finally, we discuss "indefinite integrals" and present a brief introduction to the theory of measures on σ-algebras of sets.

A Characterization Theorem

We first relate the measurability on $\mathbb{R}$ to the measurable sets in $\mathbb{R}$; this result is quite parallel to Theorem 10.4. However, for the sake of clarity, we will first give a lemma whose validity is almost obvious.

19.1 Lemma. *If J is a compact interval in $\mathbb{R}$, then*

$$\mathbb{I}(J) = \mathbb{M}(J) \subset \mathbb{I}(\mathbb{R}) \subset \mathbb{M}(\mathbb{R}).$$

Proof. It was remarked in 6.15(a) that $\mathbb{I}(J) = \mathbb{M}(J)$. Further, if $E \in \mathbb{M}(J)$, then the characteristic function $\mathbf{1}_E$ belongs to $\mathcal{M}(J)$. Since $\mathbf{1}_E$ is bounded and equals 0 outside J, it is integrable on $\overline{\mathbb{R}}$, so that $E \in \mathbb{I}(\mathbb{R}) \subset \mathbb{M}(\mathbb{R})$. Q.E.D.

19.2 Characterization of Measurability Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$. Then the following statements are equivalent:*

(a) *The function f belongs to $\mathcal{M}(\mathbb{R})$.*

(b) *For every $r \in \mathbb{R}$ the set $\{f < r\} := \{x \in \mathbb{R} : f(x) < r\}$ is in $\mathbb{M}(\mathbb{R})$.*

(c) *For every $r \in \mathbb{R}$ the set $\{f \geq r\} := \{x \in \mathbb{R} : f(x) \geq r\}$ is in $\mathbb{M}(\mathbb{R})$.*

(d) *For every $r \in \mathbb{R}$ the set $\{f \leq r\} := \{x \in \mathbb{R} : f(x) \leq r\}$ is in $\mathbb{M}(\mathbb{R})$.*

(e) *For every $r \in \mathbb{R}$ the set $\{f > r\} := \{x \in \mathbb{R} : f(x) > r\}$ is in $\mathbb{M}(\mathbb{R})$.*

Proof. (a) $\Rightarrow$ (b) Let $r \in \mathbb{R}$ be given and let I be a compact interval. If $f \in \mathcal{M}(\mathbb{R})$, then the restriction $f|I$ is in $\mathcal{M}(I)$, so Theorem 10.4 implies that the set $\{x \in I : f(x) < r\}$ belongs to $\mathbb{M}(I)$. Since it is clear that

$$(19.\alpha) \qquad I \cap \{x \in \mathbb{R} : f(x) < r\} = \{x \in I : f(x) < r\},$$

and since every such intersection belongs to $\mathbb{M}(I)$, we conclude that the set $\{f < r\}$ belongs to $\mathbb{M}(\mathbb{R})$.

(b) $\Rightarrow$ (a) If $\{f < r\} \in \mathbb{M}(\mathbb{R})$ then it follows from $(19.\alpha)$ that the set $\{x \in I : f(x) < r\} \in \mathbb{M}(I)$. Since this holds for all $r \in \mathbb{R}$, Theorem 10.4 implies that $f|I$ is measurable on I. Since this holds for every compact interval $I \subset \mathbb{R}$, then $f \in \mathcal{M}(\mathbb{R})$.

(b) $\Leftrightarrow$ (c) The complement of a set in $\mathbb{M}(\mathbb{R})$ belongs to $\mathbb{M}(\mathbb{R})$.

(b) $\Leftrightarrow$ (d) The set $\{f \leq r\}$ is the intersection of $\{f < r + 1/n\}$, $n \in \mathbb{N}$, and the set $\{f < r\}$ is the union of $\{f \leq r - 1/n\}$, $n \in \mathbb{N}$.

(d) $\Leftrightarrow$ (e) This follows by complementation. Q.E.D.

The following strengthened form of Theorem 19.2(b) is sometimes useful.

19.3 Corollary. *A function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable if and only if the set $\{f > q\}$ is in $\mathbb{M}(\mathbb{R})$ for every rational number $q \in \mathbb{Q}$.*

Proof. ($\Rightarrow$) This direction is immediate from 19.2.

($\Leftarrow$) Given $c \in \mathbb{R}$, let (q_n) be a strictly decreasing sequence of rational numbers converging to c. If $f(x) > c$, then there exists a rational q_n such that $f(x) > q_n$. Therefore $\{f > c\} = \bigcup_{n=1}^{\infty}\{f > q_n\} \in \mathbb{M}(\mathbb{R})$, and the conclusion follows from Theorem 18.5(c). Q.E.D.

It is an exercise to show that we can use sets of any of the four types of sets in Theorem 19.2 with $r \in \mathbb{Q}$ to establish the measurability of a function. It also follows from Theorem 19.2 that we could define a function $f : \mathbb{R} \to \mathbb{R}$ to be measurable in case it satisfies any of the conditions (b)–(e) of that theorem. There are other equivalent conditions that we could use, based on the properties of the inverse images of functions, which we now review.

Direct and Inverse Images

First we recall some facts that are probably well known to the reader, but which will be needed. Here we let X, Y be any abstract sets and consider a function $f : X \to Y$.

19.4 Definition. If $E \subseteq X$, then the **direct image** of E under f is the set

$$f(E) := \{f(x) \in Y : x \in E\} \subseteq Y. \tag{19.β}$$

Similarly, the **inverse image** of $H \subseteq Y$ under f is the set

$$f^{-1}(H) := \{x \in X : f(x) \in H\} \subseteq X. \tag{19.γ}$$

Note. It has frequently been pointed out that these notations are not very good, and that something like $f^{\rightarrow}(E)$ and $f^{\leftarrow}(H)$ would be preferable; however, these suggested notations have not been widely adopted, so we will use the conventional ones.

It is well known that the mapping $E \mapsto f(E)$ of subsets of X into subsets of Y is not as well-behaved as the mapping $H \mapsto f^{-1}(H)$ of subsets of Y into subsets of X. For example, for any sets $C, E \subseteq X$, the direct image mapping satisfies the conditions:

$$\begin{aligned} f(C \cap E) &\subseteq f(C) \cap f(E), \\ f(C \cup E) &= f(C) \cup f(E), \\ f(C) - f(E) \subseteq f(C - E) &\subseteq f(C), \end{aligned} \tag{19.δ}$$

and, in general, the inclusions cannot be replaced by equalities. However, for any sets $H, K \subseteq Y$, the inverse image mapping satisfies the conditions:

$$(19.\varepsilon)\qquad \begin{aligned} f^{-1}(H \cap K) &= f^{-1}(H) \cap f^{-1}(K),\\ f^{-1}(H \cup K) &= f^{-1}(H) \cup f^{-1}(K),\\ f^{-1}(H - K) &= f^{-1}(H) - f^{-1}(K). \end{aligned}$$

In fact, if $\{H_\alpha\}$ is *any* collection of subsets in Y, then one can show that:

$$(19.\zeta)\qquad f^{-1}\Big(\bigcap_\alpha H_\alpha\Big) = \bigcap_\alpha f^{-1}(H_\alpha) \quad\text{and}\quad f^{-1}\Big(\bigcup_\alpha H_\alpha\Big) = \bigcup_\alpha f^{-1}(H_\alpha).$$

For example, $x \in f^{-1}\big(\bigcup_\alpha H_\alpha\big) \Leftrightarrow f(x) \in \bigcup_\alpha H_\alpha \Leftrightarrow$ there exists α such that $f(x) \in H_\alpha \Leftrightarrow$ there exists α such that $x \in f^{-1}(H_\alpha) \Leftrightarrow x \in \bigcup_\alpha f^{-1}(H_\alpha)$.

If $f : \mathbb{R} \to \mathbb{R}$, it is probably known to the reader that f is continuous on $\mathbb{R}$ if and only if the set $f^{-1}(G)$ is open for every open set $G \subseteq \mathbb{R}$ (see [B-S; p. 324]). In fact, in topology courses, this property is usually taken as the definition of a continuous function on $\mathbb{R}$. We now show that a measurable function could be defined in a very similar way, using inverse images.

We recall from Definition 18.14 that the collection of all Borel sets $\mathbb{B}(\mathbb{R})$ is the smallest σ-algebra of sets containing the open sets in $\mathbb{R}$.

19.5 Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$.*

(a) *The function f is Lebesgue measurable if and only if $f^{-1}(G) \in \mathbb{M}(\mathbb{R})$ for every open set $G \subseteq \mathbb{R}$.*

(b) *The function f is Lebesgue measurable if and only if $f^{-1}(B) \in \mathbb{M}(\mathbb{R})$ for every Borel set $B \subseteq \mathbb{R}$.*

Proof. (a) ($\Rightarrow$) If $a < b$, then since $(a, b) = \{y > a\} \cap \{y < b\}$, it follows from the first equation in $(19.\varepsilon)$ that

$$\begin{aligned} f^{-1}\big((a,b)\big) = f^{-1}\big(\{y > a\} \cap \{y < b\}\big) &= f^{-1}\big(\{y > a\}\big) \cap f^{-1}\big(\{y < b\}\big)\\ &= \{f > a\} \cap \{f < b\}. \end{aligned}$$

Therefore, if f is measurable on $\mathbb{R}$, Theorem 19.2 implies that $f^{-1}\big((a,b)\big)$ belongs to $\mathbb{M}(\mathbb{R})$. We have seen in Section 18 that every open set $G \subseteq \mathbb{R}$ is the union of a sequence of open intervals which are in $\mathbb{M}(\mathbb{R})$, so $(19.\zeta)$ implies that $f^{-1}(G)$ is in $\mathbb{M}(\mathbb{R})$.

($\Leftarrow$) Since $(b, \infty) = \{y > b\}$ is an open set, the condition implies that $\{f > b\} = f^{-1}\big((b, \infty)\big) \in \mathbb{M}(\mathbb{R})$ for every $b \in \mathbb{R}$. Theorem 19.2 now implies that f is measurable.

(b) ($\Rightarrow$) Suppose that f is measurable and let $\mathcal{A}$ be the collection of all sets $H \subseteq \mathbb{R}$ such that $f^{-1}(H) \in \mathbb{M}(\mathbb{R})$. Part (a) implies that $\mathcal{A}$ contains every open set.

We claim that $\mathcal{A}$ is a σ-algebra of sets in $\mathbb{R}$. Indeed, $f^{-1}(\emptyset) = \emptyset \in \mathbb{M}(\mathbb{R})$ so that $\emptyset \in \mathcal{A}$, and similarly $\mathbb{R} \in \mathcal{A}$. Further, if $H \in \mathcal{A}$, then since $f^{-1}(H^c) = f^{-1}(\mathbb{R} - H) = \mathbb{R} - f^{-1}(H) \in \mathbb{M}(\mathbb{R})$, it follows that $H^c \in \mathcal{A}$. Finally, if (H_n) is a sequence in $\mathcal{A}$, then by (19.ζ) we have

$$f^{-1}\Big(\bigcup_{n=1}^{\infty} H_n\Big) = \bigcup_{n=1}^{\infty} f^{-1}(H_n) \in \mathbb{M}(\mathbb{R}),$$

whence $\bigcup_{n=1}^{\infty} H_n \in \mathcal{A}$, showing that $\mathcal{A}$ is a σ-algebra. Since $\mathcal{A}$ contains the open sets, then $\mathcal{A}$ contains the Borel sets (which is the smallest σ-algebra containing the open sets).

($\Leftarrow$) Since every open set G is a Borel set, the condition implies that $f^{-1}(G) \in \mathbb{M}(\mathbb{R})$. Thus f is measurable by part (a). Q.E.D.

The General Notion of Measurability

It was seen in Theorems 6.3 and 6.6 that various combinations (such as sums, products, composition by continuous functions, maxima and minima, etc.) of measurable functions in $\mathcal{M}(I)$ again yield measurable functions. However, it was difficult to prove that the limit of a sequence of measurable functions on I is always measurable. (Indeed, that fact was not proved until Theorem 9.2, and its proof used the Dominated Convergence Theorem 8.8.)

We will now show that, if the characterization of measurable functions given in Theorem 19.5 is taken as the *definition* of measurability, then the result concerning the limit is much more readily established.

In fact, we will make use of the ideas in Theorem 19.5 to define a more general notion of measurability for a function with respect to an arbitrary σ-algebra. Although we are primarily interested in functions defined on $\mathbb{R}$, we will formulate this definition for a real-valued function defined on an abstract set X.

19.6 Definition. A **measurable space** $(X, \mathcal{A})$ is a pair consisting of a set X and a σ-algebra $\mathcal{A}$ of subsets of X.

19.7 Examples. (a) Let X be an arbitrary set and let $\mathcal{A}$ denote the collection of all subsets of X. Then $\mathcal{A}$ is a σ-algebra, so $(X, \mathcal{A})$ is a measurable space.

(b) Let X be an uncountable set and let $\mathcal{C}$ denote the collection of all subsets of X that are either countable, or have countable complements. Then (see Exercise 18.C) $\mathcal{C}$ is a σ-algebra, so $(X, \mathcal{C})$ is a measurable space.

(c) It was seen in Theorem 18.5 that $\mathbb{M}(\mathbb{R})$ is a σ-algebra of sets in $\mathbb{R}$. Thus $(\mathbb{R}, \mathbb{M}(\mathbb{R}))$ is a measurable space and it is called the **Lebesgue measurable space** on $\mathbb{R}$.

(d) By Definition 18.14, the collection $\mathbb{B}(\mathbb{R})$ of all Borel subsets of $\mathbb{R}$ is a σ-algebra of sets in $\mathbb{R}$. The pair $(\mathbb{R}, \mathbb{B}(\mathbb{R}))$ is called the **Borel measurable space** on $\mathbb{R}$.

We now define what is meant by the measurability of a real-valued function $f : X \to \mathbb{R}$ with respect to an arbitrary σ-algebra $\mathcal{A}$ of sets in X.

19.8 Definition. Let $(X, \mathcal{A})$ be a measurable space. A function $f : X \to \mathbb{R}$ is said to be **$\mathcal{A}$-measurable** if $f^{-1}(G) \in \mathcal{A}$ for every open set $G \subseteq \mathbb{R}$.

We now give some examples; some proofs are left as exercises.

19.9 Examples. (a) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then f is Borel measurable.

For, if G is open and f is continuous, then $f^{-1}(G)$ is open and therefore belongs to $\mathbb{B}(\mathbb{R})$.

(b) If $g : \mathbb{R} \to \mathbb{R}$ is monotone, then g is Borel measurable.

Indeed, if g is increasing, then the set $\{g > c\} = g^{-1}((c, \infty))$ is an interval having one of the forms (γ, ∞) or $[\gamma, \infty)$. [Show that both cases can occur.] It follows (why?) that $g^{-1}(G)$ is in $\mathbb{B}(\mathbb{R})$ for each open set G.

(c) Every Borel measurable function is Lebesgue measurable.

This follows from the fact that $\mathbb{B}(\mathbb{R}) \subset \mathbb{M}(\mathbb{R})$.

(d) If $E \subset \mathbb{R}$ is a set that is not a Borel set, then the characteristic function $\mathbf{1}_E$ is not Borel measurable.

Note that $(\mathbf{1}_E)^{-1}((0, 2)) = E$.

(e) If $E \subset \mathbb{R}$, then $\mathbf{1}_E$ is Lebesgue measurable if and only if $E \in \mathbb{M}(\mathbb{R})$.

(f) A function $\varphi : X \to \mathbb{R}$ is said to be an **$\mathcal{A}$-simple function** if it is a finite linear combination of characteristic functions of sets in $\mathcal{A}$. Thus, φ has the form

$$(19.\eta) \qquad \varphi = \sum_{i=1}^{n} c_i \mathbf{1}_{E_i}, \qquad \text{where} \quad c_i \in \mathbb{R},\ E_i \in \mathcal{A}.$$

It is an exercise to show that we may assume that the sets E_i are pairwise disjoint; in this case, the set $\{\varphi > c\}$ is the union of the sets E_i for which $c_i > c$.

The next result has some interest and will be used in Theorem 19.12.

19.10 Theorem. *Let $(X, \mathcal{A})$ be a measurable space. Then $f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable if and only if $f^{-1}(B) \in \mathcal{A}$ for every Borel set $B \in \mathbb{B}(\mathbb{R})$.*

Proof. ($\Leftarrow$) Since an open set is in $\mathbb{B}(\mathbb{R})$, this follows immediately from Definition 19.8.

($\Rightarrow$) Let $\mathcal{B}$ be the collection of all sets $H \subseteq \mathbb{R}$ such that $f^{-1}(H) \in \mathcal{A}$. The hypothesis that f is $\mathcal{A}$-measurable implies that every open set is in $\mathcal{B}$.

We claim that $\mathcal{B}$ is a σ-algebra of sets in $\mathbb{R}$. Indeed, $f^{-1}(\emptyset) = \emptyset \in \mathcal{A}$, so that $\emptyset \in \mathcal{B}$, and similarly $\mathbb{R} \in \mathcal{B}$. Further, if $H \in \mathcal{B}$, then since $f^{-1}(H^c) = f^{-1}(\mathbb{R} - H) = X - f^{-1}(H) \in \mathcal{A}$, it follows that $H^c \in \mathcal{B}$. Finally, if $(H_n)_{n=1}^\infty$ is a sequence in $\mathcal{B}$, then by (19.ζ) we have

$$f^{-1}\Big(\bigcup_{n=1}^\infty H_n\Big) = \bigcup_{n=1}^\infty f^{-1}(H_n) \in \mathcal{A},$$

whence $\bigcup_{n=1}^\infty H_n \in \mathcal{B}$, showing that $\mathcal{B}$ is a σ-algebra. Since $\mathcal{B}$ is a σ-algebra containing the open sets in $\mathbb{R}$, then $\mathcal{B}$ contains $\mathbb{B}(\mathbb{R})$. Therefore $f^{-1}(B) \in \mathcal{A}$ for every $B \in \mathbb{B}(\mathbb{R})$. Q.E.D.

19.11 Corollary. *A function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable if and only if $f^{-1}(B) \in \mathbb{B}(\mathbb{R})$ for all $B \in \mathbb{B}(\mathbb{R})$.*

Proof. Take $\mathcal{A} = \mathbb{B}(\mathbb{R})$ in Theorem 19.10. Q.E.D.

Combinations of Measurable Functions

We will now see that measurable functions can be combined in many ways to yield other measurable functions. We will first consider compositions of a continuous or a Borel measurable function with an $\mathcal{A}$-measurable function.

19.12 Theorem. *Suppose $f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable and $\varphi : \mathbb{R} \to \mathbb{R}$.*

(a) *If φ is continuous, then $\varphi \circ f$ is $\mathcal{A}$-measurable.*

(b) *If φ is Borel measurable, then $\varphi \circ f$ is $\mathcal{A}$-measurable.*

Proof. In view of 19.9(a), it suffices to prove (b). We note that

$$(\varphi \circ f)^{-1}(H) = f^{-1}\big(\varphi^{-1}(H)\big)$$

for any set $H \subseteq \mathbb{R}$. If $G \subseteq \mathbb{R}$ is open and φ is Borel measurable, then $B_1 := \varphi^{-1}(G)$ belongs to $\mathbb{B}(\mathbb{R})$. Therefore it follows from Theorem 19.10 that $f^{-1}(\varphi^{-1}(G)) = f^{-1}(B_1) \in \mathcal{A}$. Therefore $(\varphi \circ f)^{-1}(G) \in \mathcal{A}$ for every open set $G \subseteq \mathbb{R}$, so that $\varphi \circ f$ is $\mathcal{A}$-measurable. Q.E.D.

19.13 Theorem. **(a)** *If $f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable and $c \in \mathbb{R}$, then the following functions are also $\mathcal{A}$-measurable:*

$$cf, \quad f^+, \quad f^-, \quad |f|.$$

(b) *If $f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable and if $p > 0$ and $f(x) \geq 0$ for $x \in X$, then f^p is $\mathcal{A}$-measurable.*

(c) *If $f, g : X \to \mathbb{R}$ are $\mathcal{A}$-measurable, then the following functions are also $\mathcal{A}$-measurable:*

$$f + g, \quad fg, \quad f \vee g, \quad f \wedge g.$$

Proof. (a) The functions $\varphi_1(x) := cx$, $\varphi_2(x) := \max\{x, 0\}$, $\varphi_3(x) := \max\{-x, 0\}$ and $\varphi_4(x) := |x|$ are continuous on $\mathbb{R}$. Now apply Theorem 19.12(a).

(b) The function $\psi(x) := x^p$ is continuous on $\{x \in \mathbb{R} : x \geq 0\}$.

(c) We note that $f(x) + g(x) > c$ if and only if there exist rational numbers p, q such that $f(x) > p$, $g(x) > q$ and $p + q > c$. It follows from this that

$$\{f + g > c\} = \bigcup_{(p,q)} \{f > p\} \cup \{g > q\},$$

where the countable union is taken over all pairs (p, q) of rational numbers such that $p + q > c$. Since f and g are measurable, it follows from this countable union and the fact that $\mathcal{A}$ is a σ-algebra that $f + g$ is measurable.

We note that $4fg = (f + g)^2 - (f - g)^2$. Since $f \pm g$ are measurable, it follows from part (a) that $(f \pm g)^2$ are measurable; hence fg is also measurable.

If we use the relations in Lemma 6.5, we obtain the measurability of the functions $f \vee g$ and $f \wedge g$. Q.E.D.

19.14 Theorem. **(a)** *Let (f_k) be a sequence of $\mathcal{A}$-measurable functions on $X \to \mathbb{R}$. If the functions f and F, defined by*

$$f(x) := \inf\{f_k(x) : k \in \mathbb{N}\} \qquad \text{and} \qquad F(x) := \sup\{f_k(x) : k \in \mathbb{N}\},$$

are finite on X, then f and F are $\mathcal{A}$-measurable.

(b) *If the functions f_* and F^*, defined by*

$$f_*(x) := \liminf\{f_k(x) : k \in \mathbb{N}\} \quad \text{and} \quad F^*(x) := \limsup\{f_k(x) : k \in \mathbb{N}\},$$

are finite on X, then f_* and F^* are $\mathcal{A}$-measurable.

(c) *If the sequence (f_k) converges on X to a function $f : X \to \mathbb{R}$, then f is $\mathcal{A}$-measurable.*

Proof. (a) Observe that

$$\{f \geq c\} = \bigcap_{k=1}^{\infty} \{f_k \geq c\} \qquad \text{and} \qquad \{F > c\} = \bigcup_{k=1}^{\infty} \{f_k > c\}.$$

In fact, $f(x) \geq c$ if and only if $f_k(x) \geq c$ for all $k \in \mathbb{N}$. Similarly, $F(x) > c$ if and only if there exists $k \in \mathbb{N}$ such that $f_k(x) > c$.

(b) We recall that the functions f_* and F^* are defined by

$$f_*(x) = \sup_{n \geq 1} \left\{ \inf_{k \geq n} f_k(x) \right\} \qquad \text{and} \qquad F^*(x) = \inf_{n \geq 1} \left\{ \sup_{k \geq n} f_k(x) \right\}.$$

Therefore the assertion follows from part (a).

(c) It is an exercise to show that the limit function $\lim f_k(x)$ exists in X if and only if $\liminf f_k(x) = \limsup f_k(x)$, in which case $\lim f_k(x)$ is this common value. Since f_* and F^* are $\mathcal{A}$-measurable, then so is the limit function $f(x) := \lim f_k(x)$. Q.E.D.

It follows from Theorem 19.14(c) and Example 19.9(f) that the limit of a sequence of $\mathcal{A}$-simple functions is an $\mathcal{A}$-measurable function. We now show that the converse of this assertion is also true.

19.15 Theorem. *If $f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable, then there exists a sequence (φ_k) of $\mathcal{A}$-simple functions such that $f(x) = \lim \varphi_k(x)$ for all $x \in X$. If $f(x) \geq 0$ for all $x \in X$, the sequence (φ_k) can be chosen to be increasing on X.*

Proof. (This proof is essentially the same as an argument in Theorem 10.4.) We first consider the case that $f(x) \geq 0$ for all $x \in \mathbb{R}$.

For each $k \in \mathbb{N}$, we divide the interval $[0, \infty)$ of values of f into the $4^k + 1$ pairwise disjoint sets consisting of the subintervals

$$[r/2^k, (r+1)/2^k), \qquad r = 0, 1, \cdots, 4^k - 1,$$

and the unbounded interval $[2^k, \infty)$. If $r = 0, 1, \cdots, 4^k - 1$, we let

$$E_{r,k} := f^{-1}\big([r/2^k, (r+1)/2^k)\big),$$

and if $r = 4^k$, we let

$$E_{4^k,k} := f^{-1}\big([2^k, \infty)\big).$$

We observe that the sets $\{E_{r,k} : r = 0, 1, \cdots, 4^k\}$ belong to $\mathcal{A}$, are pairwise disjoint and that their union is X. We define φ_k to be equal to $r/2^k$ on the set $E_{r,k}$ so that φ_k is an $\mathcal{A}$-simple function on X. Finally, it is easily seen that the sequence (φ_k) is monotone increasing on X and converges on X to f.

If f takes some negative values, then we let $X^+ := f^{-1}\big([0, \infty)\big)$ and $X^- := f^{-1}\big((-\infty, 0)\big)$ so that $X = X^+ \cup X^-$ and $X^+ \cap X^- = \emptyset$. We apply

the argument in the preceding paragraph to X^+ and a similar argument to X^-. We leave the details to the reader. Q.E.D.

19.16 Remarks. **(a)** There is an aspect about the general notion of measurability, given in Definition 19.8, that deserves further discussion.

If X and Y are topological spaces with the families $\mathcal{T}$ and $\mathcal{U}$ of open sets (i.e., the "topologies"), respectively, then a function $f : X \to Y$ is often said to be $(\mathcal{T},\mathcal{U})$-*continuous* if $f^{-1}(U) \in \mathcal{T}$ for all $U \in \mathcal{U}$. Similarly, if $\mathcal{T}$ is a σ-algebra in X and $\mathcal{U}$ is a σ-algebra in Y, it would be natural to define a function $f : X \to Y$ to be $(\mathcal{T},\mathcal{U})$-*measurable* if $f^{-1}(U) \in \mathcal{T}$ for all $U \in \mathcal{U}$.

We see from Corollary 19.11 that Borel measurability of $f : \mathbb{R} \to \mathbb{R}$ has this general character. It might be expected that Lebesgue measurability of $f : \mathbb{R} \to \mathbb{R}$ also would have this character; in other words, that f is Lebesgue measurable if an only if $f^{-1}(E) \in \mathbb{M}(\mathbb{R})$ for all $E \in \mathbb{M}(\mathbb{R})$. However, that expectation is *not* the case.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function, then $f^{-1}(C)$ may *not* be a Lebesgue measurable set for every Lebesgue measurable set C.

To see this, in Exercise 19.O we will construct a strictly increasing continuous function Ψ of $\mathbb{R}$ onto $\mathbb{R}$ and show that there exists a Lebesgue measurable set V such that $W := \Psi(V)$ is not Lebesgue measurable. If Φ is the strictly increasing continuous function inverse to Ψ, then Φ is Lebesgue measurable, but $W = \Phi^{-1}(V)$ is not a Lebesgue measurable set.

(c) The composition $\omega \circ f$ of two Lebesgue measurable functions may *not* be Lebesgue measurable.

Indeed, let Ψ, V and W be as in part (b) and let $\omega := \mathbf{1}_V$. Since V is a Lebesgue measurable set, its characteristic function ω is a Lebesgue measurable function. Now consider the composition $\omega \circ \Phi$. Although the singleton set $\{1\}$ is a Borel set, we see that

$$\big(\omega \circ \Phi\big)^{-1}(\{1\}) = \Phi^{-1}\big(\omega^{-1}(\{1\})\big) = \Psi(V) = W,$$

so that $\omega \circ \Phi$ is not a Lebesgue measurable function.

Almost Everywhere Properties

We now leave the discussion of general measurability, and return to Lebesgue and Borel measurable functions.

It is an exercise to show that if $f \in \mathcal{M}(\mathbb{R})$ and $g = f$ a.e. on $\mathbb{R}$, then $g \in \mathcal{M}(\mathbb{R})$. However, if f is Borel measurable and $g = f$ a.e., then g may not be Borel measurable. Although a Lebesgue measurable function may not be Borel measurable, it is a.e. equal to a Borel measurable function, as we now show.

19.17 Theorem. *If $f \in \mathcal{M}(\mathbb{R})$, then there exist a Borel null set Z and a Borel measurable function h such that $f(x) = h(x)$ for all $x \in Z^c$. Hence $f = h$ a.e.*

Proof. For each rational number $q \in \mathbb{Q}$ let $A_q := \{f > q\}$. By Theorem 18.18(c), there exist a G_δ-set (and hence Borel set) H_q and a null set $Z_q \subseteq H_q$ such that $A_q = H_q - Z_q$. The set $\bigcup\{Z_q : q \in \mathbb{Q}\}$ is a null set and, by Exercise 18.I, there exists a Borel null set Z containing this union.

We now define $h(x) := f(x)$ for $x \notin Z$ and $h(x) := 0$ for $x \in Z$, so that $f(x) = h(x)$ almost everywhere. To see that h is Borel measurable, we note that if $r \in \mathbb{Q}$, $r \geq 0$, then $\{h > r\} = H_r - Z$, while if $r \in \mathbb{Q}$, $r < 0$, then $\{h > r\} = H_r \mathbin{\dot\cup} Z$. Thus, $\{h > r\} \in \mathbb{B}(\mathbb{R})$ when $r \in \mathbb{Q}$. If $s \in \mathbb{R}$, let (r_n) be a decreasing sequence in $\mathbb{Q}$ that converges to s, and note that $\{h > s\} = \bigcup_{n=1}^{\infty}\{h > r_n\} \in \mathbb{B}(\mathbb{R})$. Therefore, h is Borel measurable. Q.E.D.

Luzin's Theorem

We have seen that a continuous function on $\mathbb{R}$ is Lebesgue measurable. We now prove a partial converse of that result due to N. N. Luzin [= Lusin] (1883–1950). This result is a strengthened version of Theorem 11.4.

19.18 Luzin's Theorem. *If $f \in \mathbb{M}(\mathbb{R})$ and $\varepsilon > 0$, then there exist a continuous function $g_\varepsilon : \mathbb{R} \to \mathbb{R}$ and a closed set C_ε with $|\mathbb{R} - C_\varepsilon| \leq \varepsilon$ such that $g_\varepsilon(x) = f(x)$ for $x \in C_\varepsilon$.*

Proof. Let (s_n) be a sequence of step functions that converges to f a.e. on $\mathbb{R}$. Let Z be the set of all points x such that either the sequence $(s_n(x))$ does not converge to $f(x)$, or one of the functions s_n is not continuous at x, or $x \in \mathbb{Z}$; then Z is a null set. For each $n \in \mathbb{Z}$, let

$$E_n := \{x \in \mathbb{R} : n - 1 < |x| < n\} - Z,$$

and let $C_n \subseteq E_n$ be a compact set such that $|E_n - C_n| \leq \varepsilon/2^{n+1}$ and (using Egorov's Theorem 11.3) such that the sequence (s_n) converges to f uniformly on C_n. Since each s_n is continuous on C_n, it follows that the restriction $f|C_n$ is continuous. Now, let $C := \bigcup_{-\infty}^{\infty} C_n$, so that C is closed, $f|C$ is continuous, and $|\mathbb{R} - C| \leq \sum_{-\infty}^{\infty} |E_n - C_n| \leq \varepsilon$. Now define $g_\varepsilon(x) := f(x)$ for $x \in C$; if (a_k, b_k) is a component of $\mathbb{R} - C$, define g_ε to be linear joining the point $(a_k, f(a_k))$ to $(b_k, f(b_k))$, while if $a_k = -\infty$ or $b_k = \infty$, we define g_ε to be constant. Q.E.D.

19.19 Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$. Then f is Lebesgue measurable if and only if there exists a sequence $(g_n)_{n=1}^{\infty}$ of continuous functions on $\mathbb{R}$ such that $g_n \to f$ a.e. on $\mathbb{R}$.*

Proof. ($\Leftarrow$) This follows immediately from the fact that each g_n is measurable and the extension of Theorem 9.2(b) to $\mathbb{R}$.

($\Rightarrow$) For each $n \in \mathbb{N}$, Luzin's Theorem implies that there exist a continuous function g_n on $\mathbb{R}$ and a closed set C_n such that $f(x) = g_n(x)$ for $x \notin C_n$ and $|C_n| \leq 1/2^n$. Let $F := \limsup C_n$ be the set of points that belong to infinitely many of the sets C_n. Hence, if $x \notin F$, the point x belongs to only a finite number of sets C_n, and it follows that $f(x) = \lim g_n(x)$ for $x \notin F$. It follows from the Borel-Cantelli Lemma (as in Lemma 10.3(b), or Exercise 20.B(c)) that $|F| = 0$, so that (g_n) converges to f a.e. Q.E.D.

Indefinite Integrals

We will now see that a function $f \in \mathcal{L}(\mathbb{R})$ gives rise to a function ν_f defined on the σ-algebra $\mathbb{M}(\mathbb{R})$ with values in $\mathbb{R}$ that has very special properties. The function ν_f can be regarded as an extension to arbitrary measurable sets of the familiar indefinite integral function

$$F(x) := \int_a^x f,$$

which can be considered to be a function of intervals $[a, x]$, $x \in \mathbb{R}$. As in Section 10, it is conventional to refer to the function ν_f as the "indefinite integral" of f. The double use of this term will not be a source of confusion.

19.20 Definition. If $f \in \mathcal{L}(\mathbb{R})$, then we define $\nu_f : \mathbb{M}(\mathbb{R}) \to \mathbb{R}$ by

$$(19.\theta) \qquad \nu_f(E) := \int_E f = \int_{-\infty}^{\infty} f \cdot \mathbf{1}_E \qquad \text{for} \quad E \in \mathbb{M}(\mathbb{R}).$$

The function ν_f will also be called the **indefinite integral** of f.

Since $f \in \mathcal{L}(\mathbb{R})$ and $E \in \mathbb{M}(\mathbb{R})$, both functions $f, \mathbf{1}_E$ and hence their product are measurable. Since $-|f| \leq f \cdot \mathbf{1}_E \leq |f|$, it follows from the extension of Theorem 9.2 that the function $f \cdot \mathbf{1}_E$ belongs to $\mathcal{L}(\mathbb{R})$ so the integral in (19.θ) exists. We now obtain some properties of ν_f.

19.21 Theorem. *If $f \in \mathcal{L}(\mathbb{R})$, then the indefinite integral ν_f satisfies the properties:*

(a) $\nu_f(\emptyset) = 0$.

(b) *If $(E_k)_{k=1}^{\infty}$ is any sequence in $\mathbb{M}(\mathbb{R})$ that is pairwise disjoint, then*

$$(19.\iota) \qquad \nu_f\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} \nu_f(E_k),$$

where the series on the right is absolutely convergent.

Proof. (a) Since $f \cdot \mathbf{1}_\emptyset = 0$, it is clear that $\nu_f(\emptyset) = 0$.

(b) First note that if $E, F \in \mathbb{M}(\mathbb{R})$ with $E \cap F = \emptyset$, then $\mathbf{1}_{E \cup F} = \mathbf{1}_E + \mathbf{1}_F$, so that it follows from the extension of Theorem 3.1(a) that

$$\nu_f(E \cup F) = \int_{-\infty}^{\infty} f \cdot \mathbf{1}_{E \cup F} = \int_{-\infty}^{\infty} f \cdot \mathbf{1}_E + \int_{-\infty}^{\infty} f \cdot \mathbf{1}_F = \nu_f(E) + \nu_f(F).$$

Now let $(E_k)_{k=1}^{\infty}$ be a pairwise disjoint sequence in $\mathbb{M}(\mathbb{R})$ with $E := \bigcup_{k=1}^{\infty} E_k$ and let $B_n := E_1 \cup \cdots \cup E_n$ for $n \in \mathbb{N}$. Since $f \cdot \mathbf{1}_{B_n} = \sum_{k=1}^{n} f \cdot \mathbf{1}_{E_k}$, an induction argument shows that

$$\nu_f(B_n) = \sum_{k=1}^{n} \nu_f(E_k).$$

Since $|f \cdot \mathbf{1}_{B_n}| \leq |f|$ and $f \cdot \mathbf{1}_{B_n}(x) \to f \cdot \mathbf{1}_E(x)$ for all $x \in \mathbb{R}$, an application of the extension of the Dominated Convergence Theorem 8.8 implies that

$$\begin{aligned}\nu_f(E) &= \int_{-\infty}^{\infty} f \cdot \mathbf{1}_E = \lim_{n\to\infty} \int_{-\infty}^{\infty} f \cdot \mathbf{1}_{B_n} \\ &= \lim_{n\to\infty} \nu_f(B_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \nu_f(E_k) = \sum_{k=1}^{\infty} \nu_f(E_k).\end{aligned}$$

Since the union $E = \bigcup_{k=1}^{\infty} E_k$ is independent of the ordering of the sets E_k, the equality (19.ι) shows that the convergence of $\sum_{k=1}^{\infty} \nu_f(E_k)$ is also independent of this ordering; therefore, this series converges absolutely. Q.E.D.

It is immediate from the equation (19.θ) that if $f \in \mathcal{L}(\mathbb{R})$ and if $E \in \mathbb{M}(\mathbb{R})$ satisfies $\lambda(E) = 0$, then $\nu_f(E) = 0$. Indeed, if $\lambda(E) = 0$, then $f \cdot \mathbf{1}_E$ is a null function and so its integral $\nu_f(E)$ equals 0.

We will now see that the decomposition $f = f^+ - f^-$ induces a decomposition of ν_f into the difference $\nu_{f^+} - \nu_{f^-}$ of positive (actually nonnegative) functions on $\mathbb{M}(\mathbb{R})$.

19.22 Theorem. **(a)** *If $f \in \mathcal{L}(\mathbb{R})$ and $f = f^+ - f^-$, where $0 \leq f^\pm \in \mathcal{L}(\mathbb{R})$, and if ν_{f^+}, ν_{f^-} and $\nu_{|f|}$ are the indefinite integrals of f^+, f^- and $|f|$, respectively, then $\nu_f = \nu_{f^+} - \nu_{f^-}$ and $\nu_{|f|} = \nu_{f^+} + \nu_{f^-}$.*

(b) *Moreover, we have*

$$\nu_{|f|}(E) = \sup\Big\{\sum_{i=1}^{n} |\nu_f(E_i)|\Big\}, \tag{19.κ}$$

where the supremum is extended over all finite collections $(E_i)_{i=1}^{n}$ of pairwise disjoint sets in $\mathbb{M}(\mathbb{R})$ with $E = \bigcup_{i=1}^{n} E_i$.

Proof. (a) Evidently the functions $\nu_{f^\pm}$ are positive. Since $f = f^+ - f^-$ and $|f| = f^+ + f^-$, the corresponding decompositions of ν_f and $\nu_{|f|}$ are immediate.

(b) Since $\nu_f(E_i) = \nu_{f^+}(E_i) - \nu_{f^-}(E_i)$, it is clear that

$$|\nu_f(E_i)| \leq \nu_{f^+}(E_i) + \nu_{f^-}(E_i) = \nu_{|f|}(E_i),$$

from which we have

$$\sum_{i=1}^{n} |\nu_f(E_i)| \leq \sum_{i=1}^{n} \nu_{|f|}(E_i) = \nu_{|f|}\Big(\bigcup_{i=1}^{n} E_i\Big) = \nu_{|f|}(E).$$

But, if $E^+ := E \cap \{f \geq 0\}$ and $E^- := E \cap \{f < 0\}$, then $E = E^+ \cup E^-$ and $E^+ \cap E^- = \emptyset$. Since we have

$$|\nu_f(E^+)| + |\nu_f(E^-)| = \nu_{|f|}(E^+) + \nu_{|f|}(E^-) = \nu_{|f|}(E),$$

then $\nu_{|f|}(E)$ is also $\leq$ this supremum. **Q.E.D.**

The quantity on the right of (19.κ) is sometimes called the **total variation** of ν_f over the set $E \in \mathbb{M}(\mathbb{R})$. We have shown that this total variation equals $\nu_{|f|}(E)$. The next result is an analogue of Theorem 10.10(a). It also asserts that $\nu_f(E)$ is small when $\lambda(E)$ is sufficiently small.

19.23 Theorem. *Let $f \in \mathcal{L}(\mathbb{R})$. If $\varepsilon > 0$ is given, there exists $\eta_\varepsilon > 0$ such that if $E \in \mathbb{M}(\mathbb{R})$ and $\lambda(E) \leq \eta_\varepsilon$ then $|\nu_f(E)| \leq \nu_{|f|}(E) \leq \varepsilon$.*

Proof. The proof of Theorem 10.10(a) applies and shows that there exists $\eta_\varepsilon > 0$ such that if $E \in \mathbb{M}(\mathbb{R})$ and $\lambda(E) \leq \eta_\varepsilon$ then $\nu_{|f|}(E) = \int_E |f| \leq \varepsilon$. But since $-|f| \leq f \leq |f|$, it follows from Corollary 3.5 that $|\nu_f(E)| = |\int_E f| \leq \int_E |f| = \nu_{|f|}(E) \leq \varepsilon$. **Q.E.D.**

Measures and Measure Spaces

The Lebesgue measure function λ and the indefinite integrals ν_f for $f \in \mathcal{L}(\mathbb{R})$ are important examples of measures and charges on measurable spaces. Although we will not go into an extended discussion, it is appropriate that we introduce the notion of a general "measure" and of a "measure space" here. It is traditional to allow measures to take their values in the infinite interval $[0, \infty]$.

19.24 Definition. (a) If $(X, \mathcal{A})$ is a measurable space, then a **measure** is a function $m : \mathcal{A} \to [0, \infty]$ such that $m(\emptyset) = 0$, and which is **countably**

additive in the sense that if $(E_n)_{n=1}^{\infty}$ is any sequence in $\mathcal{A}$ that is pairwise disjoint, then

$$(19.\lambda) \qquad m\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} m(E_n) \le \infty,$$

where the series is either convergent in $\mathbb{R}$, or is properly divergent to ∞.

(b) A **measure space** is a triple $(X, \mathcal{A}, m)$ consisting of a set X, a σ-algebra $\mathcal{A}$ of subsets of X, and a measure m on $\mathcal{A}$.

A measure m is said to be **finite** if $m(E) \in [0, \infty)$ for all $E \in \mathcal{A}$; a measure m is said to be **σ-finite** if there exists an increasing sequence $F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$ with $X = \bigcup_{n=1}^{\infty} F_n$ such that $m(F_n) < \infty$. It is an easy exercise to show that a measure m is **monotone** in the sense that if $E, F \in \mathcal{A}$ satisfy $E \subseteq F$, then $m(E) \le m(F) \le \infty$. Thus a measure m is finite if and only if $m(X) < \infty$. Moreover, a measure is seen to be **subtractive** in the sense that if $E \subseteq F$ and $m(E) < \infty$, then $m(F - E) = m(F) - m(E)$. (For other properties of measures, see Exercises 20.A and 20.B.)

19.25 Examples. **(a)** Let X be any nonempty set and let $\mathcal{A}$ be the σ-algebra of all subsets of X.

If we define $m_0(E) := 0$ for all $E \in \mathcal{A}$, then m_0 is a measure, called the **zero measure** on $\mathcal{A}$. Obviously, m_0 is a finite measure.

If we define m_1 on $\mathcal{A}$ by $m_1(\emptyset) := 0$ and $m_1(E) := \infty$ for $\emptyset \ne E \in \mathcal{A}$, then m_1 is also a measure on $\mathcal{A}$. Certainly m_1 is not a finite measure; in fact, it is about as infinite as a measure can be.

(b) If X is as in (a), if $p \in X$ is fixed, and if we define $m_p(E) := 0$ when $p \notin E \in \mathcal{A}$ and $m_p(E) = 1$ when $p \in E \in \mathcal{A}$, then m_p is a measure; it is called the **unit (or Dirac) measure concentrated** at p. Obviously, m_p is a finite measure.

(c) Let X be an infinite set and let $\mathcal{A}$ be the σ-algebra of all subsets of X. Define $\#(E)$ to be the number of elements in E if E is a finite subset of X, and $\#(E) := \infty$ if E is an infinite subset of X. It is an exercise to show that $\#$ is a measure on $\mathcal{A}$; it is called the **counting measure** on $\mathcal{A}$. Evidently $\#$ is not a finite measure, but it is σ-finite if and only X is countable.

(d) If $X := [a, b] \subset \mathbb{R}$ and $\mathcal{A} := \mathbb{I}([a, b]) = \mathbb{M}([a, b])$, then it was seen in Theorem 10.2(d) that the function $m(E) := |E| = \int_a^b \mathbf{1}_E$ is a measure on $\mathbb{M}([a, b])$. It is obviously a finite measure, and is called the **Lebesgue measure** on $\mathbb{M}([a, b])$.

(e) If $X := \mathbb{R}$ and $\mathcal{A} := \mathbb{M}(\mathbb{R})$, then it was seen in Theorem 18.8 that the function λ (defined by $\lambda(E) := |E|$ if $E \in \mathbb{I}(\mathbb{R})$ and $\lambda(E) := \infty$ if E is a measurable but not an integrable set) is a measure on $\mathbb{M}(\mathbb{R})$ and is called the **Lebesgue measure** on $\mathbb{M}(\mathbb{R})$. Although λ is not a finite measure, it is σ-finite.

(f) Let $X := \mathbb{R}$ and let g be an increasing function on $\mathbb{R}$; for convenience, we will assume that g is right continuous at every point. It can be shown (see, for example, [BBT; p. 142], [Ni; p. 163]) that there exists a σ-algebra $\mathbb{M}_g$ containing the Borel sets of $\mathbb{R}$ and a measure m_g on $\mathbb{M}_g$ such that $m_g((a,b]) = g(b) - g(a)$. Such a measure is called a **Lebesgue-Stieltjes measure** generated by g. The measure m_g is always σ-finite and is finite if and only if g is a bounded function.

Charges

It is often useful to consider real-valued (or even complex-valued) functions defined on a σ-algebra $\mathcal{A}$ that are countably additive.

19.26 Definition. If $\mathcal{A}$ is a σ-algebra of subsets of a set X, then a function $\gamma : \mathcal{A} \to \mathbb{R}$ (or $\gamma : \mathcal{A} \to \mathbb{C}$) is said to be a **charge** in case $\gamma(\emptyset) = 0$ and γ is countably additive in the sense that for every sequence $(E_n)_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{A}$, we have

$$\gamma\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \gamma(E_n). \qquad (19.\mu)$$

Remarks. (a) Since the left-hand side of $(19.\mu)$ is independent of the order of the terms, the series on the right of $(19.\mu)$ must be absolutely convergent.

(b) Sometimes, in dealing with real-valued charges, other authors allow γ to take on *one* (but not both) of the values $-\infty$ or ∞. In this case, the term "signed measure" is used, despite the fact that γ is not a measure.

Theorem 19.21 shows that the indefinite integrals ν_f of functions $f \in \mathcal{L}(\mathbb{R})$ are charges on the measurable space $(\mathbb{R}, \mathbb{M}(\mathbb{R}))$. Other charges, analogous to Examples 19.25(a)–(c) can be constructed.

Integration with respect to a General Measure

It is possible to construct interesting charges on a measure space $(X, \mathcal{A}, m)$ by the process of integrating certain $\mathcal{A}$-measurable functions with respect to m. In fact, if $f \geq 0$ is an $\mathcal{A}$-measurable function, then it is an easy matter to define the integral $\int_E f\,dm$, and obtain a measure on $\mathcal{A}$. Indeed,

if $\varphi : X \to \mathbb{R}$ is the $\mathcal{A}$-measurable simple function given by formula (19.η) with $c_i \geq 0$, then we define

$$\int_X \varphi\, dm := \sum_{i=1}^{n} c_i m(A_i),$$

where the value ∞ is permitted and where we use the convention that $0 \cdot \infty = 0$. (It is a (messy) exercise to show that the value of the integral is independent of the representation of φ in the form (19.η).) If $f \geq 0$ is an arbitrary $\mathcal{A}$-measurable function on X, we define the integral

$$\int_X f\, dm := \sup\Big\{ \int_X \varphi\, dm \Big\} \leq \infty, \tag{19.ν}$$

where the supremum is extended over all $\mathcal{A}$-measurable simple functions φ such that $0 \leq \varphi \leq f$.

When $f = f^+ - f^-$ for two positive $\mathcal{A}$-measurable functions $f^\pm$ such that $\int_X f^\pm\, dm < \infty$, we define $\int_X f\, dm := \int_X f^+\, dm - \int_X f^-\, dm$. For a full presentation of the details of this theory of integration, see, for example, [B-1]. It is straightforward to show that if $f \geq 0$ is m-integrable, then the indefinite integral

$$\nu_f(E) := \int_E f\, dm \qquad \text{for} \quad E \in \mathcal{A}$$

is a measure on $\mathcal{A}$, and if f is such that both f^+, f^- have finite integrals, then the corresponding indefinite integral is a charge on $\mathcal{A}$.

If γ is a charge on $(X, \mathcal{A})$, we define the **total variation** of γ on $E \in \mathcal{A}$ by

$$|\gamma|(E) := \sup\Big\{ \sum_{i=1}^{n} |\gamma(E_i)| \Big\}, \tag{19.ξ}$$

where the supremum is extended over all finite collections $(E_i)_{i=1}^n$ of pairwise disjoint sets in $\mathcal{A}$ having $E = \bigcup_{i=1}^n E_i$. It can be shown that $|\gamma|$ is a positive charge (i.e., finite measure). We define γ^+, γ^- by

$$\gamma^+ := \tfrac{1}{2}(|\gamma| + \gamma) \qquad \text{and} \qquad \gamma^- := \tfrac{1}{2}(|\gamma| - \gamma),$$

so that γ^+ and γ^- are positive charges and

$$\gamma = \gamma^+ - \gamma^- \qquad \text{and} \qquad |\gamma| = \gamma^+ + \gamma^-.$$

19.27 Definition. If γ is a charge on $(X, \mathcal{A}, m)$, we say that γ is **absolutely continuous** with respect to m if for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $E \in \mathcal{A}$ is such that $m(E) \leq \eta_\varepsilon$, then $|\gamma|(E) \leq \varepsilon$.

Remarks. **(a)** A similar definition can be given for a measure μ to be absolutely continuous with respect to m.

(b) It can be shown that a charge γ is absolutely continuous with respect to m if and only if $m(E) = 0$ implies that $|\gamma|(E) = 0$, or if and only if $m(E) = 0$ implies that $\gamma(E) = 0$.

We conclude this section by stating one of the most important theorems in the abstract theory of integration. Earlier versions of it were proved in 1913 by Johann Radon (1887–1956) and in 1930 by Otton M. Nikodým (1889–1974).

19.28 Radon-Nikodým Theorem. *Let $(X, \mathcal{A}, m)$ be a σ-finite measure space and let γ be a charge that is absolutely continuous with respect to m. Then there exists an m-integrable function $f : X \to \mathbb{R}$ such that*

$$\gamma(E) = \int_E f\,dm \qquad \textit{for} \quad E \in \mathcal{A}.$$

For proofs of this theorem, see, for example, [B-1; p. 87], [BBT; p. 237], [Dd; p. 134], [Fo-1; p. 370], [Ni; p. 253], [Sw; p. 131], etc. There are also many extensions of this important theorem to the case that γ is a measure, or where m and γ are not countably additive, or where one or both of them take values in a Banach space.

The Radon-Nikodým Theorem is important for many applications; for example, it plays a crucial role in the proof of the Riesz Representation Theorem for continuous linear functionals on various spaces of functions.

Exercises

19.A (a) Establish the relations in $(19.\delta)$.

(b) Give examples to show that the inclusions in $(19.\delta)$ may be strict.

19.B Establish the relations in $(19.\varepsilon)$ and $(19.\zeta)$.

19.C If $f : \mathbb{R} \to \mathbb{R}$, we define the **oscillation** of f at $c \in \mathbb{R}$ to be

$$\omega_f(c) := \inf_{\delta > 0} \sup\big\{|f(x) - f(y)| : x, y \in (c - \delta, c + \delta)\big\} \leq \infty.$$

(a) Show that f is continuous at c if and only if $\omega_f(c) = 0$.

(b) If $g : \mathbb{R} \to \mathbb{R}$ is the Dirichlet function, show that $\omega_g(c) = 1$ for all $c \in \mathbb{R}$.

(c) If $h(x) := (1/x)\sin(1/x)$ for $x \neq 0$ and $h(0) := 0$, show that $\omega_h(c) = 0$ if $c \neq 0$ and $\omega_h(0) = \infty$.

19.D Let f and ω_f be as in Exercise 19.C.

(a) If $r > 0$, show that the set $\{x \in \mathbb{R} : \omega_f(x) < r\}$ is an open set in $\mathbb{R}$.

(b) Show that the set of points where f is discontinuous is an F_σ-set.

(c) Show that the set of points where f is continuous is a G_δ-set.

19.E Suppose that (f_n) is a sequence of functions on $\mathbb{R}$ to $\mathbb{R}$ and that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in \mathbb{R}$. If $r \in \mathbb{R}$, show that

$$\{f > r\} = \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{f_n \geq r+1/m\} = \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{f_n > r+1/m\}.$$

19.F Suppose that (f_n) is a sequence of functions on $\mathbb{R}$ to $\mathbb{R}$. Show that the set C of points at which this sequence converges to a number in $\mathbb{R}$ is

$$C = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \{|f_n - f_m| \leq 1/k\}.$$

19.G Suppose that (f_n) is a sequence of continuous functions on $\mathbb{R}$ to $\mathbb{R}$. Show that the set of points at which the sequence converges is an $F_{\sigma\delta}$-set.

19.H (a) Show that the smallest σ-algebra of sets in $\mathbb{R}$ containing all open intervals (r, s) with $r, s \in \mathbb{Q}$ is the collection of Borel sets.

(b) Show that the smallest σ-algebra of sets in $\mathbb{R}$ containing all closed intervals $[a, b]$ is the collection of Borel sets.

19.I (a) Suppose that X is an uncountable set. In Examples 19.7(a, b), show that every $\mathcal{C}$-measurable function is $\mathcal{A}$-measurable.

(b) Give an example of an $\mathcal{A}$-measurable function that is not $\mathcal{C}$-measurable.

(c) Give an example to show that both cases can occur in Example 19.9(b).

19.J (a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is not Lebesgue measurable, but is such that $|f|$ is Lebesgue measurable.

(b) If $f : X \to \mathbb{R}$ is Lebesgue measurable, show that $\operatorname{sgn} \circ f$ is also Lebesgue measurable.

19.K Let $(X, \mathcal{A})$ be a measurable space. A function $f : X \to \mathbb{C}$ is said to be $\mathcal{A}$-measurable if $f^{-1}(G) \in \mathcal{A}$ for every open set $G \subseteq \mathbb{C}$. Show that f is $\mathcal{A}$-measurable if and only if its real and imaginary parts are $\mathcal{A}$-measurable as functions on X to $\mathbb{R}$.

19.L Let $f : X \to Y$. If $\mathcal{C}$ is a σ-algebra of subsets of Y, show that $\{f^{-1}(H) : H \in \mathcal{C}\}$ is a σ-algebra of subsets of X.

19.M Let $f : X \to Y$. If $\mathcal{A}$ is a σ-algebra of subsets of X and $\mathcal{C} := \{H \subseteq Y : f^{-1}(H) \in \mathcal{A}\}$, show that $\mathcal{C}$ is a σ-algebra of subsets of Y.

19.N Let $(X, \mathcal{A})$ be a measurable space and $f : X \to Y$. Let $\mathcal{C}_1$ be a collection of subsets $H \subseteq Y$ such that $f^{-1}(H) \in \mathcal{A}$. Show that $f^{-1}(K) \in \mathcal{A}$ for any set K that belongs to the smallest σ-algebra containing $\mathcal{C}_1$.

19.O In this exercise we will establish the existence of a Lebesgue measurable subset V of $\mathbb{R}$ and a Lebesgue measurable function Φ such that $\Phi^{-1}(V)$ is not a Lebesgue measurable set. We will also see that there exist Lebesgue measurable sets that are not Borel sets.

Let Λ denote the extension of the Cantor-Lebesgue function in 4.17 to $\mathbb{R}$ by defining $\Lambda(x) := 0$ for $x < 0$ and $\Lambda(x) = 1$ for $x > 1$, and define $\Psi(x) := x + \Lambda(x)$ for $x \in \mathbb{R}$.

(a) Show that Ψ is a strictly increasing and one-one map of $\mathbb{R}$ onto $\mathbb{R}$, and maps $[0, 1]$ onto $[0, 2]$.

(b) Show that if B is a Borel set in $\mathbb{R}$, then both $\Psi(B)$ and $\Psi^{-1}(B)$ are Borel sets.

(c) If Γ is the Cantor set (see 4.15), show that $|\Psi(\Gamma)| = 1$.

(d) Since $\Psi(\Gamma)$ has positive measure, Exercise 18.W implies that there exists a set $W \subseteq \Psi(\Gamma)$ that is not Lebesgue measurable. Show that the set $V := \Psi^{-1}(W) \subseteq \Gamma$ is Lebesgue measurable.

(e) The function $\Phi := \Psi^{-1}$ is Lebesgue measurable and V is a Lebesgue measurable set, but $W = \Phi^{-1}(V)$ is not a Lebesgue measurable set.

(f) The Lebesgue measurable set V is not a Borel set.

19.P (a) The direct image $f(E)$ of a Lebesgue measurable set under a continuous function f may not be Lebesgue measurable.

(b) The inverse image $f^{-1}(H)$ of a Lebesgue measurable set H under a continuous function f may not be Lebesgue measurable.

(c) If f is Lebesgue measurable and g is continuous. the composition $f \circ g$ may not be Lebesgue measurable.

19.Q A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **upper semicontinuous at** $c \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - c| < \delta$ then $f(x) < f(c) + \varepsilon$. Show that the following statements are equivalent.

(a) f is upper semicontinuous at every point $c \in \mathbb{R}$.

(b) $\{x \in \mathbb{R} : f(x) < r\}$ is open in $\mathbb{R}$ for every $r \in \mathbb{R}$.

(c) $\{x \in \mathbb{R} : f(x) \geq r\}$ is closed in $\mathbb{R}$ for every $r \in \mathbb{R}$.

(d) For any $c \in \mathbb{R}$, we have $\limsup_{x\to c} f \leq f(c)$, where the **(deleted) limit superior** is defined by

$$\inf_{\delta>0} \sup\{f(x) : 0 < |x - c| < \delta\} = \lim_{\delta\to 0+} \sup\{f(x) : 0 < |x - c| < \delta\}.$$

(e) For any $c \in \mathbb{R}$, we have $\operatorname{Lim\,sup}_{x\to c} f = f(c)$, where the **(non-deleted) limit superior** is defined by

$$\inf_{\delta>0} \sup\{f(x) : |x - c| < \delta\} = \lim_{\delta\to 0+} \sup\{f(x) : |x - c| < \delta\}.$$

19.R (a) If $\mathbf{1}_J$ is the characteristic function of an interval $J \subset \mathbb{R}$, show that $\mathbf{1}_J$ is upper semicontinuous if and only if J is a closed interval. (It is lower semicontinuous if and only if J is an open interval.)

(b) Show that the Dirichlet function on $\mathbb{R}$ is upper semicontinuous at every rational number. (It is lower semicontinuous at every irrational number.)

(c) Show that an upper semicontinuous function is Borel measurable.

19.S Let $\{f_n : n \in \mathbb{N}\}$ be a countable set of functions that are continuous on $\mathbb{R} \to \mathbb{R}$ and such that $\{f_n(x) : n \in \mathbb{N}\}$ is bounded below for each $x \in \mathbb{R}$. Let $f(x) := \inf\{f_n(x) : n \in \mathbb{N}\}$.

(a) Give an example to show that f may be discontinuous at infinitely many points.

(b) Show that f is upper semicontinuous on $\mathbb{R}$.

19.T Let $\mathcal{A}$ be the collection of all subsets of $\mathbb{N}$. In this exercise we define measures and charges on $\mathcal{A}$ using a sequence $(a_k)_{k=1}^{\infty}$ in $\mathbb{R}$.

(a) Let $a_k \geq 0$ for $k \in \mathbb{N}$. Define $\alpha(\emptyset) := 0$ and if $E \in \mathcal{A},\ E \neq \emptyset$, we define

$$\alpha(E) := \lim_{N\to\infty} \sum\{a_k : k \in E,\ k \leq N\} \leq \infty.$$

Show that α is well defined and is a (possibly infinite) measure on $\mathcal{A}$.

(b) If $a_k \geq 0$ and if the series $\sum_{k=1}^{\infty} a_k$ is convergent, show that α is a finite measure on $\mathcal{A}$.

(c) If $a_k \in \mathbb{R}$ and the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, show that α is a charge on $\mathcal{A}$.

(d) If $a_k \in \mathbb{R}$ and the series $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, show that α is not well defined.

19.U (a) Let C be a closed subset of $\mathbb{R}$ and $f : C \to \mathbb{R}$ be continuous. Show that there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x)$ for all $x \in C$. Moreover, if $|f(x)| \leq M$ for all $x \in C$, then we can require that $|g(x)| \leq M$ for all $x \in \mathbb{R}$. (Such a function g is called a **continuous extension of** f. This result is a very special case of **Tietze's Theorem** in topology.)

(b) Can this extension always be constructed such that $\lim_{x\to\pm\infty} g(x) = 0$?

(c) Show that the continuous function $h(x) := \sin(1/x)$ on $(0, 1]$ does not have a continuous extension to $\mathbb{R}$.

19.V Let γ be a charge on the measure space $(X, \mathcal{A})$ that is bounded (in the sense that $|\gamma(E)| \leq M$ for all $E \in \mathcal{A}$). We define the total variation $|\gamma|$ of γ as in equation (19.ξ).

(a) Show that $|\gamma|$ is additive in the sense that $|\gamma|(E \cup F) = |\gamma|(E) + |\gamma|(F)$ whenever $E, F \in \mathcal{A}$ and $E \cap F = \emptyset$.

(b) Show that $|\gamma|$ is monotone and bounded; in fact, $|\gamma|(E) \leq 2M$ for all $E \in \mathcal{A}$. Thus γ is finitely additive.

(c) Now show that $|\gamma|$ is countably additive.

19.W Let $(X, \mathcal{A}, m)$ be a measure space.

(a) If μ is a finite measure on $\mathcal{A}$, show that μ is absolutely continuous with respect to m in the sense of Definition 19.27 if and only if $E \in \mathcal{A}$ and $m(E) = 0$ imply that $\mu(E) = 0$. [Hint: If $A_k \in \mathcal{A},\ k \in \mathbb{N}$, are such that $\mu(A_k) \geq \varepsilon_0$ and $m(A_k) \leq 1/2^k$, consider $A := \limsup A_k$.]

(b) If γ is a charge on $\mathcal{A}$, show that γ is absolutely continuous with respect to m if and only if $E \in \mathcal{A}$ and $m(E) = 0$ imply that $|\gamma|(E) = 0$.

(c) Let $X := \mathbb{N}$ and $\mathcal{A}$ be the collection of all subsets of $\mathbb{N}$. Let m, μ be defined on $\mathcal{A}$ by:

$$m(E) := \sum\{2^k : k \in E\} \qquad \text{and} \qquad \mu(E) := \sum\{2^{-k} : k \in E\}.$$

Show that m, μ are measures with the sole null set $\emptyset$; thus $m(E) = 0$ if and only if $E = \emptyset$ if and only if $\mu(E) = 0$. Now show that μ is absolutely continuous with respect to m, but m is not absolutely continuous with respect to μ.

19.X Let λ_0 denote the restriction of Lebesgue measure to the Borel sets $\mathbb{B}_0$ in $[0, B]$ and let γ be a finite measure on $\mathbb{B}_0$. Let $g(x) := \gamma([0, x])$ for $x \in [0, B]$. Show that the measure γ is absolutely continuous with respect to λ_0 (in the sense of Definition 19.27) if and only if the function g is absolutely continuous (in the sense of Definition 14.4).

Sequences of Functions

In Section 11 we introduced the notion of almost uniform convergence for a sequence (f_k) of measurable functions on a finite interval and we established the important theorem of Egorov that states that, on a finite interval, almost everywhere convergence implies almost uniform convergence and therefore convergence in measure. It will be seen that Egorov's Theorem does not hold on all of $\mathbb{R}$. However, we will see that there are some additional conditions that will allow one to obtain versions of Egorov's Theorem for infinite intervals.

We will also discuss convergence in measure and in mean and will revisit the Vitali Convergence Theorems.

Almost Uniform Convergence

Although we are primarily interested in Lebesgue measurable functions on $\mathbb{R}$, we will carry out the discussion for an arbitrary measure space $(X, \mathcal{A}, m)$, and will use properties analogous to those established for Lebesgue measure in Section 18 (see Exercises 20.A and 20.B). On such a space, a sequence (f_k) in $\mathcal{M}(\mathcal{A})$ is said to be **almost uniformly convergent** to a function f if, for every $\gamma > 0$ there exists $E_\gamma \in \mathcal{A}$ with $m(E_\gamma) \leq \gamma$ such that (f_k) converges to f uniformly on $E_\gamma^c := X - E_\gamma$.

Similarly, we say that a sequence (f_k) in $\mathcal{M}(\mathcal{A})$ is **almost uniformly Cauchy** on X if, for every $\gamma > 0$, there exists $E_\gamma \in \mathcal{A}$ with $m(E_\gamma) \leq \gamma$ such that (f_k) is a uniformly Cauchy sequence on E_γ^c.

20.1 Lemma. *Let $(X, \mathcal{A}, m)$ be a measure space.*

(a) *If (f_k) in $\mathcal{M}(\mathcal{A})$ is almost uniformly convergent to $f \in \mathcal{M}(\mathcal{A})$ on X, then it is almost uniformly Cauchy on X.*

(b) *If a sequence (f_k) in $\mathcal{M}(\mathcal{A})$ is almost uniformly Cauchy on X, then there exists a function $f \in \mathcal{M}(\mathcal{A})$ such that (f_k) converges almost uniformly (and hence almost everywhere) to f.*

We leave the proof of this result as an exercise for the reader; it requires only minor changes in the proof of Lemma 11.2.

We will now show that Egorov's Theorem does not hold when the measure is infinite.

20.2 Example. Let $f_k(x) := \mathbf{1}_{[k,k+1)}$ on the measure space $(\mathbb{R}, \mathbb{B}, \lambda)$.

It is evident that $f_k(x) \to 0$ as $k \to \infty$ for all $x \in \mathbb{R}$. But, since $\{f_k > 0\} = [k, k+1)$, if E_γ is any set with $m(E_\gamma) < 1$, then $\sup\{|f_k(x) - 0| : x \in E_\gamma^c\} = 1$, so that the sequence (f_k) does not converge uniformly on E_γ^c. Thus (f_k) does not converge almost uniformly on $\mathbb{R}$.

Tails of Sequences

We will now present some notions that will be useful in obtaining a more general form of Egorov's Theorem.

20.3 Definition. Let $(f_k)_{k=1}^\infty$ and f be $\mathcal{A}$-measurable functions on X and let $n \in \mathbb{N}$, $r > 0$. We define the **nth r-tail** $T_n(r)$ to be the set:

$$T_n(r) := \bigcup_{k=n}^{\infty} \{|f_k - f| > r\}. \tag{20.α}$$

It is clear that the sets $T_n(r)$ belong to $\mathcal{A}$. They also are monotone decreasing sequences in both variables.

20.4 Lemma. (a) *For fixed $r > 0$, the map $n \mapsto T_n(r)$ is decreasing on $\mathbb{N}$.*

(b) *For fixed $n \in \mathbb{N}$, the map $r \mapsto T_n(r)$ is decreasing on $(0, \infty)$.*

We leave the proofs as exercises for the reader.

We now define the "tail properties" for the sequence (f_k). These properties will enable us to give a unified discussion of some convergence properties.

20.5 Definition. Suppose that $(f_k)_{k=1}^\infty$ is in $\mathcal{M}(\mathcal{A})$ and that $n \in \mathbb{N}$, $r > 0$.

(a) We say that f_k, f have the **empty tail (= ET) property** if for every $r > 0$ there exists $n(r) \in \mathbb{N}$ such that $T_{n(r)}(r) = \emptyset$.

(b) We say that f_k, f have the **vanishing tail (= VT) property** if for every $r > 0$, we have $m(T_n(r)) \to 0$ as $n \to \infty$.

(c) We say that f_k, f have the **finite tail (= FT) property** if for every $r > 0$ there exists $n(r) \in \mathbb{N}$ such that $T_{n(r)}(r)$ has finite m-measure.

We will first show that the ET property characterizes uniform convergence.

20.6 Theorem. *A sequence $(f_k)_{k=1}^{\infty}$ in $\mathcal{M}(\mathcal{A})$ converges uniformly to f on X if and only if f_k, f have the ET property.*

Proof. ($\Rightarrow$) If (f_k) converges uniformly to f, then given $r > 0$ there exists $n(r)$ such that if $k \geq n(r)$, $x \in X$, then $|f_k(x) - f(x)| \leq r$. Therefore if $k \geq n(r)$ then we have $T_{n(r)}(r) = \emptyset$, showing that f_k, f have the ET property.

($\Leftarrow$) If f_k, f have the ET property, then given $r > 0$ there exists $n(r) \in \mathbb{N}$ such that $T_{n(r)}(r) = \emptyset$. Therefore, if $k \geq n(r)$, $x \in X$, then we have $|f_k(x) - f(x)| \leq r$. Since this holds for all $r > 0$, the sequence (f_k) converges uniformly to f. Q.E.D.

We now see that the VT property characterizes almost uniform convergence.

20.7 Theorem. *A sequence $(f_k)_{k=1}^{\infty}$ in $\mathcal{M}(\mathcal{A})$ is almost uniformly convergent to f if and only if f_k, f have the VT property.*

Proof. ($\Rightarrow$) Let $r > 0$ be given. If $\varepsilon > 0$ is also given, there exists $B_\varepsilon \in \mathcal{A}$ such that $m(B_\varepsilon) \leq \varepsilon$ and (f_k) converges to f uniformly on B_ε^c. Therefore there exists $n(r) \in \mathbb{N}$ such that $T_{n(r)}(r) \subseteq B_\varepsilon$ so that $m(T_{n(r)}(r)) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we infer that $m(T_n(r)) \to 0$ as $n \to \infty$.

($\Leftarrow$) Suppose that f_k, f have the VT property and let $\varepsilon > 0$. For each $p \in \mathbb{N}$ there exists $n_p \in \mathbb{N}$ such that $m(T_{n_p}(1/p)) \leq \varepsilon/2^p$. Let B_ε be defined by

$$B_\varepsilon := \bigcup_{p=1}^{\infty} T_{n_p}(1/p),$$

so that $B_\varepsilon \in \mathcal{A}$ and $m(B_\varepsilon) \leq \sum_{p=1}^{\infty} \varepsilon/2^p = \varepsilon$. Since $T_{n_p}(1/p) \subseteq B_\varepsilon$, it follows that if $k \geq n_p$ and $x \notin B_\varepsilon$ then

$$|f_k(x) - f(x)| \leq 1/p.$$

Thus the sequence (f_k) converges to f uniformly on B_ε^c. Consequently, (f_k) converges to f almost uniformly on X. Q.E.D.

The next result gives two characterizations for almost uniform convergence.

20.8 Generalized Egorov Theorem. *Suppose that f_k, f belong to $\mathcal{M}(\mathcal{A})$. The following statements are equivalent:*

(a) *The sequence (f_k) converges almost uniformly to f on $\mathbb{R}$.*

(b) *f_k, f have the VT property.*

(c) *f_k, f have the FT property and $f_k(x) \to f(x)$ almost everywhere.*

Proof. (a) $\Leftrightarrow$ (b) This equivalence is Theorem 20.7.

(a, b) $\Rightarrow$ (c) If f_k, f have the VT property, they also have the FT property.

If $n \in \mathbb{N}$, let $E_n \in \mathcal{A}$ be such that $m(E_n) < 1/2^n$ and (f_k) converges to f uniformly on E_n^c. Let $F := \limsup E_n$ so that by the Borel-Cantelli Lemma (see Exercise 20.B(c)) we have $m(F) = 0$. If $x \notin F$, then x belongs to only a finite number of the sets E_n, so that x belongs to the sets E_n^c for all $k \geq n_x$, whence $f_k(x) \to f(x)$ for $x \in F^c$.

(c) $\Rightarrow$ (b) Let D be the subset of X where $(f_k(x))$ does not converge to $f(x)$; by hypothesis, there exists an m-null set $Z \in \mathcal{A}$ with $D \subseteq Z$. Fix $r > 0$; we claim that $\bigcap_{n=1}^\infty T_n(r) \subseteq D$. For, if $x \in \bigcap_{n=1}^\infty T_n(r)$, then for every $n \in \mathbb{N}$ there exists $k \geq n$ such that $|f_k(x) - f(x)| > r$, whence it follows that $x \in D \subseteq Z$. Since $(T_n(r))_{n=1}^\infty$ is a decreasing sequence in $\mathcal{A}$ and $T_{n(r)}(r)$ has finite measure, it follows from Exercise 20.A(b) that

$$0 \leq \lim_{n\to\infty} m\big(T_n(r)\big) = m\Big(\bigcap_{n=1}^{\infty} T_n(r)\Big) \leq m(Z) = 0.$$

Hence f_k, f have property VT. Q.E.D.

Dominating Functions

We now recast the foregoing material slightly. In the following it will be assumed that: *The sets $\{|f_k(x) - f(x)| : k \in \mathbb{N}\}$ are bounded for each $x \in X$.*

20.9 Definition. Let $(f_k)_{k=1}^\infty$ and f belong to $\mathcal{M}(\mathcal{A})$ and let $n \in \mathbb{N}$. We define the **dominating functions** ψ_n on X for $n \in \mathbb{N}$ by

$$\psi_n(x) := \sup_{k\geq n} |f_k(x) - f(x)|. \tag{20.β}$$

It is clear that $0 \le \psi_n(x) < \infty$ for all $x \in X$ and it follows from Theorems 19.13 and 19.14 that $\psi_n \in \mathcal{M}(\mathcal{A})$. In addition, $(\psi_n(x))$ is a decreasing sequence for each $x \in X$.

The next result collects useful properties of these dominating functions. The proofs of these assertions are quite straightforward and are left to the reader.

20.10 Lemma. *Let $(f_k)_{k=1}^{\infty}$ and f belong to $\mathcal{M}(\mathcal{A})$.*

(a) *If $x \in X$, then $f_k(x) \to f(x)$ if and only if $\psi_n(x) \to 0$.*

(b) *If $r > 0$, $n \in \mathbb{N}$, then $T_n(r) = \{\psi_n > r\}$.*

(c) *f_k, f have the ET property if and only if for every $r > 0$, there exists $n(r) \in \mathbb{N}$ such that $\{\psi_{n(r)} > r\} = \emptyset$ if and only if $0 \le \psi_{n(r)}(x) \le r$ for all $x \in X$.*

(d) *f_k, f have the VT property if and only if for every $r > 0$, we have $m(\{\psi_n > r\}) \to 0$ as $n \to \infty$.*

(e) *f_k, f have the FT property if and only if for every $r > 0$, there exists $n(r)$ such that $\{\psi_{n(r)} > r\}$ has finite measure.*

We now reformulate Theorem 20.8 in terms of dominating functions. The proof of this result is left as an exercise.

20.11 Theorem. *Suppose that f_k, f belong to $\mathcal{M}(\mathcal{A})$. The following statements are equivalent:*

(a) *The sequence (f_k) converges almost uniformly to f on X,*

(b) *For every $r > 0$, we have $m(\{\psi_n > r\}) \to 0$ as $n \to \infty$.*

(c) *For every $r > 0$ there exists $n(r) \in \mathbb{N}$ such that $\{\psi_{n(r)} > r\}$ has finite measure, and $\lim_{k\to\infty} f_k(x) = f(x)$ almost everywhere.*

The next result gives a form of Egorov's Theorem for sequences of functions in $\mathcal{L}(\mathbb{R})$ that are dominated by a function in $\mathcal{L}(\mathbb{R})$.

20.12 Corollary. *Suppose that $f_k, g \in \mathcal{L}(\mathbb{R})$ and that $f_k(x) \to f(x)$ a.e. on $\mathbb{R}$ and that $|f_k(x)| \le |g(x)|$ for a.e. $x \in \mathbb{R}$. Then the convergence is almost uniform.*

Proof. With no loss of generality, we assume that g is finite everywhere on X. Taking the limit as $k \to \infty$, we conclude that $|f(x)| \le |g(x)|$ a.e. Therefore we have that $|f_k(x) - f(x)| \le 2|g(x)|$ so that $0 \le \psi_n \le 2|g|$ for all $n \in \mathbb{N}$ and so $\{\psi_n > r\} \subseteq \{|g| > r/2\}$. But since $m(\{|g| > r/2\}) \le (2/r)\int_{-\infty}^{\infty} |g| < \infty$, we conclude from 20.11(c) that $f_k \to f$ almost uniformly. Q.E.D.

Convergence in Measure

As in Definition 11.5, we say that a sequence (f_k) of measurable functions on a measure space $(X, \mathcal{A}, m)$ **converges in measure to** $f \in \mathcal{M}(\mathcal{A})$ if for every $r > 0$, we have

$$m(\{|f_k - f| > r\}) \to 0 \qquad \text{as} \quad k \to \infty.$$

Similarly, we say that the sequence (f_k) is **Cauchy in measure** if for every $r > 0$, we have

$$m(\{|f_j - f_k| > r\}) \to 0 \qquad \text{as} \quad j, k \to \infty.$$

As in 11.6 it follows from the subadditivity of m that a sequence (f_k) that converges in measure to f is Cauchy in measure. Also, as in 11.7(a), if a sequence (f_k) converges almost uniformly to f, then it converges in measure to f.

In the case of a compact interval, convergence in measure does not imply a.e. or a.u. convergence (see Exercise 8.F). Therefore these implications cannot hold for a general measure space.

One of the most important theorems in this connection is the Riesz Subsequence Theorem 11.9. Much, but not all, of the proof given in 11.9 goes without any change. However the argument there used Egorov's Theorem, and this part needs to be replaced. We will do that now.

20.13 Riesz Subsequence Theorem. *Let $(X, \mathcal{A}, m)$ be a measure space. If $(f_n) \subset \mathcal{M}(\mathcal{A})$ is Cauchy in measure, then there exist a subsequence (f_{n_k}) and a function $f \in \mathcal{M}(\mathcal{A})$ such that $f_{n_k} \to f$ a.e., a.u., and in measure on X.*

In fact, the entire sequence (f_n) converges in measure to f.

Proof. (Completion) The construction of the sequence $g_k = f_{n_k}$ proceeds as before and we have $g_k \to f$ a.e. on X and $f \in \mathcal{M}(\mathcal{A})$.

To see that (g_k) converges a.u. to f, let $\gamma > 0$ be given and let K be such that $1/2^{K-1} \leq \gamma$. If $F_K := \bigcup_{j=K}^{\infty} E_j$, then $m(F_K) \leq 1/2^{K-1} \leq \gamma$. Also, if $x \in F_K^c$, then $x \in E_j^c$ for all $j \geq K$. If $j > i \geq K$, then the argument in inequality (11.γ) shows that $|g_j(x) - g_i(x)| < 1/2^{i-1}$. Passing to a limit as $j \to \infty$, we have $|f(x) - g_i(x)| \leq 1/2^{i-1} \leq 1/2^{K-1}$ for all $x \in F_K^c$ and $i \geq K$. Consequently, the sequence (g_i) converges uniformly to f on F_K^c. Since $\gamma > 0$ is arbitrary, we conclude that $g_i \to f$ a.u. on X.

An application of Lemma 11.7(a) then shows that $g_i \to f$ in measure on X.

To see that the original sequence (f_n) converges in measure to f on X, we use the fact that

$$\{|f - f_n| > r\} \subseteq \{|f - f_{n_i}| > \tfrac{1}{2}r\} \cup \{|f_{n_i} - f_n| > \tfrac{1}{2}r\},$$

so that we have

$$m\Big(\{|f - f_n| > r\}\Big) \le m\Big(\{|f - f_{n_i}| > \tfrac{1}{2}r\}\Big) + m\Big(\{|f_{n_i} - f_n| > \tfrac{1}{2}r\}\Big).$$

Now the first term approaches 0 since the subsequence $(f_{n_i}) = (g_i)$ converges in measure to f, and the second term approaches 0 since the sequence (f_n) is Cauchy in mean. Q.E.D.

Convergence in Mean

Since the notion of convergence in mean is based on an integral, and we have not obtained the properties of the integral with respect to a general measure, we now return to the concrete measure space $(\mathbb{R}, \mathbb{M}, \lambda)$. As in Definition 9.6, if $f \in \mathcal{L}(\mathbb{R})$, we denote the **seminorm** $\|f\|$ of f by

$$\|f\| := \int_{-\infty}^{\infty} f.$$

We say that a sequence (f_k) in $\mathcal{L}(\mathbb{R})$ is **Cauchy in mean** if $\|f_j - f_k\| \to 0$ as $j, k \to \infty$. Similarly, a sequence (f_k) in $\mathcal{L}(\mathbb{R})$ **converges in mean to** $f \in \mathcal{L}(\mathbb{R})$ if $\|f_k - f\| \to 0$ as $k \to \infty$.

As usual, a sequence that converges in mean is Cauchy in mean. Also, the extension to $\mathbb{R}$ of the Completeness Theorem 9.12 states: *Every sequence (f_k) in $\mathcal{L}(\mathbb{R})$ that is Cauchy in mean actually converges in mean to some $f \in \mathcal{L}(\mathbb{R})$; moreover, this limit f is the* a.u. *limit of a subsequence of (f_k).* The proof of this result needs no change.

We also claim: *If (f_k) in $\mathcal{L}(\mathbb{R})$ converges in mean to f, then $f_k \to f$ in measure.* For, if $r > 0$ and $F_k := \{|f_k - f| > r\} \in \mathbb{M}(\mathbb{R})$, then since $0 \le r \cdot \mathbf{1}_{F_k} \le |f_k - f|$ and $|f_k - f| \in \mathcal{L}(\mathbb{R})$, it follows that $r \cdot \mathbf{1}_{F_k} \in \mathcal{L}(\mathbb{R})$ so that

$$r \cdot \lambda(F_k) \le \int_{-\infty}^{\infty} |f_k - f| = \|f_k - f\|.$$

Since $\|f_k - f\| \to 0$, we infer that $\lambda(F_k) \to 0$ as $k \to \infty$ for each fixed $r > 0$; therefore, $f_k \to f$ in measure.

However, if $f_k \to f$ in measure and $f_k, f \in \mathcal{L}(\mathbb{R})$, then it need not follow that $f_k \to f$ in mean (see Example 11.8(a)).

We will close this brief discussion of convergence in mean by stating two results that apply to sequences in $\mathcal{R}^*(\mathbb{R})$, rather than to $\mathcal{L}(\mathbb{R})$. The first is

a version of the Mean Convergence Theorem 8.9, but applies to sequences that are Cauchy in measure and satisfy a familiar domination condition.

20.14 Dominated Convergence Theorem. *Suppose that $(f_n) \subset \mathcal{R}^*(\mathbb{R})$ is Cauchy in measure on $\mathbb{R}$ and that $\alpha, \omega \in \mathcal{R}^*(\mathbb{R})$ are such that for each $n \in \mathbb{N}$,*

$$\alpha(x) \le f_n(x) \le \omega(x) \qquad \textit{for a.e.} \quad x \in \mathbb{R}.$$

Then there exists $f \in \mathcal{R}^*(\mathbb{R})$ such that $\|f - f_n\| \to 0$.

The proof given in 11.10 applies here.

20.15 Theorem. *Suppose that $(f_n) \subset \mathcal{R}^*(\mathbb{R})$ is Cauchy in mean and converges in measure to a function f. Then $f \in \mathcal{R}^*(\mathbb{R})$ and $\|f - f_n\| \to 0$.*

The proof given in 11.11 applies here.

Remark. The functions f_n in Theorems 20.14 and 20.15 are assumed to be in $\mathcal{R}^*(\mathbb{R})$, rather than in $\mathcal{L}(\mathbb{R})$. However, since we conclude that the difference $f - f_n \to 0$ in mean, the functions f_n must differ from f by a function in $\mathcal{L}(\mathbb{R})$, so the generality in the statement is a bit misleading.

Two Diagrams

Diagram 11.1 is not valid for $\mathbb{R}$, since the implications that follow from Egorov's Theorem 11.3 are not generally valid for $\mathbb{R}$. Thus we obtain the following diagram for infinite intervals. We leave it to the reader to verify that all of these implications hold and that no other ones are valid without additional hypotheses.

$$\begin{tikzcd}
{[\text{a.e.}]} & & {[\text{a.u.}]} \arrow[ll] \arrow[d, shift right] \\
{[\text{mean}]} \arrow[u, dashed] \arrow[urr, dashed] \arrow[rr] & & {[\text{meas}]} \arrow[u, dashed, shift right] \arrow[ull, dashed]
\end{tikzcd}$$

Diagram 20.1 Infinite interval.

When we assume that the sequence (f_k) is dominated by a function g in $\mathcal{L}(\mathbb{R})$ in the sense that $|f_k(x)| \leq |g(x)|$ for a.e. $x \in \mathbb{R}$, then a number of new implications are added. We indicate the implications available for this case in Diagram 20.2. Again, we leave it to the reader to verify these implications.

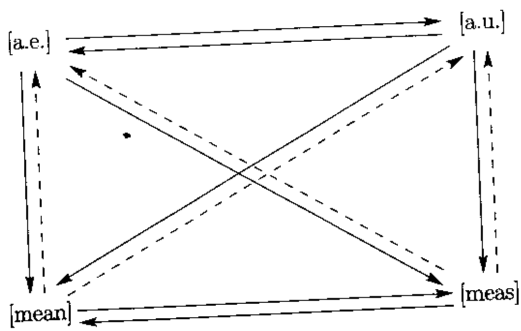

Diagram 20.2 Dominated convergence.

Vitali's Convergence Theorems

We will conclude this section with some necessary and sufficient conditions for a sequence (f_n) in $\mathcal{L}(\mathbb{R})$ to be convergent in mean. The theorems are similar to the Vitali Theorems 11.13 and 11.14; however, some changes are needed.

Remarks. (a) Neither Theorem 11.13 nor 11.14 is true for $\mathbb{R}$ without additional hypotheses. For, let $f_n(x) := 1/n$ for $x \in [0, n]$ and $f_n(x) := 0$ elsewhere. Then (f_n) converges uniformly (and hence a.e., a.u., and in measure) to the 0-function. However, the convergence is not in mean. On the other hand, since $|f_n(x)| \leq 1$ for all $x \in \mathbb{R}$, it is clear that $\|f_n\|_E \leq \lambda(E)$ for all $E \in \mathbb{M}$, so that the collection (f_n) is uniformly absolutely continuous in the sense of Definition 11.12(a). Further, if $K > 1$, then $H_{f_n,K} := \{|f_n| > K\} = \emptyset$ for all $n \in \mathbb{N}$, so the collection (f_n) is also uniformly integrable in the sense of Definition 11.12(b).

(b) The reader will observe that Egorov's Theorem was used in the proof that (b) $\Rightarrow$ (a) in both Theorems 11.13 and 11.14. Also, in the proof that (b) $\Rightarrow$ (c), it was used that a compact interval is the union of a *finite* number of subintervals that have arbitrarily small measure.

We will need the fact that Theorem 10.11 can be extended to $\mathbb{R}$, with the same proof. For convenience, we will state this result explicitly. We recall that $\|f\|_E := \int_E |f|$ for $f \in \mathcal{L}(\mathbb{R}),\ E \in \mathbb{M}$.

20.16 Theorem. *Let f belong to $\mathcal{L}(\mathbb{R})$.*

(a) *Given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $E \in \mathbb{M}(\mathbb{R})$ and $|E| \le \delta_\varepsilon$ then $\|f\|_E \le \varepsilon$.*

(b) *If $H_k := \{|f| \ge k\}$ for $k > 0$, then $\lim_{k\to\infty} |H_k| = 0$.*

(c) *If H_k is as in (b), then $\lim_{k\to\infty} \|f\|_{H_k} = 0$.*

There is another property that will be needed.

20.17 Lemma. *If $f \in \mathcal{L}(\mathbb{R})$ and $\varepsilon > 0$, there exists a set $B = B_\varepsilon \in \mathbb{M}$ with $|B| < \infty$ and such that $\|f\|_{B^c} \le \varepsilon$.*

Proof. The Density Theorem 9.15(a) implies that there exists a step function s_ε such that $\|f - s_\varepsilon\| \le \varepsilon$. Since s_ε vanishes outside a set $B \in \mathbb{M}$ with $|B| < \infty$, we conclude that $\|f\|_{B^c} = \int_{B^c} |f| = \int_{B^c} |f - s_\varepsilon| \le \|f - s_\varepsilon\| \le \varepsilon$. Q.E.D.

It will be clear to the reader what is meant by saying that a collection $\mathcal{F}$ in $\mathcal{L}(\mathbb{R})$ is **uniformly absolutely continuous**. We will need another concept (which might more properly be termed "nearly equifinitely supported").

20.18 Definition. We say that a collection $\mathcal{F} \subset \mathcal{L}(\mathbb{R})$ is **equifinite** on $\mathbb{R}$ if for every $\varepsilon > 0$ there exists a set $B = B_\varepsilon \in \mathbb{M}(\mathbb{R})$ with $|B| < \infty$ such that if $f \in \mathcal{F}$ then $\|f\|_{B^c} \le \varepsilon$.

It is obvious that a finite set in $\mathcal{L}(\mathbb{R})$ is equifinte, and that the union of two equifinite collections is equifinite.

We are now prepared to present the Vitali Convergence Theorems, one for convergence in measure, and one for a.e. convergence. For the sake of simplicity, we will not consider sequences that are uniformly integrable.

20.19 Vitali Convergence Theorem, I. *Let (f_n) belong to $\mathcal{L}(\mathbb{R})$. Then $\|f_n - f\| \to 0$ if and only if the following three conditions are satisfied.*

(a) *The sequence (f_n) converges in measure to f.*

(b) *The collection $\{f_n\}$ is equifinite on $\mathbb{R}$.*

(c) *The collection $\{f_n\}$ is uniformly absolutely continuous.*

Proof. ($\Rightarrow$) We have seen that mean convergence implies convergence in measure. If $\varepsilon > 0$, let $N = N_\varepsilon$ be such that if $n \geq N$, then $\|f_n - f\| \leq \varepsilon$. For any $E \in \mathbb{M}(\mathbb{R})$ we have

$$\big|\|f_n\|_E - \|f\|_E\big| \leq \|f_n - f\|_E = \int_E |f_n - f| \leq \|f_n - f\|,$$

from which it follows that if $n \geq N$ then

$$\|f_n\|_E \leq \|f\|_E + \|f_n - f\| \leq \|f\|_E + \varepsilon. \tag{20.γ}$$

By Lemma 20.17 there exists $B \in \mathbb{M}(\mathbb{R})$ with $|B| < \infty$ such that $\|f\|_{B^c} \leq \varepsilon$, whence it follows that if $n \geq N$ then $\|f_n\|_{B^c} \leq 2\varepsilon$. If we apply Lemma 20.17 to $f_1, \cdots, f_n$, we obtain the equifinite condition (b).

To obtain (c), we apply Theorem 20.16(a) to conclude that if $E \in \mathbb{M}(\mathbb{R})$ and $|E| \leq \delta_\varepsilon$ then $\|f\|_E \leq \varepsilon$. Therefore, if $n \geq N$, we conclude from (20.γ) that if $n \geq N$, then $\|f_n\|_{B^c} \leq 2\varepsilon$. If we apply Theorem 20.16(a) to $f_1, \cdots, f_N$, we obtain the uniform absolute continuity property (c).

($\Leftarrow$) Let $\varepsilon > 0$ be given. From the equifiniteness condition (b), there exists $B \in \mathbb{M}(\mathbb{R})$ with $|B| < \infty$ such that $\|f_n\|_{B^c} \leq \varepsilon$. It is seen that

$$\begin{aligned} \|f_n - f_m\| &\leq \|f_n - f_m\|_B + \|f_n\|_{B^c} + \|f_m\|_{B^c} \\ &\leq \|f_n - f_m\|_B + 2\varepsilon. \end{aligned} \tag{20.δ}$$

By (c) the collection (f_n) is uniformly absolutely continuous, so there exists $\delta_\varepsilon > 0$ such that if $E \in \mathbb{M}(\mathbb{R})$, $|E| \leq \delta_\varepsilon$ and $n \in \mathbb{N}$, then $\|f_n\|_E \leq \varepsilon$.

Now, let $r := \varepsilon/|B|$ and consider $H_{nm} := \{x \in B : |f_n - f_m| > r\}$. Since $f_n \to f$ in measure, the sequence (f_n) is Cauchy in measure, so there exists K such that if $n, m \geq K$, then $|H_{nm}| \leq \delta_\varepsilon$. Consequently, $\|f_n\|_{H_{nm}} \leq \varepsilon$ for all $n \geq K$. But, since we have $|f_n - f_m| \leq \varepsilon/|B|$ on $B - H_{nm}$, we conclude that

$$\begin{aligned} \|f_n - f_m\|_B &\leq \|f_n - f_m\|_{B-H_{nm}} + \|f_n\|_{H_{nm}} + \|f_m\|_{H_{nm}} \\ &\leq (\varepsilon/|B|) \cdot |B| + \varepsilon + \varepsilon = 3\varepsilon, \end{aligned}$$

for $n, m \geq K$. If we combine this inequality with (20.δ), we have $\|f_n - f_m\| \leq 5\varepsilon$ for $n, m \geq K$. Since $\varepsilon > 0$ is arbitrary, we conclude that (f_n) is Cauchy in mean. Since it is also convergent in measure to f, we infer from Theorem 20.15 that $f_n \to f$ in mean. Q.E.D.

We now obtain the corresponding theorem for a.e. convergence. This time, a.e. convergence is a hypothesis, not a conclusion.

20.20 Vitali Convergence Theorem, II. *Let (f_n) be a sequence in $\mathcal{L}(\mathbb{R})$ such that $f_n \to f$ a.e. on $\mathbb{R}$. Then $\|f_n - f\| \to 0$ if and only if the following two conditions are satisfied.*

(b) *The collection $\{f_n\}$ is equifinite on $\mathbb{R}$.*

(c) *The collection $\{f_n\}$ is uniformly absolutely continuous.*

Proof. ($\Rightarrow$) This part is exactly as in the proof of 20.18.

($\Leftarrow$) As in the proof of 20.19, given $\varepsilon > 0$ it follows from (b) that there exists a set $B \in \mathbb{M}(\mathbb{R})$ with $|B| < \infty$ such that

$$\|f_n - f_m\| \le \|f_n - f_m\|_B + 2\varepsilon.$$

By property (c) there exists $\delta_\varepsilon > 0$ such that if $E \in \mathbb{M}$, $|E| \le \delta_\varepsilon$ and $n \in \mathbb{N}$, then $\|f_n\|_E \le \varepsilon$.

By hypothesis, $f_n(x) \to f(x)$ a.e. on B. Moreover (f_n) has the FT property on B. Therefore, the Generalized Egorov Theorem 20.8 implies that (f_n) converges almost uniformly to f on B. Consequently, (f_n) is almost uniformly Cauchy on B so there exists $K = K_\varepsilon$ such that if $n, m \ge K$ then the set $H_{nm} := \{x \in B : |f_n - f_m| > \varepsilon/|B|\}$ has $|H_{nm}| \le \delta_\varepsilon$. Therefore $\|f_n\|_{H_{nm}} \le \varepsilon$ for all $n \ge K$ and so

$$\begin{aligned}\|f_n - f_m\|_B &\le \|f_n - f_m\|_{B-H_{nm}} + \|f_n\|_{H_{nm}} + \|f_m\|_{H_{nm}} \\ &\le \big(\varepsilon/|B|\big)\cdot|B| + \varepsilon + \varepsilon = 3\varepsilon,\end{aligned}$$

from which it follows that $\|f_n - f_m\| \le 5\varepsilon$ for $n, m \ge K$. Thus (f_n) is Cauchy in mean; thus it converges in mean and therefore in measure to some $g \in \mathcal{L}(\mathbb{R})$. But since a subsequence of (f_n) also converges a.e. to g, we infer that $f = g$ a.e. and so $f_n \to f$ in mean. Q.E.D.

Exercises

20.A Let $(X, \mathcal{A}, m)$ be a measure space. (The reader has met the results in this and the next exercise for $(\mathbb{R}, \mathbb{M}(\mathbb{R}), \lambda)$, but some proofs need to be changed.)

(a) If $E, F \in \mathcal{A}$ and $E \subseteq F$, show that $m(E) \le m(F)$. If we also have $m(E) < \infty$, show that $m(F - E) = m(F) - m(E)$.

(b) If $(A_k)_{k=1}^\infty$ is any sequence in $\mathcal{A}$, show that we have $m(\bigcup_{k=1}^\infty A_k) \le \sum_{k=1}^\infty m(A_k) \le \infty$.

(c) If $(E_k)_{k=1}^\infty$ is an increasing sequence in $\mathcal{A}$ (that is, $E_k \subseteq E_{k+1}$ for $k \in \mathbb{N}$), show that $m(\bigcup_{k=1}^\infty E_k) = \lim m(E_k) \le \infty$.

(d) If $(F_k)_{k=1}^{\infty}$ is a decreasing sequence in $\mathcal{A}$ (that is, $F_k \supseteq F_{k+1}$ for $n \in \mathbb{N}$) and if $m(F_1) < \infty$, show that $m(\bigcap_{k=1}^{\infty} F_k) = \lim m(F_k)$.

(e) Give an example of a decreasing sequence (F_k) such that $m(\bigcap_{k=1}^{\infty} F_k) \neq \lim m(F_k)$.

20.B (a) If $(A_k)_{k=1}^{\infty}$ is a sequence in $\mathcal{A}$ such that $m(\bigcup_{k=1}^{\infty} A_k) < \infty$, show that $\limsup m(A_k) \leq m(\limsup A_k) < \infty$.

(b) Show that the conclusion in (a) may fail if the hypothesis is dropped.

(c) If $(A_k)_{k=1}^{\infty}$ is a sequence in $\mathcal{A}$ such that $\sum_{k=1}^{\infty} m(A_k) < \infty$, show that $m(\limsup A_k) = 0$. (This is part of the Borel-Cantelli Lemma.)

(d) Show that the conclusion in (c) may fail if the hypothesis is dropped.

20.C Prove assertion (b) in Lemma 20.1.

20.D The following functions are defined to be 0 for $x < 0$. All of these sequences converge to 0 on $\mathbb{R}$. Determine whether the convergence is almost uniform, in measure, or in mean. Determine the tails $T_n(r)$ for $0 < r < 1$ for these sequences. Which have the ET, VT, or FT properties?

(a) $f_k(x) := x/k$ for $x \geq 0$.

(b) $f_k(x) := (1/k)\mathbf{1}_{[0,k]}$.

(c) $f_k(x) := (1/k^2)\mathbf{1}_{[0,k]}$.

(d) $f_k(x) := \mathbf{1}_{[k,k+1]}$.

(e) $f_k(x) := \mathbf{1}_{[k,k+1/k]}$.

(f) $f_k(x) := k\mathbf{1}_{[k,k+1/k]}$.

(g) $f_k(x) := \mathbf{1}_{[k,k+1/k^2]}$.

(h) $f_k(x) := k\mathbf{1}_{[k,k+1/k^2]}$.

20.E If (f_k) converges almost uniformly to f on a set $E \in \mathbb{M}(\mathbb{R})$ and if $\gamma > 0$, show that there exists a *closed* set $F \subseteq E$ with $|E - F| < \gamma$ such that (f_k) converges to f uniformly on F.

20.F Let $(f_k)_{k=1}^{\infty}$ and f be $\mathcal{A}$-measurable functions on X and let $n \in \mathbb{N}$, $r > 0$. Define the n**th Cauchy** r**-tail** to be $\tilde{T}_n(r) := \bigcup_{k,j=n}^{\infty}\{|f_k - f_j| > r\}$.

(a) Show that for fixed $r > 0$, the map $n \mapsto \tilde{T}_n(r)$ is decreasing on $\mathbb{N}$.

(b) Show that for fixed $n \in \mathbb{N}$, the map $r \mapsto \tilde{T}_n(r)$ is decreasing on $(0, \infty)$.

(c) Show that $\tilde{T}_n(r) \subseteq T_n(r/2)$.

(d) Show that if $f_k(x) \to f(x)$ for all $x \in X$, then $T_n(r) \subseteq \tilde{T}_n(r)$.

20.G We say that the sequence (f_k) has the **empty Cauchy tail (ECT) property** if for every $r > 0$ there exists $n(r) \in \mathbb{N}$ such that $\tilde{T}_{n(r)}(r) = \emptyset$. Show that (f_k) is a uniform Cauchy sequence if and only if it has the ECT property.

20.H We say that the sequence (f_k) has the **vanishing Cauchy tail (VCT) property** if for every $r > 0$ we have $m(\tilde{T}_n(r)) \to 0$ as $n \to 0$.

(a) Show that (f_k) is an almost uniform Cauchy sequence if and only if it has the VCT property.

(b) Show that if (f_k) has the VCT property, then there exists an $\mathcal{A}$-measurable function f such that $f_k \to f$ a.u.

(c) Show that if (f_k) has the VCT property and $f_k(x) \to f(x)$ for $x \in X$, then f_k, f have the VT property.

20.I We say that (f_k) has the **finite Cauchy tail (FCT) property** if for every $r > 0$ there exists $n(r) \in \mathbb{N}$ such that $\tilde{T}_{n(r)}(r)$ has finite measure. Show that if (f_k) has the FCT property and $f_k(x) \to f(x)$ for $x \in X$, then f_k, f have the FT property.

20.J Establish the assertions in Lemma 20.10.

20.K Establish the assertions in Theorem 20.11.

20.L Show that the sequence (f_k) converges to f a.u. if and only if the sequence (ψ_n) converges to 0 in measure.

20.M Let (f_k) be $\mathcal{A}$-measurable and suppose the sets $\{|f_k(x) - f_j(x)| : k, j \geq 1\}$ are bounded for all $x \in X$. For $n \in \mathbb{N}$ we define the **Cauchy dominating function** $\tilde{\psi}_n(x) := \sup_{k,j \geq n} |f_k(x) - f_j(x)|$.

(a) Show that for all $x \in X$, the sequence $(f_k(x))$ is a Cauchy sequence if and only if $\lim_{n\to\infty} \tilde{\psi}_n(x) = 0$.

(b) If $r > 0$, $n \in \mathbb{N}$, show that $\tilde{T}_n(r) = \{\tilde{\psi}_n > r\}$.

(c) Show that f_k has the ECT property if and only if for every $r > 0$ there exists $n(r) \in \mathbb{N}$ such that $\{\tilde{\psi}_{n(r)} > r\} = \emptyset$.

(d) Show that f_k has the VCT property if and only if for every $r > 0$ we have $m(\{\tilde{\psi}_n > r\}) \to 0$ as $n \to \infty$.

(e) Show that f_k has the FCT property if and only if for every $r > 0$ there exists $n(r) \in \mathbb{N}$ such that $\{\tilde{\psi}_n > r\}$ has finite measure.

20.N If $f_k, f \in \mathcal{M}([a,b])$, it was seen in Exercise 11.J that (f_k) converges to f in measure if and only if every subsequence of (f_n) has a further subsequence that converges a.e. to f. Show that this result is not true for $\mathcal{M}(\mathbb{R})$.

20.O (a) If (f_k) and (g_k) converge in measure to f and g, respectively, show that $(f_k + g_k)$ converges in measure to $f + g$.

(b) Give an example of a sequence (f_k) in $\mathcal{M}(\mathbb{R})$ that converges in measure to f and a $g \in \mathcal{M}(\mathbb{R})$ such that $(f_k g)$ does not converge in measure to fg.

(c) Give an example of a sequence (f_k) in $\mathcal{M}(\mathbb{R})$ that converges uniformly to f and a function $g \in \mathcal{M}(\mathbb{R})$ such that $(f_k g)$ does not converge uniformly to fg.

20.P (a) If $f_k \to f$ in measure, and f_k, f have the VCT property, show that $f_k \to f$ almost uniformly.

(b) If $f_k \to f$ in mean, and f_k, f have the VCT property, show that $f_k \to f$ almost uniformly.

20.Q Let $f_k \to f$ a.e. on $\mathbb{R}$. Show that there exists a sequence $(B_n)_{n=0}^{\infty}$ of pairwise disjoint measurable sets with union $\mathbb{R}$ such that B_0 is a null set and (f_k) converges to f uniformly on each set B_n, $n \geq 1$. [Hint: First apply Egorov's Theorem to an interval $[a, b)$.]

20.R We say that a sequence (f_k) in $\mathcal{M}(\mathcal{A})$ is **M-convergent** to $f \in \mathcal{M}(\mathcal{A})$ if for every $r > 0$ we have $\{|f_k - f| \leq r\} \subseteq \{|f_j - f| \leq r\}$ whenever $k \leq j$.

(a) If (f_k) is a monotone sequence and converges to f on X, show that (f_n) is M-convergent.

(b) Show that (f_k) is M-convergent if and only if $T_n(r) = \{|f_n - f| > r\}$ for all $n \in \mathbb{N}$, $r > 0$.

(c) Give an example of a sequence (f_k) that converges everywhere to f and is M-convergent, but such that (f_k) does not converge a.u. or in measure.

(d) Show that if $f_k \to f$ in measure and is M-convergent to f, then $f_k \to f$ a.u.

20.S Suppose that f_k, f are in $\mathcal{L}(\mathbb{R})$ and satisfy $\sum_{k=1}^{\infty} \|f_k - f\| < \infty$. Prove that $f_k \to f$ in mean, in measure, and a.u.

20.T Let $0 \le \omega \in \mathcal{L}(\mathbb{R})$ and $\mathcal{F}_\omega := \{f \in \mathcal{L}(\mathbb{R}) : |f| \le \omega \text{ a.e.}\}$. (See Exercise 11.T.)

(a) Show that $\mathcal{F}_\omega$ is equifinite on $\mathbb{R}$ in the sense of Definition 20.18.

(b) Give an example of an equifinite set $\mathcal{F} \subset \mathcal{L}(\mathbb{R})$ that is not of the form $\mathcal{F}_\omega$ for some $\omega \in \mathcal{L}(\mathbb{R})$.

20.U Let $(X, \mathcal{A}, m)$ be a probability space (that is, a measure space in which $m(X) = 1$). In this exercise, one establishes the other part of the **Borel-Cantelli Lemma**.

(a) A pair of sets A_1, A_2 in $\mathcal{A}$ is said to be **independent** if we have $m(A_1 \cap A_2) = m(A_1)m(A_2)$. In this case, show that the pair A_1^c, A_2 is independent, as are A_1, A_2^c and A_1^c, A_2^c.

(b) A sequence $(A_n)_{n=1}^\infty$ in $\mathcal{A}$ is said to be **independent** if

$$m(A_{n_1} \cap \cdots \cap A_{n_k}) = m(A_{n_1}) \cdots m(A_{n_k})$$

for all finite subsets of the (A_n). If the sequence $(A_n)_{n=1}^\infty$ is independent, and if $(B_n)_{n=1}^\infty$ is any sequence of sets where B_n is either A_n or A_n^c, show that (B_n) is also independent.

(c) If $(A_n)_{n=1}^\infty$ is an independent sequence in $\mathcal{A}$ and if $\sum_{n=1}^\infty m(A_n)$ is divergent, show that $m\big(\liminf_{n\to\infty} A_n^c\big) = 0$ and $m\big(\limsup_{n\to\infty} A_n\big) = 1$. [Hint: If $t \ge 0$, then $1 - t \le e^{-t}$.]

20.V This exercise is related to Exercise 11.Y.

(a) If $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and if (f_n) is a sequence in $\mathcal{M}(\mathbb{R})$ such that $f_n \to f$ a.e., show that $\varphi \circ f_n \to \varphi \circ f$ a.e.

(b) If φ is not continuous on $\mathbb{R}$, show that there exists a sequence (f_n) in $\mathcal{M}(\mathbb{R})$ that converges uniformly to f, but such that $(\varphi \circ f_n)$ does not converge at any point or in measure to $\varphi \circ f$.

(c) If φ is uniformly continuous on $\mathbb{R}$ and if (f_n) is a sequence in $\mathcal{M}(\mathbb{R})$ that converges uniformly, almost uniformly, or in measure to f, show that $(\varphi \circ f_n)$ converges uniformly, almost uniformly, or in measure to $\varphi \circ f$.

(d) If φ is not uniformly continuous on $\mathbb{R}$, show that there exists a sequence (f_n) in $\mathcal{M}(\mathbb{R})$ that converges uniformly and in measure to $f \in \mathcal{M}(\mathbb{R})$, but such that $(\varphi \circ f_n)$ does not converge uniformly or in measure to $\varphi \circ f$.

20.W This exercise is related to Exercise 11.Z.

(a) Show that there exist a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying

$$(*) \qquad |\varphi(t)| \le 1 + |t| \qquad \text{for all} \quad t \in \mathbb{R},$$

and an $f \in \mathcal{L}(\mathbb{R})$ such that $\varphi \circ f \notin \mathcal{L}(\mathbb{R})$.

(b) Suppose that $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$(\dagger) \qquad |\varphi(x)| \le P|t| \quad \text{for all} \quad t \in \mathbb{R},$$

for some $P > 0$. Show that if $f \in \mathcal{L}(\mathbb{R})$, then $\varphi \circ f \in \mathcal{L}(\mathbb{R})$.

(c) If φ is continuous on $\mathbb{R}$ but does not satisfy ($\dagger$), show that there exists $f \in \mathcal{L}(\mathbb{R})$ such that $\varphi \circ f \notin \mathcal{L}(\mathbb{R})$.

(d) If φ is uniformly continuous and satisfies ($\dagger$) and if $(f_n) \subset \mathcal{L}(\mathbb{R})$ converges in mean to f, show that $(\varphi \circ f_n)$ converges in mean to $\varphi \circ f$.

(e) If φ is uniformly continuous but does not satisfy ($\dagger$), show that there exists a sequence (f_n) in $\mathcal{L}(\mathbb{R})$ that converges in mean to f, but such that $(\varphi \circ f_n)$ does not converge in mean to $\varphi \circ f$.

Appendix A

Limits Superior and Inferior

In this appendix we will discuss the notion of the limit superior and the limit inferior of a *bounded* sequence in $\mathbb{R}$. In Appendix B we will treat the case of an unbounded sequence.

If a sequence (x_n) in $\mathbb{R}$ is convergent, then it is bounded and every subsequence converges to $\lim x_n$. Of course, there are many examples of bounded sequences in $\mathbb{R}$ that are not convergent. Two important result in this direction are (1) the Monotone Convergence Theorem (see [B-S; p. 69]), which asserts: *A monotone sequence in* $\mathbb{R}$ *is convergent if and only if it is bounded*, and (2) the Monotone Subsequence Theorem [B-S; p. 78] which asserts: *Every bounded sequence in* $\mathbb{R}$ *has a monotone subsequence.* These results have as an immediate corollary the fundamental Bolzano-Weierstrass Theorem: *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence.*

In fact, bounded sequences in $\mathbb{R}$ that are not convergent have various subsequences that converge to different limits. We could define the "limit superior" of a bounded sequence (x_n) to be the largest real number that is the limit of some subsequence of (x_n), and define the "limit inferior" to be the smallest real number that is the limit of some subsequence of (x_n). Indeed, that is often a useful way of thinking about these two generalized limits. Unfortunately, however, it is not usually a very good way of calculating these limits. Instead, we shall see that the tails of a bounded sequence in $\mathbb{R}$ give rise to a pair of bounded monotone (and therefore convergent) sequences.

A.1 Definition. **(a)** If (x_n) is a bounded sequence in $\mathbb{R}$ and $m \in \mathbb{N}$, let $v_m := \sup\{x_n : n \geq m\}$. Then the sequence (v_m) is bounded and decreasing, and therefore convergent. We define the **limit superior of** (x_n) to be the limit of the sequence (v_m). Thus

$$\limsup_{n\to\infty} x_n := \lim_{m\to\infty} v_m.$$

(b) Similarly, we define $u_m := \inf\{x_n : n \geq m\}$ for $m \in \mathbb{N}$. Then the sequence (u_m) is bounded and increasing, and therefore convergent. We define the **limit inferior of** (x_n) to be

$$\liminf_{n\to\infty} x_n := \lim_{m\to\infty} u_m.$$

Generally, we will denote the limit superior of (x_n) by $\limsup x_n$, and the limit inferior of (x_n) by $\liminf x_n$. [Some authors denote the limit superior and the limit inferior by

$$\overline{\lim}\, x_n \qquad \text{and} \qquad \underline{\lim}\, x_n,$$

but we will not use these notations in this book.]

We will now give several other equivalent possible definitions for the limit superior. The reader is strongly urged to attempt to prove this equivalence before reading the proof, and should also formulate and prove the result corresponding to the limit inferior.

A.2 Theorem. *If (x_n) is a bounded sequence in $\mathbb{R}$, then the following statements about a number $x^* \in \mathbb{R}$ are equivalent:*

(a) $x^* = \limsup x_n$.

(b) *If $v_m := \sup\{x_n : n \geq m\}$, then $x^* = \inf\{v_m : m \in \mathbb{N}\}$.*

(c) *If L is the set of $v \in \mathbb{R}$ such that there exists a subsequence of (x_n) that converges to v, then $x^* = \sup L$.*

(d) *If V is the set of $v \in \mathbb{R}$ such that there are at most a finite number of $n \in \mathbb{N}$ such that $v < x_n$, then $x^* = \inf V$.*

(e) *The number x^* has the property that if $\varepsilon > 0$, then there are at most a finite number of $n \in \mathbb{N}$ such that $x^* + \varepsilon < x_n$, but there are an infinite number such that $x^* - \varepsilon < x_n$.*

Proof. (a) $\Leftrightarrow$ (b) Since (v_m) is a bounded decreasing sequence, then $\lim v_m = \inf\{v_m : m \in \mathbb{N}\}$.

(a) $\Rightarrow$ (c) If (x_{n_k}) is a convergent subsequence of (x_n), then since $n_k \geq k$, we have $x_{n_k} \leq v_k$ for all $k \in \mathbb{N}$. Hence $\lim x_{n_k} \leq \lim v_k = x^*$.

To see that there exists a subsequence converging to x^*, note that there exists $m_1 \in \mathbb{N}$ such that $v_1 - 1 < x_{m_1} \leq v_1$; inductively we choose $m_k > m_{k-1}$ such that $v_k - 1/k < x_{m_k} \leq v_k$. Since $x^* = \lim v_k$, then $x^* = \lim x_{m_k}$.

(c) $\Rightarrow$ (d) If $x^* := \sup L$ and $\varepsilon > 0$ is given, there can be at most a finite number of $n \in \mathbb{N}$ with $x^* + \varepsilon < x_n$ (for otherwise the Bolzano-Weierstrass Theorem implies that there is a subsequence converging to a limit $\geq x^* + \varepsilon$). Therefore $x^* + \varepsilon \in V$ and $\inf V \leq x^* + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\inf V \leq x^*$.

If $\inf V < x^*$, then there exists $v_0 \in V$ with $v_0 < x^*$. Since x^* is the limit of a subsequence of (x_n), there are infinitely many values of $n \in \mathbb{N}$ with $v_0 < x_n$, contradicting that $v_0 \in V$. Therefore we must have $x^* = \inf V$.

(d) $\Rightarrow$ (e) Let $x^* := \inf V$. Hence, if $\varepsilon > 0$ is given, there exists $v \in V$ with $v \leq x^* + \varepsilon$, whence there exist at most a finite number of $n \in \mathbb{N}$ with $x^* + \varepsilon < x_n$. Since $x^* - \varepsilon \notin V$, there are an infinite number of $n \in \mathbb{N}$ such that $x^* - \varepsilon < x_n$.

(e) $\Rightarrow$ (a) Let x^* have the stated property. Then, given $\varepsilon > 0$, there exists $m_\varepsilon \in \mathbb{N}$ such that if $m \geq m_\varepsilon$, then $x_m \leq x^* + \varepsilon$, whence $v_m \leq x^* + \varepsilon$. But since there are an infinite number of $n \in \mathbb{N}$ with $x^* - \varepsilon < x_n$, then $x^* - \varepsilon \leq v_m$ for all $m \in \mathbb{N}$. Thus, if $m \geq m_\varepsilon$, we have that $x^* - \varepsilon \leq v_m \leq x^* + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $x^* = \lim v_m$. **Q.E.D.**

We now establish the basic properties of the limit superior and the limit inferior of bounded sequences. These properties are often used in calculations.

A.3 Theorem. *Let (x_n) and (y_n) be bounded sequences in $\mathbb{R}$. Then:*

(a) $\liminf x_n \leq \limsup x_n$.

(b) *If $c \geq 0$ and $\gamma \leq 0$, then*

$$\liminf(c \cdot x_n) = c \cdot \liminf x_n \quad \textit{and} \quad \limsup(c \cdot x_n) = c \cdot \limsup x_n,$$

$$\liminf(\gamma \cdot x_n) = \gamma \cdot \limsup x_n \quad \textit{and} \quad \limsup(\gamma \cdot x_n) = \gamma \cdot \liminf x_n.$$

(c) $\liminf x_n + \liminf y_n \leq \liminf(x_n + y_n)$.

(d) $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$.

(e) *If (x_n) is convergent, then*

$$\liminf(x_n + y_n) = \lim x_n + \liminf y_n,$$
$$\limsup(x_n + y_n) = \lim x_n + \limsup y_n.$$

(f) *If $x_n \geq 0$ and $y_n \geq 0$, then*

$$\liminf(x_n \cdot y_n) \geq (\liminf x_n) \cdot (\liminf y_n),$$
$$\limsup(x_n \cdot y_n) \leq (\limsup x_n) \cdot (\limsup y_n).$$

(g) *If $x_n \geq 0$ and $y_n \geq 0$ and if (x_n) is convergent, then*

$$\liminf(x_n \cdot y_n) = (\lim x_n) \cdot (\liminf y_n),$$
$$\limsup(x_n \cdot y_n) = (\lim x_n) \cdot (\limsup y_n).$$

(h) *If $x_n \leq y_n$ for all $n \in \mathbb{N}$, then*

$$\liminf x_n \leq \liminf y_n \qquad \textit{and} \qquad \limsup x_n \leq \limsup y_n.$$

Proof. (a) Since $\liminf x_n$ is the smallest limit of a convergent subsequence of (x_n) and $\limsup x_n$ is the largest value limit of such a subsequence, this is clear.

(b) If $c \geq 0$, then $\sup\{c \cdot x_n : n \geq m\} = c \cdot \sup\{x_n : n \geq m\}$ for $m \in \mathbb{N}$, whence it follows that $\limsup(c \cdot x_n) = c \cdot \limsup x_n$.

If $\gamma \leq 0$, then $\inf\{\gamma \cdot x_n : n \geq m\} = \gamma \cdot \sup\{x_n : n \geq m\}$ for $m \in \mathbb{N}$, whence the statement follows.

(d) Since $\sup\{x_n + y_n : n \geq m\} \leq \sup\{x_n : n \geq m\} + \sup\{y_n : n \geq m\}$ for all $m \in \mathbb{N}$, the statement follows from elementary properties of limits of sequences. The proof of (c) is similar.

(e) Let (y_{n_k}) be a subsequence of (y_n) that converges to $\limsup y_n$. Then

$$\lim(x_{n_k} + y_{n_k}) = \lim x_{n_k} + \lim y_{n_k} = \lim x_n + \limsup y_n.$$

Since $(x_{n_k} + y_{n_k})$ is a convergent subsequence of $(x_n + y_n)$, we conclude that its limit is $\leq \limsup(x_n + y_n)$, so that, using (d), we have

$$\lim x_n + \limsup y_n \leq \limsup(x_n + y_n) \leq \lim x_n + \limsup y_n,$$

from which the second equality holds. The other proof is similar.

(f) Since $\sup\{x_n \cdot y_n : n \geq m\} \leq \sup\{x_n : n \geq m\} \cdot \sup\{y_n : n \geq m\}$ for all $m \in \mathbb{N}$, the second statement follows from properties of convergent sequences.

(g) The proof is similar to that of (e).

(h) The hypothesis implies that $\sup\{x_n : n \geq m\} \leq \sup\{y_n : n \geq m\}$ for each $m \in \mathbb{N}$, from which it follows that $\limsup x_n \leq \limsup y_n$. Q.E.D.

In general the inequalities in Theorem A.3(c, d, f, g) cannot be replaced by equalities. For, if $x_n := (-1)^n$ and $y_n := (-1)^{n+1}$ for $n \in \mathbb{N}$, then

$$\begin{aligned}\liminf x_n + \liminf y_n &= -2 < 0 = \liminf(x_n + y_n) \\ &\leq \limsup(x_n + y_n) = 0 < 2 = \limsup x_n + \limsup y_n.\end{aligned}$$

Also, if $u_{2n-1} := 0$, $u_{2n} := 1$ and $v_{2n-1} := 1$, $v_{2n} := 0$ for $n \in \mathbb{N}$, then

$$\limsup(u_n \cdot v_n) = 0 < 1 = (\limsup u_n) \cdot (\limsup v_n).$$

The next theorem is often useful.

A.4 Theorem. *Let (x_n) be a bounded sequence in $\mathbb{R}$. Then (x_n) is convergent in $\mathbb{R}$ if and only if $\liminf x_n = \limsup x_n$, in which case*

$$\lim x_n = \liminf x_n = \limsup x_n.$$

Proof. ($\Rightarrow$) If $x^* := \lim x_n$, then given $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$x^* - \varepsilon \leq x_n \leq x^* + \varepsilon \qquad \text{for} \quad n \geq N(\varepsilon).$$

Theorem A.3(h, a) implies that $x^* - \varepsilon \leq \liminf x_n \leq \limsup x_n \leq x^* + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, then $\lim x_n = \liminf x_n = \limsup x_n$.

($\Leftarrow$) If the equality holds and $\varepsilon > 0$, Theorem A.2(e) implies that there exists $M_1(\varepsilon)$ such that if $n \geq M_1(\varepsilon)$, then $x_n \leq x + \varepsilon$. By the analogous property for the limit inferior, there exists $M_2(\varepsilon)$ such that if $n \geq M_2(\varepsilon)$, then $x^* - \varepsilon \leq x_n$. Now let $M(\varepsilon) := \max\{M_1(\varepsilon), M_2(\varepsilon)\}$, so that if $n \geq M(\varepsilon)$ then $|x^* - x_n| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $x^* = \lim x_n$. Q.E.D.

Unbounded Sets and Sequences

We now modify the notions in Appendix A to apply to *unbounded* sets and sequences in $\mathbb{R}$ and the extended real numbers $\overline{\mathbb{R}}$.

A set $S \subseteq \overline{\mathbb{R}}$ is said to be **unbounded above** if for every $k \in \mathbb{N}$ there exists $s_k \in S$ such that $s_k \geq k$. A sequence (x_n) in $\overline{\mathbb{R}}$ is said to be **unbounded above** if the corresponding set of values $\{x_n : n \in \mathbb{N}\}$ is unbounded above; that is, if for every $k \in \mathbb{N}$ there exists an element x_{n_k} such that $x_{n_k} \geq k$.

If a nonempty set $S \subseteq \overline{\mathbb{R}}$ is unbounded above, we sometimes say that $\infty\,(= +\infty)$ **is the supremum of** S and write

$$\sup S = \infty.$$

If (x_n) is a sequence in $\overline{\mathbb{R}}$, we say that ∞ **is the limit of** (x_n) if for every $K \in \mathbb{N}$ there exists $n_K \in \mathbb{N}$ such that if $n \geq n_K$, then $x_n \geq K$; in this case we write

$$\lim x_n = \infty.$$

It will be clear to the reader what it means for a nonempty set or sequence in $\overline{\mathbb{R}}$ to be **unbounded below**, and what it means when we write

$$\inf S = -\infty \qquad \text{or} \qquad \lim x_n = -\infty.$$

If we allow $\pm\infty$ to be limits of sequences, the Monotone Convergence Theorem can be reformulated: *Every monotone sequence in $\overline{\mathbb{R}}$ is convergent in $\overline{\mathbb{R}}$.* For, let (x_n) be an increasing sequence in $\overline{\mathbb{R}}$. If this sequence is bounded above, the ordinary form of the Monotone Convergence Theorem applies. If it is unbounded above, then $\lim x_n = \infty = \sup\{x_n : n \in \mathbb{N}\}$.

Similarly, if we allow $\pm\infty$ to be limits of sequences, then the Bolzano-Weierstrass Theorem can be reformulated: *Every sequence in $\overline{\mathbb{R}}$ has a subsequence that converges to some element of $\overline{\mathbb{R}}$.*

B.1 Definition. If (x_n) is a sequence in $\overline{\mathbb{R}}$ and if $m \in \mathbb{N}$, we let $v_m := \sup\{x_n : n \geq m\}$. Then (v_m) is a decreasing sequence in $\overline{\mathbb{R}}$ and therefore converges to an element $\inf\{v_m\} \in \overline{\mathbb{R}}$. We then define the **limit superior** of (x_n) to be the limit of the sequence (v_m).

Thus we have

$$\limsup_{n\to\infty} x_n := \lim_{m\to\infty} v_m = \inf_m \sup_{n\geq m} x_n.$$

It is important to note that: *Every sequence in $\overline{\mathbb{R}}$ has a limit superior in $\overline{\mathbb{R}}$.*

We leave it as an exercise to show that if (x_n) is any sequence in $\overline{\mathbb{R}}$, then $x^* := \limsup x_n$ can also be characterized by:

(1) If L is the set of all $v \in \overline{\mathbb{R}}$ such that there exists a subsequence of (x_n) that converges to v, then $x^* = \sup L$.

(2) If V is the set of $v \in \overline{\mathbb{R}}$ such that there are at most a finite number of $n \in \mathbb{N}$ such that $v < x_n$, then $x^* = \inf V$.

The definition of the limit inferior of a sequence (x_n) in $\overline{\mathbb{R}}$ is entirely similar, and the properties of $\liminf x_n$ are analogous to the ones just given.

The following result is useful in connection with the limits superior and inferior. It proof is similar to that of Theorems A.3 and A.4.

B.2 Theorem. *Let (x_n) and (y_n) be sequences in $\overline{\mathbb{R}}$. Then:*

(a) $\liminf x_n \leq \limsup x_n$.

(b) *If* $0 \leq c \leq \infty$ *and* $-\infty \leq \gamma \leq 0$, *then*

$$\liminf(c \cdot x_n) = c \cdot \liminf x_n \quad \text{and} \quad \limsup(c \cdot x_n) = c \cdot \limsup x_n,$$
$$\liminf(\gamma \cdot x_n) = \gamma \cdot \limsup x_n \quad \text{and} \quad \limsup(\gamma \cdot x_n) = \gamma \cdot \liminf x_n.$$

(c) $\liminf x_n + \liminf y_n \leq \liminf(x_n + y_n)$.

(d) $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$.

(e) *If* $x_n \leq y_n$ *for all* $n \in \mathbb{N}$, *then*

$$\liminf x_n \leq \liminf y_n \quad \text{and} \quad \limsup x_n \leq \limsup y_n.$$

(f) *A sequence (x_n) is convergent in $\overline{\mathbb{R}}$ if and only if* $\liminf x_n = \limsup x_n$, *in which case*

$$\lim x_n = \liminf x_n = \limsup x_n.$$

The Arctangent Lemma

We introduce a result that is useful in extending certain results to an unbounded interval in $\overline{\mathbb{R}}$. We will consider the function Arctan to be defined on all of $\overline{\mathbb{R}} = [-\infty, \infty]$ by defining $\operatorname{Arctan}(\pm\infty) := \pm\frac{1}{2}\pi$. We recall that

$$(C.\alpha) \qquad \operatorname{Arctan} x = \int_0^x \frac{dt}{1+t^2} \quad \text{for} \quad x \in \overline{\mathbb{R}},$$

and that $(\operatorname{Arctan} x)' = 1/(1+x^2)$ for $x \in \mathbb{R}$.

C.1 Lemma. *If $t \in \mathbb{R}$, there exists $\gamma(t) > 0$ such that if $u, v \in \mathbb{R}$ satisfy $t \in [u, v] \subseteq [t - \gamma(t), t + \gamma(t)]$, then we have*

$$(C.\beta) \qquad 0 \leq \frac{v-u}{2\pi(1+t^2)} \leq \frac{1}{\pi}[\operatorname{Arctan} v - \operatorname{Arctan} u].$$

Proof. If we take $F(x) := \operatorname{Arctan} x$ and $\varepsilon := 1/[2(1+t^2)]$, the Straddle Lemma 4.4 implies that there exists $\gamma(t) > 0$ such that if $t \in [u, v] \subseteq [t - \gamma(t), t + \gamma(t)]$, then

$$\left| \operatorname{Arctan} v - \operatorname{Arctan} u - \frac{v-u}{1+t^2} \right| \leq \frac{v-u}{2(1+t^2)}.$$

Hence we have

$$-\frac{v-u}{2(1+t^2)} \leq \operatorname{Arctan} v - \operatorname{Arctan} u - \frac{v-u}{1+t^2}.$$

We add $(v-u)/(1+t^2)$ to both sides and divide by π, to obtain $(C.\beta)$. Q.E.D.

C.2 Definition. If $I := [a,b]$ is a closed interval in $\overline{\mathbb{R}}$, then we define the **ϑ-length** of I to be

$$(C.\gamma) \qquad \vartheta(I) := \frac{1}{\pi}[\operatorname{Arctan} b - \operatorname{Arctan} a] = \frac{1}{\pi}\int_a^b \frac{dt}{1+t^2}$$

(where we understand $\operatorname{Arctan}(-\infty) = -\frac{1}{2}\pi$ and $\operatorname{Arctan}(\infty) = \frac{1}{2}\pi$).

It is clear that $0 \le \vartheta(I) \le 1$ for any closed interval $I \subseteq \overline{\mathbb{R}}$. In particular, we have

$$(C.\delta) \qquad \vartheta([0,\infty]) = \tfrac{1}{2} \qquad \text{and} \qquad \vartheta([-\infty,\infty]) = 1.$$

C.3 Lemma. *If $I \subseteq \overline{\mathbb{R}}$ is a closed interval and $\{I_i\}_{i=1}^n$ is a partition of I, then*

$$(C.\varepsilon) \qquad \vartheta(I) = \sum_{i=1}^{n} \vartheta(I_i).$$

Proof. Use the extension of Corollary 3.9 applied to $f(t) := 1/[\pi(1+t^2)]$ for $t \in \mathbb{R}$ and $f(\pm\infty) := 0$, which is in $\mathcal{R}^*([-\infty,\infty])$. Q.E.D.

C.4 Corollary. *If $t \in \mathbb{R}$, there exists $\gamma(t) > 0$ such that if $u, v \in \mathbb{R}$ satisfy $t \in [u,v] \subseteq [t-\gamma(t), t+\gamma(t)]$, then*

$$(C.\zeta) \qquad 0 \le \frac{l([u,v])}{2\pi(1+t^2)} \le \vartheta([u,v]).$$

Proof. This inequality is an immediate consequence of Lemma C.1 and of Definition C.2. Q.E.D.

Outer Measure

We have seen in Definition 6.14 and Section 10 that, for *certain* subsets A of a compact interval $I := [a, b]$, the measure $|A|$ of A is defined. It would be desirable to have this function defined for arbitrary subsets of I, but it was also seen in Theorem 18.22 that there are subsets of $[0, 2]$ that are not measurable.

However, we will now define what is called the *outer measure* $|A|_e$ of an *arbitrary* subset A of I. Although the mapping $A \mapsto |A|_e$ does not have all of the properties of the measure function (given in 10.1 and 10.2), it has enough of them to be useful for many purposes. We will make use of the outer measure in Appendix E. While we will not use the notion of the *inner measure*, we will define it here.

D.1 Definition. (a) If $A \subseteq I$, the **outer (or exterior) measure** $|A|_e$ of A is defined by

$$(D.\alpha) \qquad |A|_e := \inf\{|G| : G \text{ is open and } A \subseteq G\}.$$

(b) If $A \subseteq I$, the **inner (or interior) measure** $|A|_i$ of A is defined by

$$(D.\beta) \qquad |A|_i := \sup\{|F| : F \text{ is compact and } F \subseteq A\}.$$

D.2 Theorem. *The outer measure function $A \mapsto |A|_e$ is defined for all subsets of $I := [a, b]$ and satisfies:*

(a) $0 \le |A|_e \le b - a$ *for all* $A \subseteq I$.

(b) *If* $A \subseteq B$, *then* $|A|_e \le |B|_e$.

(c) *If $(A_n)_{n=1}^{\infty}$ is any sequence of subsets of I, then*

$$(D.\gamma) \qquad \Big|\bigcup_{n=1}^{\infty} A_n\Big|_e \le \sum_{n=1}^{\infty} |A_n|_e.$$

(d) *If $A - B \subseteq C$, then $|A|_e \le |B|_e + |C|_e$.*

Proof. (a) If $\varepsilon > 0$, the set A is contained in the open interval $(a-\varepsilon, b+\varepsilon)$, so that $|A|_e \le (b-a) + 2\varepsilon$. But $\varepsilon > 0$ is arbitrary.

(b) If G is an open set containing B, then G also contains A.

(c) Let $\varepsilon > 0$ and for each $n \in \mathbb{N}$, let G_n be an open set with $A_n \subseteq G_n$ and $|G_n| \le |A_n|_e + \varepsilon/2^n$. Then $G := \bigcup_{n=1}^{\infty} G_n$ is an open set containing $\bigcup_{n=1}^{\infty} A_n$ and the countable subadditivity of the measure function (Theorem 10.2(e)) implies that $|G| \le \sum_{n=1}^{\infty} |G_n|$. Therefore we have

$$\begin{aligned}\Big|\bigcup_{n=1}^{\infty} A_n\Big|_e &\le |G| \le \sum_{n=1}^{\infty} |G_n| \\ &\le \sum_{n=1}^{\infty}\Big(|A_n|_e + \varepsilon/2^n\Big) = \sum_{n=1}^{\infty} |A_n|_e + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, inequality $(D.\gamma)$ holds.

(d) If $A - B \subseteq C$, then $A \subseteq B \cup (A-B) \subseteq B \cup C$, whence the inequality follows from parts (b) and (c). Q.E.D.

D.3 Theorem. **(a)** *If $A \subseteq I$ is measurable, then $|A|_e = |A|$.*

(b) *A set $A \subseteq I$ is a null set if and only if $|A|_e = 0$.*

Proof. (a) If $A \in \mathcal{M}(I)$ and $G \supseteq A$ is a bounded open set, then $|A| \le |G|$ whence it follows that $|A| \le |A|_e$. On the other hand, the Measurable-Open Set Theorem 18.18 implies that given $\varepsilon > 0$ there exists an open set G containing A with $|G - A| \le \varepsilon$. But since $G = A \cup (G - A)$, we infer that $|A|_e \le |G| \le |A| + |G - A| \le |A| + \varepsilon$, whence $|A|_e \le |A|$.

(b) $(\Rightarrow)$ If A is a null set, then Example 2.6(a) implies that $\mathbf{1}_A$ is integrable, so that $A \in \mathcal{M}(I)$, and $|A| = 0$. Part (a) then implies that $|A|_e = 0$.

$(\Leftarrow)$ If $|A|_e = 0$ and $\varepsilon > 0$, then there exists an open set G with $A \subseteq G$ such that $|G| \le \varepsilon$. Since G is an open set, by [B-S; p. 315] it is the union of a countable collection of pairwise disjoint open intervals (J_n). Therefore, by Theorem 10.2(d), we have $\sum_{n=1}^{\infty} l(J_n) = |G| \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies that A is a null set. Q.E.D.

D.4 Theorem. **(a)** *If $A \subseteq I$ is measurable, then $|A|_i = |A|$.*

(b) *The set $A \subseteq I$ is measurable if and only if $|A|_i = |A|_e$.*

Proof. (a) If F is a compact subset of A, then $|F| \leq |A|$, whence it follows that $|A|_i \leq |A|$. On the other hand, the Measurable-Closed Set Theorem 18.19 implies that given $\varepsilon > 0$ there exists a compact set $F \subseteq A$ with $|A - F| \leq \varepsilon$. It follows from Theorem 18.3(d) that $|A| - |F| \leq \varepsilon$, so that $|A| \leq |F| + \varepsilon \leq |A|_i + \varepsilon$. But, since $\varepsilon > 0$ is arbitrary, this implies that $|A| \leq |A|_i$.

(b) ($\Rightarrow$) From D.4(a) and D.3(a) we infer that $|A|_i = |A| = |A|_e$.

($\Leftarrow$) Let $\alpha := |A|_i = |A|_e$. From the definitions, for every $n \in \mathbb{N}$ there exist a compact set $F_n \subseteq A$ and an open set $G_n \supseteq A$ such that

$$\alpha - 1/n = |A|_i - 1/n \leq |F_n| \qquad \text{and} \qquad |G_n| \leq |A|_e + 1/n = \alpha + 1/n.$$

Since $F_n \subseteq A \subseteq G_n$, we have that

$$|G_n - F_n| = |G_n| - |F_n| \leq 2/n.$$

Now let $K := \bigcup_{n=1}^{\infty} F_n \subseteq A$ and $H := \bigcap_{m=1}^{\infty} G_m$, so that

$$H - K \subseteq G_n - F_n \qquad \text{for all} \quad n \in \mathbb{N}.$$

Therefore, we have $|H - K| \leq |G_n - F_n| \leq 2/n$ for all $n \in \mathbb{N}$, whence $H - K$ is a null set. Therefore $A - K \subseteq H - K$ is also a null set, and so $A = K \cup (A - K)$ is a measurable set. **Q.E.D.**

The Vitali Covering Theorem

We now reformulate the statement of the Vitali Covering Theorem 5.8 in terms of the outer measure.

D.5 Theorem. *Let $E \subseteq [a, b]$ and let $\mathcal{F}$ be a Vitali cover for E.*

(a) *Given $\varepsilon > 0$ there exist pairwise disjoint intervals $I_1, \cdots, I_p$ in $\mathcal{F}$ with*

$$\Big|E - \bigcup_{i=1}^{p} I_i\Big|_e \leq \varepsilon.$$

(b) *There exists a sequence $(I_i)_{i=1}^{\infty}$ of pairwise disjoint intervals in $\mathcal{F}$ with*

$$\Big|E - \bigcup_{i=1}^{\infty} I_i\Big|_e = 0.$$

(c) *There exists a sequence $(I_i)_{i=1}^{\infty}$ of pairwise disjoint intervals in $\mathcal{F}$ with*

$$|E|_e \leq \sum_{i=1}^{\infty} |I_i|.$$

Proof. (a) Indeed, it follows from (5.ζ) that

$$\left|E - \bigcup_{i=1}^{p} I_i\right|_e \leq \left|\bigcup_{i=p+1}^{\infty} J_i\right|_e \quad \text{and} \quad \sum_{i=p+1}^{\infty} |J_i| \leq \varepsilon.$$

If we apply $(D.\gamma)$, we get $|\bigcup_{i=p+1}^{\infty} J_i|_e \leq \sum_{i=p+1}^{\infty} |J_i|_e$. Since $|J_i|_e = |J_i|$, the assertion follows.

(b) The sequence of pairwise disjoint intervals $(I_i)_{i=1}^{\infty}$ is constructed in the proof of Theorem 5.8. Since $E - \bigcup_{i=1}^{\infty} I_i \subseteq E - \bigcup_{i=1}^{p} I_i$, part (a) implies that

$$\left|E - \bigcup_{i=1}^{\infty} I_i\right|_e \leq \varepsilon.$$

But $\varepsilon > 0$ is arbitrary, so the assertion in (b) results.

(c) If we let $C := E - \bigcup_{i=1}^{\infty} I_i$, then $|C|_e = 0$, and it follows from Theorem D.2(d) that

$$|E|_e \leq \left|\bigcup_{i=1}^{\infty} I_i\right|_e + |C|_e.$$

Since the I_i are pairwise disjoint intervals, then $|\bigcup_{i=1}^{\infty} I_i|_e = \sum_{i=1}^{\infty} |I_i|$, so that $|E|_e \leq \sum_i |I_i| + 0$. Q.E.D.

We now give an easy (but not obvious) consequence of the Vitali Covering Theorem.

D.6 Theorem. *Let $\mathcal{K}$ be a collection of nondegenerate closed intervals in $I = [a, b]$. Then the set $K := \bigcup\{F : F \in \mathcal{K}\}$ is a measurable set.*

Proof. The statement is clear if $\mathcal{K}$ is finite or countable. In general, let $\mathcal{F}$ be the collection of all nondegenerate compact intervals that are contained in sets in $\mathcal{K}$. It is obvious that $\mathcal{F}$ is a Vitali Cover for K. Therefore, Theorem D.5(b) implies that there exists a sequence $(I_i)_{i=1}^{\infty}$ of intervals in $\mathcal{F}$ with $\bigcup_{i=1}^{\infty} I_i \subseteq K$ such that $K - \bigcup_{i=1}^{\infty} I_i$ is a null set, whence K is measurable. Q.E.D.

Lebesgue's Differentiation Theorem

Let $I := [a, b]$ and let $f : I \to \mathbb{R}$. We will show here that if f belongs to $BV(I)$, then f has a (finite) derivative at a.e. point in I. This is a famous theorem proved by Henri Lebesgue in 1904. The proof given here is based on the Vitali Covering Theorem in the formulation given in the preceding appendix, and follows [N-1; pp. 207–212]. For a proof based on the "Rising Sun Lemma" of F. Riesz, see [A-B; pp. 266–279].

It will be convenient for us to introduce the notion of a "derived number". If $c \in I$, then an extended real number $\xi \in \overline{\mathbb{R}}$ is said to be a **derived number** for f at c if there exists a sequence (h_n) of nonzero numbers such that

$$\lim_{n\to\infty} \frac{f(c+h_n) - f(c)}{h_n} = \xi.$$

Remarks. **(1)** A function has *at least one* derived number $\xi \in \overline{\mathbb{R}}$ at each point $c \in I$. For, a bounded sequence has a convergent subsequence in $\mathbb{R}$, and an unbounded sequence has a subsequence that converges to $\pm\infty$.

(2) The function f has a (finite) derivative at $c \in I$ if and only if $f'(c)$ is the *only* derived number for f at c.

(3) A function may have infinitely many derived numbers at a point. For example, the function $\varphi(x) := x \sin(1/x)$ and $\varphi(0) := 0$ has *every* real number $\xi \in [-1, 1]$ as a derived number at $c = 0$. (To see this, draw a graph.

Also, note that the equation $\sin(1/x) = \xi$ has infinitely many roots near 0 for such ξ.)

(4) In fact, a function may have *every* extended real number as a derived number at a point. For example, consider $g(x) := |x|^{1/2} \sin(1/x)$ and $g(0) := 0$.

If the function f is strictly increasing, then the image $f([c,d])$ of any nondegenerate compact interval $[c,d]$ is contained in the nondegenerate compact interval $[f(c), f(d)]$. We also note that the ratio of the lengths of these intervals is the difference quotient

$$\frac{f(d) - f(c)}{d - c}.$$

Thus, the derived numbers of f at a point c are limits of such difference quotients. We will use the Vitali Covering Theorem to obtain information about the outer measure of certain sets $f(E)$ in terms of the outer measure of the sets E.

E.1 Lemma. *Suppose that $f : I \to \mathbb{R}$ is strictly increasing on I and that $p \in \mathbb{R}$ with $p > 0$. Let $A_p \subseteq I$ satisfy*

(i) *every point of A_p is a point of continuity of f, and*

(ii) *if $c \in A_p$, then there exists a derived number ξ of f at c with $\xi < p$.*

Then $|f(A_p)|_e \le p|A_p|_e$.

Proof. Given $\varepsilon > 0$, let G be an open set with $A_p \subseteq G$ such that $|G| \le |A_p|_e + \varepsilon$. If $c \in A_p$, there exists a nonzero sequence (h_n) with $h_n \to 0$ with

$$0 < \frac{f(c+h_n) - f(c)}{h_n} < p \qquad \text{for all} \quad n \in \mathbb{N}.$$

If $h_n > 0$, let $J_n(c) := [c, c+h_n]$ and $\tilde{J}_n(c) := [f(c), f(c+h_n)]$, while if $h_n < 0$, let $J_n(c) := [c+h_n, c]$ and $\tilde{J}_n(c) := [f(c+h_n), f(c)]$. In either case we have

$$|\tilde{J}_n(c)| = |f(c+h_n) - f(c)| < p|h_n| = p|J_n(c)|.$$

Since $h_n \to 0$ we may assume that $J_n(c) \subseteq G$, and since f is continuous at c, we may assume that $|\tilde{J}_n(c)| \le 1/n$ for all $n \in \mathbb{N}$. Now let

$$\mathcal{F} := \{J_n(c) : c \in A_p,\ n \in \mathbb{N}\} \qquad \text{and} \qquad \tilde{\mathcal{F}} := \{\tilde{J}_n(c) : c \in A_p,\ n \in \mathbb{N}\}.$$

If $c \in A_p$, then $f(c) \in \tilde{J}_n(c)$ and since $|\tilde{J}_n(c)| \le 1/n$, we conclude that $\tilde{\mathcal{F}}$ is a Vitali cover for $f(A_p)$. It follows from the Vitali Covering Theorem, as formulated in Appendix D.5(c), that there exists a pairwise disjoint sequence

of intervals $(\tilde{J}_k)_{k=1}^{\infty}$ such that $|f(A_p)|_e \le \sum_{k=1}^{\infty} |\tilde{J}_k|$. If J_k denotes the interval in $\mathcal{F}$ that corresponds to $\tilde{J}_k \in \tilde{\mathcal{F}}$, then the (J_k) are a pairwise disjoint sequence of intervals whose union is contained in G. Since $|\tilde{J}_k| < p|J_k|$, we conclude that

$$|f(A_p)|_e < p \sum_{k=1}^{\infty} |J_k| \le p|G| \le p(|A_p|_e + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $|f(A_p)|_e \le p|A_p|_e$. Q.E.D.

Remark. The hypothesis (i) could have been avoided in this proof.

E.2 Lemma. *Suppose that $f : I \to \mathbb{R}$ is strictly increasing on I and that $q \in \mathbb{R}$ with $q > 0$. Let $B_q \subseteq I$ satisfy*

(i) *every point of B_q is a point of continuity of f, and*

(ii) *if $c \in B_q$, then there exists a derived number ξ of f at c with $\xi > q$.*

Then $|f(B_q)|_e \ge q|B_q|_e$.

The proof of Lemma E.2 is exactly parallel to that of Lemma E.1 and will be omitted. However, hypothesis (i) is used in this case.

E.3 Lebesgue's Differentiation Theorem. *If f belongs to $BV([a,b])$, then f has a finite derivative a.e.*

Proof. By Jordan's Theorem (see Exercise 7.J) f is the difference of two increasing functions, so it suffices to prove the theorem when f is increasing. Since we can replace $f(x)$ by $x + f(x)$, it suffices to consider the case where f is strictly increasing on $I := [a,b]$. Further, we recall (see [B-S; p. 151]) that an increasing function is continuous except for a countable set D_f of points.

We let E_∞ be the set of $c \in I$ such that f is continuous at c, and f has ∞ as a derived number at c. Then we can take $B_q = E_\infty$ in Lemma E.2 for any $q > 0$ to conclude that

$$|E_\infty|_e \le (1/q)|f(E_\infty)|_e \le (1/q)\big(f(b) - f(a)\big).$$

Since this holds for all $q > 0$, then $|E_\infty|_e = 0$ and E_∞ is a null set.

Now let $0 \le p < q < \infty$, and let E_{pq} be the set of points $c \in I$ such that f is continuous at c, and there exist derived numbers $\xi_1 < p$ and $\xi_2 > q$ for f at c. If we apply Lemma E.1 with A_p replaced by E_{pq}, we infer that

$$|f(E_{pq})|_e \le p|E_{pq}|_e,$$

and if we apply Lemma E.2 with B_q replaced by E_{pq}, we infer that

$$q|E_{pq}|_e \leq |f(E_{pq})|_e.$$

If we combine these inequalities, we obtain

$$q|E_{pq}|_e \leq p|E_{pq}|_e.$$

Since $p < q$, we conclude that $|E_{pq}|_e = 0$, so that each set E_{pq} is a null set.

If f is not differentiable at a point $c \in I$, then either f is not continuous at c, or f has ∞ as a derived number at c, or f has distinct derived numbers $\xi_1 < \xi_2$ at c. In the latter case, there exist rational numbers p and q such that $\xi_1 < p < q < \xi_2$, in which case $c \in E_{pq}$. Thus the set $N_f \subseteq I$ of all points where f is not differentiable is contained in

$$D_f \cup E_\infty \cup \bigcup \{E_{pq} : p, q \in \mathbb{Q},\ p < q\},$$

where D_f is the countable set of points at which f is not continuous, and where E_∞ and E_{pq} are the sets introduced above. Since the last union is a countable union of null sets, it is also a null set. Therefore, we conclude that the set N_f of nondifferentiability points of f is a null set. Q.E.D.

Vector Spaces

We will introduce the notion of a *vector space* (or a *linear space*) in this brief appendix. It is assumed that the reader has already met this concept — at least informally in a course dealing with *Linear algebra*. For the sake of concreteness, we will limit our consideration to the case where the field $\mathbb{F}$ of scalars is either the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers, although it is possible to consider *any* field in the sense of abstract algebra.

To oversimplify, a vector space is a set V in which one can add two elements of V, and can multiply an element in V by a number in $\mathbb{F}$, in such a way that certain properties that are reminiscent of the situation in the plane or 3-space remain valid. We will now be more precise.

F.1 Definition. A **vector space** (or a **linear space**) over the field $\mathbf{F}$ is a set V (whose elements are called the **vectors**), equipped with two binary operations which are called **vector addition** and **scalar multiplication**.

If $x, y \in V$, there is an element in V denoted by $x + y$ and called the **vector sum** of x and y. This **vector addition operation** satisfies the properties:

(A1) $x + y = y + x$ for all $x, y \in V$;

(A2) $(x + y) + z = x + (y + z)$ for all $x, y, z \in V$;

(A3) there exists an element 0 in V such that $0 + x = x$ and $x + 0 = x$ for all $x \in V$;

(A4) given $x \in V$ there is an element $-x \in V$ such that $x + (-x) = 0$ and $(-x) + x = 0$.

If $\alpha \in \mathbb{F}$ and $x \in V$, there is an element $\alpha x \in V$, called the **scalar multiple** of α and x. This **scalar multiple operation** satisfies the properties:

(M1) $1x = x$ for all $x \in V$;

(M2) $\alpha(\beta x) = (\alpha\beta)x$ for all $\alpha, \beta \in \mathbb{F}$ and all $x \in V$;

(D) $\alpha(x + y) = \alpha x + \alpha y$ and $(\alpha + \beta)x = \alpha x + \alpha y$ for all $\alpha, \beta \in \mathbb{F}$ and all $x, y \in V$.

Remark: The properties **(A1)** – **(A4)** amount to the assertion that V is a *commutative group* under the addition operation $+$, with neutral element 0. It is an exercise to show that this neutral element 0 is unique; in fact, if $z \in V$ is such that $x + z = x$ for some element $x \in V$, then $z = 0$. It is also an exercise to show that the additive inverse element $-x$ is uniquely determined; in fact, if $x + w = 0$ for some x, then $w = -x$. Ordinarily, we will write $x - y$ instead of $x + (-y)$.

It is to be noticed that the element 1 in **(M1)** is the unit element in $\mathbb{F}$; that is, the real or complex number 1.

F.2 Examples. **(a)** The real number system $\mathbb{R}$ is a vector space over $\mathbb{R}$, where the addition and scalar multiplication operations are the usual addition and multiplication operations in $\mathbb{R}$.

(b) The complex number system $\mathbb{C}$ is a vector space over $\mathbb{C}$, where the addition and scalar multiplication are the usual addition and multiplication operations in $\mathbb{C}$.

(c) The complex numbers $\mathbb{C}$ is a vector space over $\mathbb{R}$, where the addition operation is the usual addition operation in $\mathbb{C}$, and where the the scalar multiplication is understood to be the multiplication of a real number α and a complex number x.

(d) Let $m \in \mathbb{N}$ and let $\mathbb{R}^m$ denote the collection of all m-tuples of real numbers. Thus, an element x in $\mathbb{R}^m$ has the form

$$x = (x^1, x^2, \cdots, x^m),$$

where $x^1, x^2, \cdots, x^m \in \mathbb{R}$. If we define vector addition by

$$(x^1, x^2, \cdots, x^m) + (y^1, y^2, \cdots, y^m) := (x^1 + y^1, x^2 + y^2, \cdots, x^m + y^m)$$

and if we define scalar multiplication by

$$\alpha(x^1, x^2, \cdots, x^m) := (\alpha x^1, \alpha x^2, \cdots, \alpha x^m),$$

where $\alpha \in \mathbb{R}$ and $x^i \in \mathbb{R}$, then it is a tedious exercise to show that these operations make $\mathbb{R}^m$ into a vector space over $\mathbb{R}$, where the zero element in $\mathbb{R}^m$ is the vector $(0, 0, \cdots, 0)$ and where $-(x^1, x^2, \cdots, x^m) = (-x^1, -x^2, \cdots, -x^m)$.

(e) Let S be any set and let $\mathbb{F}^S$ denote the collection of all functions $u : S \to \mathbb{F}$. If we define $u + v$ and αu to be the functions in $\mathbb{F}^S$ defined by

$$(u+v)(s) := u(s) + v(s) \qquad \text{and} \qquad (\alpha u)(s) := \alpha u(s)$$

for all $s \in S$, then it can readily be checked that $\mathbb{F}^S$ is a vector space over $\mathbb{F}$ under these operations, which are called the *pointwise operations.* Here 0 is the function that is identically equal to the zero element in $\mathbb{F}$, and $-u$ is the function whose value at any $s \in S$ equals $-u(s)$.

(f) Let $I := [0,1]$ and let $\mathcal{R}^*(I)$ be the collection of all generalized Riemann integrable functions $u : I \to \mathbb{R}$ with the pointwise operations defined as in (e). Then $\mathcal{R}^*(I)$ is a vector space over $\mathbb{R}$.

F.3 Definition. If V is a vector space over the field $\mathbb{F}$, then a subset $U \subseteq V$ is said to be a **subspace** of V in case whenever $u, v \in U$ and $\alpha \in \mathbb{F}$, then the vectors $u + v$ and αu also belong to U.

It is easily seen that every subspace of a vector space over $\mathbb{F}$ is a vector space over $\mathbb{F}$ in its own right. Trivially, the entire vector space V is a subspace of V, and the set $\{0\}$ consisting only of the zero vector of V is a subspace of V. More interestingly, the intersection of any collection of subspaces of V is a subspace of V. Also if $W \subseteq V$ is any *subset* of the vector space V, then there exists a smallest *subspace* of V containing W; namely, the intersection of all subspaces of V that contain the set W. It is also clear that the subset $\mathcal{R}^*(I)$ is a subspace of the vector space $\mathbb{R}^I$, as is the subset $C(I)$ of all continuous functions $u : I \to \mathbb{R}$. In addition, the set $C(I)$ is a subspace of $\mathcal{R}^*(I)$.

Linear Operators

The most important type of functions from one vector space over $\mathbb{F}$ to another such vector space, are the *linear operators*, which we will now define.

F.4 Definition. If V and U are vector spaces over the same field $\mathbb{F}$, then a function $L : V \to U$ is said to be a **linear operator** from V to U if it satisfies:

$$L(v_1 + v_2) = L(v_1) + L(v_2) \qquad \text{and} \qquad L(\alpha v) = \alpha L(v)$$

for all $v_1, v_2, v \in V$ and all $\alpha \in \mathbb{F}$.

Remark: The terms **linear transformation** and **linear mapping** are also used instead of linear operator. In the particular case where the range

space is the vector space consisting of the scalars $\mathbb{F}$, a linear operator is often called a **linear functional**.

F.5 Definition. Let V and U be vector spaces over the same field $\mathbb{F}$, and let L and M be linear operators mapping V into U. Then we define the **sum** of L and M by

$$(L+M)(v) := L(v) + M(v)$$

for all $v \in V$. If $\alpha \in \mathbb{F}$, we define the **scalar multiple** of α and L by

$$(\alpha L)(v) := \alpha L(v)$$

for all $v \in V$. It is a routine verification to show that $L+M$ and αL are also linear operators from V to U under these pointwise operations.

In fact, if $\mathcal{L}(V,U)$ denotes the collection of all linear operators from V to U, then it is an exercise to check that the space $\mathcal{L}(V,U)$ is itself a vector space under the operations specified in Definition F.5. If we take $U = \mathbb{F}$, this observation shows that the collection of all linear functionals on a vector space V is itself a vector space; it is usually called the vector space **dual** (or **adjoint** or **conjugate**) to the space V.

Semimetric Spaces

Although the notion of a *metric space* is probably familiar to the reader, the closely related notion of a *semimetric space* may not be. Therefore, we give here the definition of such a space, and a few related notions.

G.1 Definition. **(a)** If X is a set, then a function $\sigma : X \times X \to \mathbb{R}$ is said to be a **semimetric function on** X in case it satisfies:

(i) $\sigma(x,y) \geq 0$ for all $x, y \in X$;

(ii) $\sigma(x,x) = 0$ for all $x \in X$;

(iii) $\sigma(x,y) = \sigma(y,x)$ for all $x, y \in X$; and

(iv) $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y)$ for all $x, y, z \in X$.

A **semimetric space** is an ordered pair (X, σ) consisting of a set X and a semimetric function σ on X.

(b) A semimetric function on X is said to be a **metric function** on X if, instead of (ii), it satisfies

(ii*) $\sigma(x,y) = 0$ if and only if $x = y$.

A **metric space** is an ordered pair (X, σ) consisting of a set X and a metric function σ on X.

Remark. It is important to notice that, in a semimetric space we can have $\sigma(x,y) = 0$ even when $x \neq y$. If we think of $\sigma(x,y)$ as giving the "σ-distance" between the points x and y, this means that the σ-distance between distinct points can equal 0.

G.2 Examples. (a) If $X = \mathbb{R}$ and if $\sigma(x,y) := |x - y|$ for $x, y \in \mathbb{R}$, then σ is a metric function on $\mathbb{R}$.

(b) Let $X := \mathbb{R}^2$ and if, for any $x := (\xi^1, \xi^2)$ and $y := (\eta^1, \eta^2)$, we define

$$\sigma_1(x, y) := |\xi^1 - \eta^1| + |\xi^2 - \eta^2|,$$

then it is easy to show that σ_1 is a metric function on $\mathbb{R}^2$. Similarly, if we define

$$\sigma_\infty(x, y) := \max\{|\xi^1 - \eta^1|, |\xi^2 - \eta^2|\},$$

then it is easy to show that σ_∞ is a metric function on $\mathbb{R}^2$. However, if we define

$$\sigma_0(x, y) := |\xi^1 - \eta^1|,$$

then σ_0 is a semimetric (but *not* a metric) function on $\mathbb{R}^2$.

(c) Let X be the set $C([0, 1])$ of all continuous real-valued functions on $[0, 1]$. If we define

$$\sigma(f, g) := \sup\{|f(t) - g(t)| : t \in [0, 1]\},$$

then σ is a metric function on $C([0, 1])$. However, if we define

$$\sigma_0(f, g) := \sup\{|f(t) - g(t)| : t \in [0, \tfrac{1}{2}]\},$$

then σ_0 is a semimetric (but *not* a metric) function on $C([0, 1])$.

(d) If $X := \mathcal{L}([0, 1])$ and if $f, g \in X$, then

$$\sigma(f, g) := \int_0^1 |f - g|$$

is a semimetric function on X. However, it is *not* a metric function, since $\sigma(f, g) = 0$ whenever the difference $f - g$ is a null function on $[0, 1]$.

G.3 Definition. If (X, σ) is a semimetric space and (x_n) is a sequence in X, we say that (x_n) **converges to** $x \in X$ (with respect to σ) if $\lim \sigma(x_n, x) = 0$; that is, if for every $\varepsilon > 0$ there exists $K(\varepsilon)$ such that if $n \geq K(\varepsilon)$, then $0 \leq \sigma(x_n, x) \leq \varepsilon$.

G.4 Examples. In Example G.2(b) it is easy to show that a sequence $x_n = (\xi_n^1, \xi_n^2)$ converges to $x = (\xi^1, \xi^2)$ with respect to the metric functions σ_1 or σ_∞ if and only if

$$\xi^1 = \lim_{n\to\infty} \xi_n^1 \qquad \text{and} \qquad \xi^2 = \lim_{n\to\infty} \xi_n^2.$$

However, such a sequence (x_n) converges to x with respect to the semimetric function σ_0 if and only if

$$\xi^1 = \lim_{n\to\infty} \xi_n^1,$$

and there is absolutely no restriction put on the second coordinate ξ^2 of the limit. This illustrates that: *In a semimetric space, one sequence can converge to many different points.*

Some people shy away from semimetric spaces because they feel uneasy over the possibility of having nonunique limits of sequences, and having distinct points with σ-distance equal to 0. This can be "corrected" by identifying two points having σ-distance equal to 0, but it is often inconvenient (and unnatural) to make such an identification.

G.5 Definition. If (X, σ) is a semimetric space, then a sequence $(x_n)_{n=1}^{\infty}$ in X is said to be a **Cauchy sequence** (with respect to the semimetric σ) if $\lim_{m,n\to\infty} \sigma(x_m, x_n) = 0$; that is, if for every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that if $m, n \geq N(\varepsilon)$, then $0 \leq \sigma(x_m, x_n) \leq \varepsilon$.

It is an easy exercise to show that if $(x_n)_{n=1}^{\infty}$ is a convergent sequence with respect to a semimetric σ, then it is a Cauchy sequence. Indeed, if $x = \lim x_n$, then it is immediate that

$$\sigma(x_m, x_n) \leq \sigma(x_m, x) + \sigma(x, x_n) = \sigma(x_m, x) + \sigma(x_n, x).$$

The converse is not true, however; that is, there exist semimetric spaces in which some Cauchy sequences are not convergent.

G.6 Definition. A semimetric space (X, σ) is said to be **complete** if every Cauchy sequence in (X, σ) converges to some element of X.

It is clear that $(\mathbb{R}^2, \sigma_0)$ is a complete semimetric space, since if $(x_n) = \big((\xi_n^1, \xi_n^2)\big)_{n=1}^{\infty}$ is a Cauchy sequence in this space, then the sequence $(\xi_n^1)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and so converges to an element ξ^1. Moreover, it is clear that any point $x = (\xi^1, \eta)$ is a limit of the sequence (x_n) with respect to the semimetric σ_0, regardless of how we choose the coordinate $\eta \in \mathbb{R}$.

It is a consequence of Theorem 9.12 that the space $(\mathcal{L}([0,1]), \sigma)$ introduced in Example G.2(d) is a complete semimetric space.

Although we will not use this result, it is an interesting fact that every semimetric space is a subset of a *complete* semimetric space. We will state this result formally.

G.7 Completion Theorem. *If (X, σ) is a semimetric space, then there exists a complete semimetric space $(\tilde{X}, \tilde{\sigma})$ and a mapping $J : X \to \tilde{X}$ such that*

$$\tilde{\sigma}\big(J(x), J(y)\big) = \sigma(x, y) \qquad \text{for all} \quad x, y \in X.$$

Moreover, the subset $J(X)$ is dense in $\tilde{X}$ in the sense that every element of $\tilde{X}$ is the $\tilde{\sigma}$-limit of a sequence in $J(X)$.

Riemann-Stieltjes Integral

The Riemann-Stieltjes (or RS) integral is a modification of the ordinary Riemann integral obtained by replacing the length $x_i - x_{i-1}$ of the subintervals $[x_{i-1}, x_i]$ that appear in the Riemann sums by the differences $\varphi(x_i)-\varphi(x_{i-1})$, where $\varphi : I \to \mathbb{R}$ is a given function; thus the Riemann-Stieltjes integral allows for a more general "length function". This integral was introduced by the Dutch mathematician Thomas J. Stieltjes (1856–1894) in his study of continued fractions and the moment problem. This modification of the Riemann integral has proved to be of considerable utility in statistics. The RS integral also won converts when F. Riesz showed that it can be used to represent an arbitrary continuous functional on the space $C([a, b])$.

We will present here the main properties of the Riemann-Stieltjes integral, since they are used in Section 10.

H.1 Definition. Let $f, \varphi : I \to \mathbb{R}$ be bounded functions on a compact interval $I := [a, b]$. If $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition of I, then the **Riemann-Stieltjes** (or **RS**) **sum** of f with respect to φ for $\dot{\mathcal{P}}$ is

$$(H.\alpha) \qquad \Sigma(f, \varphi; \dot{\mathcal{P}}) := \sum_{i=1}^{n} f(t_i)\left[\varphi(x_i) - \varphi(x_{i-1})\right].$$

We say that f is **Riemann-Stieltjes** (or **RS**) **integrable** with respect to φ on $[a, b]$ if there exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\zeta_\varepsilon > 0$ such that if $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any partition with **mesh** $\mu(\dot{\mathcal{P}}) := \max_i\{x_i - x_{i-1}\} \le \zeta_\varepsilon$, then

$$\left|\Sigma(f, \varphi; \dot{\mathcal{P}}) - A\right| \le \varepsilon.$$

In this case, we write $A = \int_a^b f\,d\varphi$ or $A = \int_I f\,d\varphi$. The function f is called the **integrand** and φ is called the **integrator** or the **weight function**.

Remarks. **(1)** If $\varphi(x) := x$ for all $x \in [a,b]$, then the RS integral reduces to the ordinary Riemann integral.

(2) In many applications it is natural to require φ to be an increasing function, so that $\varphi(x_i) - \varphi(x_{i-1}) \geq 0$ for all i. A detailed presentation of the RS integral when φ is increasing is given in [Ni; Chapter 3], using what can be called "the Darboux approach".

In our discussion, we do not impose this restriction on φ.

(3) There are other approaches to the Riemann-Stieltjes integral that are similar, but not equivalent to Definition H.1. We will comment on three other approaches at the end of this appendix.

If f, g are Riemann-Stieltjes integrable with respect to φ on $[a,b]$ and if $k \in \mathbb{R}$, then $f+g$ and kf are RS integrable with respect to φ on $[a,b]$ and

$$\int_a^b (f+g)\,d\varphi = \int_a^b f\,d\varphi + \int_a^b g\,d\varphi,$$
$$\int_a^b kf\,d\varphi = k\int_a^b f\,d\varphi.$$

Also, if f is Riemann-Stieltjes integrable with respect to both φ and ψ on $[a,b]$ and $k \in \mathbb{R}$, then f is RS integrable with respect to $\varphi + \psi$ and $k\varphi$ on $[a,b]$ and

$$\int_a^b f\,d(\varphi+\psi) = \int_a^b f\,d\varphi + \int_a^b f\,d\psi,$$
$$\int_a^b f\,d(k\varphi) = k\int_a^b f\,d\varphi.$$

These relations are often called the **bilinear property** of the Riemann-Stieltjes integral, and are readily proved. Similarly, it is an easy exercise to establish the following **Cauchy Criterion** for the existence of the Riemann-Stieltjes integral.

H.2 Theorem. *Let f and φ be bounded functions on $I := [a,b]$. Then f is Riemann-Stieltjes integrable with respect to φ if and only if for every $\varepsilon > 0$ there exists $\theta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}_1$ and $\dot{\mathcal{P}}_2$ are any tagged partitions of $[a,b]$ having mesh less than θ_ε, then*

$$\left|\Sigma(f,\varphi;\dot{\mathcal{P}}_1) - \Sigma(f,\varphi;\dot{\mathcal{P}}_2)\right| \leq \varepsilon.$$

Existence of the RS Integral

The next result is an important existence theorem for the Riemann-Stieltjes integral. In view of its importance, we will give a proof.

H.3 Theorem. *If $f : [a,b] \to \mathbb{R}$ is continuous and $\varphi : [a,b] \to \mathbb{R}$ has bounded variation, then f is Riemann-Stieltjes integrable with respect to φ on $[a,b]$.*

Proof. Since f is uniformly continuous, given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $t, s \in [a,b]$ and $|t-s| \le 2\delta_\varepsilon$, then $|f(t) - f(s)| \le \varepsilon$.

Suppose that $\dot{\mathcal{P}}_1 := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ and $\dot{\mathcal{P}}_2 := \{([y_{j-1}, y_j], s_j)\}_{j=1}^m$ are tagged partitions of $[a,b]$ having mesh less than δ_ε, and let $\mathcal{Q} := \{[z_{k-1}, z_k]\}_{k=1}^r$ be the (untagged) partition obtained by using all of the points x_i and y_j from $\dot{\mathcal{P}}_1$ and $\dot{\mathcal{P}}_2$. Thus each of the subintervals $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$ is the union of a finite number of the subintervals $[z_{k-1}, z_k]$; moreover, if $[z_{k-1}, z_k]$ is contained in the intersection $[x_{i-1}, x_i] \cap [y_{j-1}, y_j]$, then the tags t_i, s_j satisfy $|t_i - s_j| \le 2\delta_\varepsilon$, so that $|f(t_i) - f(s_j)| \le \varepsilon$.

The Riemann-Stieltjes sum $\Sigma(f, \varphi; \dot{\mathcal{P}}_1)$ can be written as a sum over the subintervals in $\mathcal{Q}$; namely

$$\Sigma(f, \varphi; \dot{\mathcal{P}}_1) = \sum_{i=1}^n f(t_i)\big[\varphi(x_i) - \varphi(x_{i-1})\big] = \sum_{k=1}^r f(t_i)\big[\varphi(z_k) - \varphi(z_{k-1})\big],$$

where t_i is the tag in $\dot{\mathcal{P}}_1$ corresponding to the unique subinterval $[x_{i-1}, x_i]$ that contains $[z_{k-1}, z_k]$. Similarly, $\Sigma(f, \varphi; \dot{\mathcal{P}}_2)$ can be written as a sum over $\mathcal{Q}$, using points s_j that are tags corresponding to the unique subinterval $[y_{j-1}, y_j]$ that contains $[z_{k-1}, z_k]$. It is readily seen that

$$\Sigma(f, \varphi; \dot{\mathcal{P}}_1) - \Sigma(f, \varphi; \dot{\mathcal{P}}_2) = \sum_{k=1}^r \big[f(t_i) - f(s_j)\big]\big[\varphi(z_k) - \varphi(z_{k-1})\big].$$

Therefore we have

$$\Big|\Sigma(f, \varphi; \dot{\mathcal{P}}_1) - \Sigma(f, \varphi; \dot{\mathcal{P}}_2)\Big| \le \sum_{k=1}^r \varepsilon \big|\varphi(z_k) - \varphi(z_{k-1})\big| \le \varepsilon \cdot \mathrm{Var}(\varphi; [a,b]).$$

Since $\varepsilon > 0$ is arbitrary, Theorem H.2 implies that f is RS integrable with respect to φ on $[a,b]$. **Q.E.D.**

H.4 Corollary. *If f is continuous on $[a,b]$ and $\varphi \in BV([a,b])$, then*

$$(H.\beta) \qquad \left|\int_a^b f\, d\varphi\right| \le \big[\sup_{t\in[a,b]} |f(t)|\big] \cdot \mathrm{Var}(\varphi; [a,b]).$$

Proof. Let $(\dot{\mathcal{P}}_n)_{n=1}^{\infty}$ be a sequence of tagged partitions such that we have $|\Sigma(f,\varphi;\dot{\mathcal{P}}_n) - \int_a^b f\,d\varphi| \le 1/n$. Then $|\int_a^b f\,d\varphi| \le |\Sigma(f,\varphi;\dot{\mathcal{P}}_n)| + 1/n$, whence the inequality follows. Q.E.D.

Integration by Parts

A major fact about the Riemann-Stieltjes integral is that f is RS integrable with respect to φ if and only if φ is RS integrable with respect to f, in which case

$$(H.\gamma)\qquad \int_a^b f\,d\varphi + \int_a^b \varphi\,df = f\varphi\Big|_a^b.$$

We will now prove that fact.

H.5 Integration by Parts Theorem. *Let $f, \varphi \to \mathbb{R}$ be bounded functions. Then f is Riemann-Stieltjes integrable with respect to φ on $[a,b]$ if and only if φ is Riemann-Stieltjes integrable with respect to f on $[a,b]$. In this case we have formula $(H.\gamma)$.*

Proof. Suppose that f is RS integrable with respect to φ on $[a,b]$ and let $\varepsilon > 0$. Therefore there exists a number $\zeta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any partition with mesh $\mu(\dot{\mathcal{P}}) \le \zeta_\varepsilon$, then we have $|\Sigma(f,\varphi;\dot{\mathcal{P}}) - \int_a^b f\,d\varphi| \le \varepsilon$.

Now let $\dot{\mathcal{Q}} := \{([x_{i-1},x_i],t_i)\}_{i=1}^m$ be a tagged partition of $[a,b]$ such that $\mu(\dot{\mathcal{Q}}) \le \frac{1}{4}\zeta_\varepsilon$. By combining adjacent intervals if necessary, we may assume that the tags t_i in $\dot{\mathcal{Q}}$ are distinct and that $\mu(\dot{\mathcal{Q}}) \le \frac{1}{2}\zeta_\varepsilon$. The strategy of the proof is use the points in $\dot{\mathcal{Q}}$ to construct a specific partition $\dot{\mathcal{Q}}^*$ (whose subintervals have the form $[t_{i-1},t_i]$ and whose tags are the points x_i) in such a way that $\mu(\dot{\mathcal{Q}}^*) \le \zeta_\varepsilon$ and that

$$(H.\delta)\qquad \Sigma(\varphi,f;\dot{\mathcal{Q}}) - f\varphi\Big|_a^b = -\,\Sigma(f,\varphi;\dot{\mathcal{Q}}^*).$$

Supposing that this can be done, if we add $\int_a^b f\,d\varphi$ to both sides and use the fact that $\mu(\dot{\mathcal{Q}}^*) \le \zeta_\varepsilon$, we conclude that

$$\left|\Sigma(\varphi,f;\dot{\mathcal{Q}}) - \left\{f\varphi\Big|_a^b - \int_a^b f\,d\varphi\right\}\right| \le \left|\int_a^b f\,d\varphi - \Sigma(f,\varphi;\dot{\mathcal{Q}}^*)\right| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that φ is RS integrable with respect to f and that

$$\int_a^b \varphi\,df = f\varphi\Big|_a^b - \int_a^b f\,d\varphi,$$

which is $(H.\gamma)$.

To complete the proof, we need to be more precise about the construction of $\dot{Q}^*$ from $\dot{Q}$. This construction depends on whether the tags t_1 and t_n in $\dot{Q}$ coincide with the endpoints a and b or not; there are four cases.

Case 1: $a < t_1$ and $t_n < b$. Here we set $t_0 := a$ and $t_{n+1} := b$ and define $\dot{Q}^* := \{([t_i, t_{i+1}], x_i)\}_{i=0}^{n}$. We note that $\mu(\dot{Q}^*) = \sup\{t_i - t_{i-1}\} \leq 2\mu(\dot{Q}) \leq \zeta_\varepsilon$. A calculation shows that

$$\begin{aligned}
&f(a)\varphi(a) + \Sigma(\varphi, f; \dot{Q}) - f(b)\varphi(b) \\
&\qquad = \varphi(t_0)f(x_0) + \sum_{i=1}^{n} \varphi(t_i)[f(x_i) - f(x_{i-1})] - \varphi(t_{n+1})f(x_n) \\
&\qquad\qquad = \sum_{i=0}^{n} f(x_i)[\varphi(t_i) - \varphi(t_{i+1})] = -\Sigma(f, \varphi; \dot{Q}^*),
\end{aligned}$$

which is equation $(H.\delta)$.

Case 2: $a = t_1$ and $t_n < b$. Here we set $t_{n+1} := b$ and define $\dot{Q}^* := \{([t_i, t_{i+1}], x_i)\}_{i=1}^{n}$. A calculation yields equation $(H.\delta)$.

Case 3: $a < t_1$ and $t_n = b$. Here we set $t_0 := a$ and define $\dot{Q}^* := \{([t_i, t_{i+1}], x_i)\}_{i=0}^{n-1}$. A calculation yields equation $(H.\delta)$.

Case 4: $a = t_1$ and $t_n = b$. Here we define $\dot{Q}^* := \{([t_i, t_{i+1}], x_i)\}_{i=1}^{n-1}$. A calculation yields equation $(H.\delta)$.

Since every partition $\dot{Q}$ gives rise to a partition $\dot{Q}^*$ having one of these four forms, the proof that φ is RS integrable with respect to f is complete.

If we reverse the roles of f and φ, we conclude that the RS integrability of φ with respect to f implies that f is RS integrable with respect to φ. Q.E.D.

Remark. It follows from Theorems H.3 and H.5 that every function φ in $BV([a,b])$ is Riemann-Stieltjes integrable with respect to any continuous function f on $[a,b]$.

The next result is a Mean Value Theorem for Riemann-Stieltjes integrals.

H.6 Theorem. *Let f, φ be bounded functions and let f be Riemann-Stieltjes integrable with respect to φ on $[a,b]$. Suppose that $m \leq f(x) \leq M$ for all $x \in [a,b]$ and that φ is increasing on $[a,b]$. Then*

$$(H.\varepsilon) \qquad m\big(\varphi(b) - \varphi(a)\big) \leq \int_a^b f\,d\varphi \leq M\big(\varphi(b) - \varphi(a)\big).$$

Therefore, there exists μ with $m \leq \mu \leq M$ such that

$$\int_a^b f\,d\varphi = \mu\big(\varphi(b) - \varphi(a)\big).$$

In particular, if f is continuous on $[a,b]$, then there exists $\xi \in [a,b]$ such that

$$(H.\zeta) \qquad \int_a^b f\,d\varphi = f(\xi)\big(\varphi(b) - \varphi(a)\big).$$

Proof. If $\dot{\mathcal{P}}$ is any tagged partition of $[a,b]$ and φ is increasing, then it is easily seen that $m\big(\varphi(b)-\varphi(a)\big) \leq \Sigma(f,\varphi;\dot{\mathcal{P}}) \leq M\big(\varphi(b)-\varphi(a)\big)$, from which $(H.\varepsilon)$ follows. Equation $(H.\zeta)$ follows from $(H.\varepsilon)$ by applying the Bolzano Intermediate Value Theorem. Q.E.D.

Reduction to a Lebesgue Integral

It is often desirable to convert a Riemann-Stieltjes integral into a Lebesgue (or Riemann) integral. We will now show that it is possible to do so when the integrand f is continuous and the integrator φ is absolutely continuous (see Section 14).

H.7 Theorem. *Let f be continuous and φ be absolutely continuous on $[a,b]$. Then*

$$(H.\eta) \qquad \int_a^b f\,d\varphi = \int_a^b f\varphi',$$

where the integral on the right is a Lebesgue integral.

Proof. Since f is uniformly continuous, given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $|t-s| \leq \delta_\varepsilon$, then $|f(t)-f(s)| \leq \varepsilon$. Since $AC([a,b]) \subset BV([a,b])$, Theorem H.3 implies that the RS integral $\int_a^b f d\varphi$ exists. Thus, there exists $\zeta_\varepsilon \leq \delta_\varepsilon$ such that if $\mu(\dot{\mathcal{P}}) \leq \zeta_\varepsilon$, then

$$\left|\Sigma(f,\varphi;\dot{\mathcal{P}}) - \int_a^b f\,d\varphi\right| \leq \varepsilon.$$

Since φ is absolutely continuous, its derivative φ' exists a.e. and belongs to $\mathcal{L}([a,b])$. Since f is bounded and measurable, the product $f\varphi'$ also belongs to $\mathcal{L}([a,b])$. It also follows from the absolute continuity of φ that

$$\varphi(x_i) - \varphi(x_{i-1}) = \int_{x_{i-1}}^{x_i} \varphi',$$

whence we have

$$
\begin{aligned}
\Big|\Sigma(f,\varphi;\dot{\mathcal{P}}) - \int_a^b f\varphi'\Big| &= \Big|\sum_{i=1}^{n} f(t_i)\big[\varphi(x_i) - \varphi(x_{i-1})\big] - \int_a^b f\varphi'\Big| \\
&= \Big|\sum_{i=1}^{n} f(t_i)\int_{x_{i-1}}^{x_i} \varphi' - \int_a^b f\varphi'\Big| \\
&= \Big|\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \big[f(t_i) - f(x)\big]\varphi'(x)\,dx\Big| \\
&\le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \big|f(t_i) - f(x)\big|\cdot\big|\varphi'(x)\big|\,dx \\
&\le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \varepsilon\,|\varphi'| \le \varepsilon\int_a^b |\varphi'|.
\end{aligned}
$$

Combining the above two inequalities, we conclude that

$$
\begin{aligned}
\Big|\int_a^b f\,d\varphi - \int_a^b f\varphi'\Big| &\le \Big|\int_a^b f\,d\varphi - \Sigma(f,\varphi;\dot{\mathcal{P}})\Big| + \Big|\Sigma(f,\varphi;\dot{\mathcal{P}}) - \int_a^b f\varphi'\Big| \\
&\le \varepsilon + \varepsilon\int_a^b |\varphi'|.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, then equation $(H.\eta)$ is established. **Q.E.D.**

Remark. Equation $(H.\eta)$ does *not* hold if φ is only assumed to have bounded variation. To see this, let $f(x) := 1$ and let the integrator be the Cantor-Lebesgue function $\Lambda : [0,1] \to \mathbb{R}$ given in Theorem 4.17. It is easy to see that, since $\Lambda'(x) = 0$ a.e., then we have

$$1 = \Lambda(1) - \Lambda(0) = \int_0^1 1\,d\Lambda \ne \int_0^1 1\,\Lambda' = 0.$$

The "refinement" RS Integral

In Definition H.1 we defined the RS integral by taking the limit of the RS sums as the mesh of the partitions approaches 0. Although this method is the simplest one, the resulting integral has two serious defects:

(i) If f and φ have a common point of discontinuity, then the RS integral $\int_a^b f\,d\varphi$ does not exist.

(ii) The RS integral is *not* additive over intervals. This means that if $c \in (a,b)$ and if f is RS integrable with respect to φ over $[a,c]$, and also RS

integrable with respect to φ over $[c, b]$, then f may not be RS integrable with respect to φ on $[a, b]$. (An example of this phenomenon is obtained by taking $f := \mathbf{1}_{(0,1]}$ and $\varphi := \mathbf{1}_{[0,1]}$ on the interval $[-1, 1]$; note that 0 is a common point of discontinuity of f and g on $[-1, 1]$, although f is continuous at 0 from the left, and φ is continuous at 0 from the right.)

One way to remove these defects is to consider the **refinement RS** integral. Here, rather than taking the limit as the mesh of the partition approaches 0, we use a different limiting process. If $\mathcal{P}$ and $\mathcal{Q}$ are given partitions of $[a, b]$, we say that $\mathcal{P}$ is a **refinement** of $\mathcal{Q}$ in case every partition point of $\mathcal{Q}$ is a partition point of $\mathcal{P}$ (or, equivalently, that every subinterval in $\mathcal{P}$ is a subset of a subinterval in $\mathcal{Q}$). With this terminology, we say that a number B is the (refinement) RS integral of f with respect to φ if, for every $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ such that if $\dot{\mathcal{P}}$ is any tagged partition that is a refinement of $\mathcal{P}_\varepsilon$, then $|\Sigma(f, \varphi; \dot{\mathcal{P}}) - B| \leq \varepsilon$. This approach is discussed in detail in the author's book [B-2; Section 29].

Here the integral of f with respect to φ does not exist if these functions have a one-sided discontinuity at a point from the *same* side. Further, the Integration by Parts Theorem holds.

The Lebesgue-Stieltjes Integral

Neither of the two RS integrals that have been mentioned above has convergence properties that go beyond the Riemann theory. To obtain more general convergence theorems, one needs to develop the Lebesgue-Stieltjes (or LS) integral. In this case the function φ is usually monotone increasing. There are two favorite ways of obtaining this integral. One is to start with the RS integral of Definition H.1 and then extend, using some version of the Daniell process, to an integral over a larger class of functions. This is done in [Ni; Chapter 8].

Another procedure is to show that if $\varphi : \mathbb{R} \to \mathbb{R}$ is a right-continuous increasing function, then there exists a σ-algebra $\mathcal{S}_\varphi$ of subsets of $\mathbb{R}$ that contains the Borel sets in $\mathbb{R}$, and there exists a measure λ_φ on $\mathcal{S}$ such that $\lambda_\varphi((a, b]) = \varphi(b) - \varphi(a)$. The process used in the construction of this measure λ_φ (which is called the **Lebesgue-Stieltjes measure** corresponding to φ) is rather similar to the one used in constructing Lebesgue measure from the length function. It is carried out in [Ni; Chapter 12]. Once the measure λ_φ and σ-algebra $\mathcal{S}_\varphi$ have been constructed, the integral can be defined as sketched in Section 20 for an arbitrary measure.

The Generalized Riemann-Stieltjes Integral

The final approach to the Stieltjes Integral that we will mention uses a gauge to order the tagged partitions $\dot{\mathcal{P}}$. Thus, a number $C \in \mathbb{R}$ is said to be the **RS*-integral** of f with respect to φ on $I := [a, b]$ in case that for every $\varepsilon > 0$ there exists a gauge δ_ε on I such that if $\dot{\mathcal{P}}$ is any δ_ε-fine partition of I, then $|\Sigma(f, \varphi; \dot{\mathcal{P}}) - C| \leq \varepsilon$.

The generalized RS-integral is bilinear in the sense defined earlier, and is also additive over the subintervals of I. An Integration by Parts Theorem holds, but contains terms involving values of the functions f and φ at their points of discontinuity. This integral also has convergence theorems such as those we have developed for the R*-integral. The reader should refer to [McL; Chapter 7] and [Sch; Chapter 24] for the properties of this integral.

Normed Linear Spaces

It is often convenient to impose the structure of a normed (or a seminormed) space on a collection of functions. Indeed, virtually all of the collections of functions that we have encountered are seminormed spaces in a very natural way. We will collect here a few results that may be of use to the reader.

The reader will recall from Appendix F what is meant by a (real) vector space V. The spaces to be mentioned in this appendix are vector spaces of functions defined on a compact interval $I := [a, b]$, where the vector operations are defined pointwise:

$$(I.\alpha) \qquad (f+g)(x) := f(x) + g(x) \qquad \text{and} \qquad (cf)(x) := cf(x)$$

for all $x \in I$, and where the zero function is $0(x) := 0$ for all $x \in I$.

The reader will also recall from Definition 9.5 what is meant by a **seminorm** and by a **norm** on V.

Some examples

We will first recall some normed and seminormed spaces of functions that we have met.

I.1 Examples. **(a)** Let $B(I)$ denote the collection of all **bounded functions** on $I \to \mathbb{R}$ with the pointwise operations given in $(I.\alpha)$. If we define N on $B(I)$ by

$$(I.\beta) \qquad N(f) := \sup_{x \in I} |f(x)|,$$

then it is clear that N is a norm and that convergence of a sequence with respect to this norm is equivalent to **uniform convergence** on I. For that

reason this norm is often called the **uniform norm**. It is also called, for obvious reasons, the "supremum norm". Generally, this norm is denoted by $\|f\|_\infty$.

(b) Let $C(I)$ denote the collection of all **continuous functions** on $I \to \mathbb{R}$. If I is compact, then the functions in $C(I)$ are automatically bounded, so that the uniform norm in part (a) is also defined on $C(I)$.

(c) Let $c := \frac{1}{2}(a+b)$ be the midpoint of I, and define $N_1(f)$ on $C(I)$ by

$$N_1(f) := \sup\{|f(x)| : a \le x \le c\}.$$

Then it is easy to see that N_1 is a seminorm on $C(I)$. However, it is *not* a norm, since $N_1(f) = 0$ for any function $f \in C(I)$ that vanishes on $[a, c]$.

(d) Let $C^1(I)$ denote the collection of all functions $f : I \to \mathbb{R}$ having **continuous derivatives** on I with the pointwise operations in $(I.\alpha)$. If we define N' on $C^1(I)$ by

$$N'(f) := \sup_{x \in I} |f'(x)|,$$

it is an exercise to show that N' is a seminorm on $C^1(I)$, but it is not a norm since $N'(g) = 0$ for any constant function g.

It is an exercise to show that $\|f\|' := |f(a)| + (b-a)N'(f)$ gives a norm on the space $C^1(I)$ satisfying $\|f\|_\infty \le \|f\|'$.

(e) Let $BV(I)$ denote the collection of all functions $\varphi : I \to \mathbb{R}$ having **bounded variation** in the sense of Definition 7.3. It follows from Exercise 7.E(a, b) that $BV(I)$ is a real vector space and that

$$N(\varphi) := \mathrm{Var}(\varphi; I)$$

defines a seminorm on $BV(I)$. Since $\mathrm{Var}(\varphi; I) = 0$ for any constant function, the function N is not a norm on $BV(I)$; however, if we define

$$\|\varphi\|_{BV} := |\varphi(a)| + \mathrm{Var}(\varphi; I), \tag{$I.\gamma$}$$

we obtain a norm on $BV(I)$. Note that it is a consequence of Exercise 7.C that

$$\|\varphi\|_\infty \le \|\varphi\|_{BV} \qquad \text{for} \quad \varphi \in BV(I).$$

(f) Let $AC(I)$ denote the collection of all functions $\varphi : I \to \mathbb{R}$ that are **absolutely continuous** on I in the sense of Definition 14.4. It was shown in Theorem 14.5 that $AC(I) \subseteq C(I) \cap BV(I)$. The restriction of the function N defined in part (e) gives a seminorm on $AC(I)$, and the restriction of the function defined in $(I.\gamma)$ gives a norm on $AC(I)$.

(g) We have seen in Lemma 9.7 that the collection $\mathcal{L}^1(I) := \mathcal{L}(I)$ of all **absolutely integrable functions** on I is a seminormed space under

$$\|f\|_1 := \int_I |f|. \tag{I.δ}$$

In Section 9 and later, we denoted this seminorm by $\|f\|$, but since we will soon define other collections of measurable functions that are very similar, we now introduce the notation in $(I.\delta)$.

(h) Let $\mathcal{R}(I)$ denote the collection of all real-valued functions f that are **Riemann integrable** on I in the sense of Definition 1.5. Since each function in $\mathcal{R}(I)$ is bounded on I, it is natural to use the norm $\|f\|_\infty$. Alternatively, we can use the seminorm $\|f\|_1 := \int_I |f|$.

(i) Let $\mathcal{R}^*(I)$ denote the collection of all real-valued functions f that are **generalized Riemann-integrable** on I in the sense of Definitions 1.6 or 1.7. Unlike the situation in (h), the function f need not be bounded or absolutely integrable, so the seminorms $\|f\|_\infty$ and $\|f\|_1$ are no longer appropriate.

Instead we will employ a seminorm defined using the indefinite integral $F(x) := \int_a^x f$ by the formula

$$\|f\|_* := \sup_{x\in I} |F(x)| = \sup_{x\in I} \left| \int_a^x f \right|. \tag{I.ε}$$

It will be seen below that this seminorm on $\mathcal{R}^*(I)$ does not seem to be entirely appropriate. (Perhaps the reader can suggest a better one.)

The Riesz spaces $\mathcal{L}^p(I)$

In order to handle the important spaces of measurable functions that were studied by F. Riesz (when $p \neq 2$), we need a lemma.

I.2 Lemma. **(a)** *If* $0 < \alpha \le 1$ *and* $t \ge 0$, *then*

$$t^\alpha \le \alpha t + (1-\alpha). \tag{I.ζ}$$

Moreover, the equality holds in $(I.\zeta)$ *if and only if* $t = 1$.

(b) *If* A, B *are nonnegative, if* $1 < p < \infty$, *and if* $q := p/(p-1)$, *then*

$$AB \le \tfrac{1}{p}A^p + \tfrac{1}{q}B^q. \tag{I.η}$$

Moreover, the equality holds in $(I.\eta)$ *if and only if* $A^p = B^q$.

Proof. (a) Let $h(t) := \alpha t - t^\alpha$ so that $h'(t) < 0$ for $0 < t < 1$ and $h'(t) > 0$ for $t > 1$. The Mean Value Theorem implies that $h(t) > h(1)$ if $t \neq 1$, whence $(I.\zeta)$ follows.

(b) Let $t := a/b$ with $b > 0$ in $(I.\zeta)$ and multiply by b to obtain

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b,$$

where the equality holds if and only if $a = b$. Finally, let $\alpha := 1/p$ so that $1 - \alpha = (p-1)/p = 1/q$ and take $a := A^p$ and $b := B^q$. Q.E.D.

I.3 Definition. If $1 \leq p < \infty$, then $\mathcal{L}^p(I)$ denotes the collection of all measurable functions f on I such that $|f|^p$ belongs to $\mathcal{L}^1(I)$ and we define

$$(I.\theta) \qquad \|f\|_p := \left(\int_I |f|^p\right)^{1/p}.$$

We note that if $p = 1$, then $\mathcal{L}^p(I)$ reduces to $\mathcal{L}(I)$, and $\|f\|_p = \|f\|_1$.

It is clear that the mapping $f \mapsto \|f\|_p$ satisfies the properties (i)–(iii) of Definition 9.5 for a seminorm. It is also clear that $\|f\|_p = 0$ if and only if f is a null function, so that property 9.5(ii*) does *not* hold. However, it is not at all clear (when $p > 1$) that this function satisfies the Triangle Inequality 9.5(iv). We will show that it does possess this property, but first we need to establish an important inequality that is usually attributed to Otto Hölder (1859–1937), showing that the pointwise product of a function in $\mathcal{L}^p(I)$ and a function in $\mathcal{L}^q(I)$ belongs to $\mathcal{L}^1(I)$ provided that $1/p + 1/q = 1$ when $p > 1$. (See also [Dd; p. 140] for other historical remarks.)

I.4 Hölder's Inequality. *Let $f \in \mathcal{L}^p(I)$ and $g \in \mathcal{L}^q(I)$, where $p > 1$ and $q := p/(p-1)$. Then the product fg belongs to $\mathcal{L}^1(I)$ and*

$$(I.\iota) \qquad \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proof. If $\|f\|_p = 0$, then f is a null function, and so is the product fg, whence it follows that $fg \in \mathcal{L}_1(I)$ and that $\|fg\|_1 = 0$. Similarly if $\|g\|_q = 0$. Therefore we may suppose that both $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$.

Now the pointwise product fg is measurable and it follows from $(I.\eta)$ with $A := |f(x)|/\|f\|_p$ and $B := |g(x)|/\|g\|_q$ that

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} \leq \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q}.$$

Since both of the terms on the right are integrable, it follows that the product $|fg|$ is integrable. Moreover, when we integrate, we obtain

$$\frac{\|fg\|_1}{\|f\|_p\|g\|_q} \leq 1/p + 1/q = 1,$$

whence Hölder's Inequality $(I.\iota)$ follows. Q.E.D.

Using Hölder's Inequality, we can establish an inequality, usually attributed to Hermann Minkowski (1864–1909), showing that the mapping $f \mapsto \|f\|_p$ satisfies the Triangle Inequality 9.5(iii) so that $\mathcal{L}^p(I)$ is a seminormed space when $p \geq 1$. (Again, see [Dd; p. 140].)

I.5 Minkowski's Inequality. *If f and h belong to $\mathcal{L}^p(I)$ with $p \geq 1$, then $f + h$ belongs to $\mathcal{L}^p(I)$ and*

$$(I.\kappa) \qquad \|f + h\|_p \leq \|f\|_p + \|h\|_p.$$

Proof. The case $p = 1$ is known, so we suppose that $p > 1$. The sum $f + h$ is a measurable function, and since

$$|f + h|^p \leq \left(2 \cdot \sup\{|f|, |h|\}\right)^p \leq 2^p\left(|f|^p + |h|^p\right),$$

then it follows that $f + h \in \mathcal{L}^p(I)$. Moreover, we have

$$(I.\lambda) \qquad \begin{aligned} |f + h|^p &= |f + h| \cdot |f + h|^{p-1} \\ &\leq |f| \cdot |f + h|^{p-1} + |h| \cdot |f + h|^{p-1}. \end{aligned}$$

Now, since $f + h \in \mathcal{L}^p(I)$, then $|f + h|^p \in \mathcal{L}^1(I)$ and since $p = (p - 1)q$, it follows that $|f + h|^{p-1} \in \mathcal{L}^q(I)$. If we apply Hölder's Inequality to the first term in $(I.\lambda)$, we infer that

$$\begin{aligned} \int_I |f| \cdot |f + h|^{p-1} &\leq \|f\|_p \cdot \left(\int_I |f + h|^{(p-1)q}\right)^{1/q} \\ &= \|f\|_p \cdot \|f + h\|_p^{p/q}. \end{aligned}$$

If we handle the second term in $(I.\lambda)$ in the same way, we obtain

$$\begin{aligned} \|f + h\|_p^p &\leq \|f\|_p \cdot \|f + h\|_p^{p/q} + \|h\|_p \cdot \|f + h\|_p^{p/q} \\ &= \{\|f\|_p + \|h\|_p\} \cdot \|f + h\|_p^{p/q}. \end{aligned}$$

If $C := \|f + h\|_p = 0$, then $(I.\kappa)$ is trivial. If $C \neq 0$, we divide by $C^{p/q}$ and, since $p - p/q = 1$, we obtain Minkowski's Inequality. Q.E.D.

I.6 Remarks. **(a)** As suggested by Hölder's Inequality, if $p > 1$, then the space $\mathcal{L}^p(I)$ is closely related to the space $\mathcal{L}^q(I)$, where p and q satisfy

$$(I.\mu) \qquad 1/p + 1/q = 1.$$

In this case the spaces $\mathcal{L}^p(I)$ and $\mathcal{L}^q(I)$ are said to form a **conjugate pair** and the numbers $p > 1$ and $q > 1$ are said to be **conjugate indices**. In fact, sometimes q is denoted by p' when they are related by $(I.\mu)$.

(b) If $p = 2$, then also $q = 2$, so the space $\mathcal{L}^2(I)$ is **self-conjugate**. This space is the archetypal concrete **Hilbert space**, named after David Hilbert (1862–1943), a German mathematician of great profundity and extraordinary breadth. In the space $\mathcal{L}^2(I)$ we can define the semi-inner product

$$(I.\nu) \qquad \langle f, g\rangle := \int_I fg,$$

although if the functions are complex-valued, we use the complex conjugate function $\bar{g}$ in the integrand. Thus the seminorm and the semi-inner product are related by

$$\|f\|_2 = \sqrt{\langle f, f\rangle}.$$

(c) It will also be noted that if $p = 1$, then there is no *finite* number related to it as in equation $(I.\mu)$; thus the space $\mathcal{L}^1(I)$ does not have a corresponding conjugate pair. However, there is a space of measurable functions that *partially* corresponds to $\mathcal{L}^1(I)$. We will now introduce this space, which will be denoted by $\mathcal{L}^\infty(I)$.

I.7 Definition. A measurable function $f : I \to \mathbb{R}$ is said to be **essentially bounded** on I if there is a null set $Z \subset I$ such that the restriction of f to $I - Z$ is bounded. The vector space of all essentially bounded functions on I is denoted by $\mathcal{L}^\infty(I)$. We define the **essential supremum** of f to be

$$(I.\xi) \qquad \|f\|_\infty := \inf_{Z\subset I}\{\sup_{x\notin Z}|f(x)|\},$$

where the infimum is taken over all null sets $Z \subset I$.

Note. The use of the notation $\|f\|_\infty$ for both the uniform norm and the essential supremum in $(I.\xi)$ will not cause confusion, since it will always be clear from the context which is intended.

I.8 Lemma. *The essential supremum, defined in $(I.\xi)$, is a seminorm on the space $\mathcal{L}^\infty(I)$.*

Proof. It is an exercise to show that if $f \in \mathcal{L}^\infty(I)$, then there exists a null set Z_f such that $|f(x)| \le \|f\|_\infty$ for $x \notin Z_f$. Let Z_g be a corresponding null set for $g \in \mathcal{L}^\infty(I)$. Then we have

$$|f(x) + g(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty$$

for $x \notin Z_f \cup Z_g$. Therefore, $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$, and the Triangle Inequality 9.5(iv) is satisfied. The remaining properties of a seminorm are clear. Q.E.D.

Convergence and Completeness

If V is a vector space on which there is defined a seminorm N, then we say that the sequence $(f_n)_{n=1}^{\infty}$ **converges to** f (with respect to N) in case for every $\varepsilon > 0$ there exists a natural number n_ε such that if $n \geq n_\varepsilon$ then

$$N(f - f_n) \leq \varepsilon.$$

As in Definition 9.11, we say that a sequence $(f_n)_{n=1}^{\infty}$ is a **Cauchy sequence** (with respect to N) in case, for every $\varepsilon > 0$ there exists a natural mumber k_ε such that if $m > n \geq k_\varepsilon$ then

$$N(f_m - f_n) \leq \varepsilon.$$

It is an easy exercise to show that every convergent sequence is a Cauchy sequence, but the converse statement is *not* true.

We say that a seminormed space V is **complete** (with respect to the seminorm N) if every Cauchy sequence converges to some element in V. It is conventional to use the term **Banach space** for a complete normed space, but we will also use this term for a complete seminormed space; otherwise the important spaces $\mathcal{L}^p(I)$ would not be included. Banach spaces are named in honor of the outstanding Polish mathematician Stefan Banach (1892–1945), who was one of the first to study them.

Note. At this point we point out that if N is a seminorm (and not a norm), then the limit of a convergent sequence is *not* uniquely determined. However, it is an easy exercise to show that two vectors f and $\bar{f}$ will be limits of the same sequence if and only if $N(f - \bar{f}) = 0$. As was discussed after 9.7, this nonuniqueness of the limits can be eliminated by "identifying" vectors f and $\bar{f}$ such that $N(f - \bar{f}) = 0$. However, we will not make this identification, and are willing to cope with the nonuniqueness of limits.

Completeness and the Examples

We will now examine each of the examples in I.1, I.3 and I.6 to determine whether or not these spaces are complete under the specified seminorms.

I.9 Examples. (a) The space $B(I)$ is complete under the uniform norm $\|f\|_\infty$.

Let (f_n) be a Cauchy sequence with respect to this norm, so that given $\varepsilon > 0$ there exists k_ε such that if $m > n \geq k_\varepsilon$, then

$$(I.o) \qquad |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty \leq \varepsilon \qquad \text{for all} \quad x \in I.$$

It follows that each sequence $(f_n(x))_n$ is a Cauchy sequence of real numbers, and hence converges to (a unique) limit $f(x) := \lim f_n(x)$. If we let $m \to \infty$ in $(I.o)$, we obtain

$$|f(x) - f_n(x)| \le \varepsilon \qquad \text{for all} \quad x \in I,\ n \ge k_\varepsilon.$$

Therefore $f - f_n \in B(I)$, whence $f = (f - f_n) + f_n$ belongs to $B(I)$. But, since $\|f - f_n\|_\infty \le \varepsilon$ when $n \ge k_\varepsilon$, we conclude that $\|f - f_n\|_\infty \to 0$.

(b) The space $C(I)$ of continuous functions is complete under $\|f\|_\infty$.

Indeed, the argument given in (a) shows that if (f_n) is a Cauchy sequence, then there exists a unique function f to which this sequence converges uniformly on I. The continuity of f follows from the familiar fact that the uniform limit of a sequence of continuous functions is continuous.

(c) The space $C(I)$ is also complete under the seminorm N_1 of Example I.1(c).

Indeed, it follows as above that if a sequence (f_n) is a Cauchy sequence relative to N_1, then it is uniformly convergent on the subinterval $[a, c]$. We define $f(x) := \lim f_n(x)$ for $x \in [a, c]$ and $f(x) := f(c)$ for $x \in (c, b]$. Then f is continuous on $I = [a, b]$ and it is an exercise to show that $N_1(f - f_n) \to 0$. (The reader should note that there are *many* other ways we could define a limit function on $(c, b]$.)

(d) The space $C^1(I)$ is complete under the norm $\|f\|'$.

If (f_n) is a Cauchy sequence relative to this norm, then both (f_n) and the sequence of derivatives (f_n') are Cauchy sequences relative to the uniform norm, and so converge uniformly to continuous functions f and g, respectively. It is a consequence of a well-known theorem [B-S; p. 235] that f is differentiable on I and that $f' = g$, so that $\|f - f_n\|' \to 0$.

(e) The space $BV(I)$ is complete under the norm given in $(I.\gamma)$.

Let (φ_n) be a Cauchy sequence relative to this norm, so that it is also a Cauchy sequence with respect to the uniform norm and the sequence (φ_n) converges (uniformly) on I to a function φ. We need to show that $\varphi \in BV(I)$ and that $\|\varphi - \varphi_n\|_{BV} \to 0$.

Given $\varepsilon > 0$ there exists a natural number k_ε such that if $m > n \ge k_\varepsilon$, then $\operatorname{Var}(\varphi_m - \varphi_n\,; I) \le \varepsilon$. Hence, if $\mathcal{P} = (x_i)_{i=1}^r$ is any partition of I, then we have

$$(I.\pi) \qquad \sum_{i=1}^{r} \big|(\varphi_m - \varphi_n)(x_i) - (\varphi_m - \varphi_n)(x_{i-1})\big| \le \varepsilon$$

for $m > n \geq k_\varepsilon$. If we rearrange the terms in this sum and apply the Triangle Inequality, we obtain

$$\begin{aligned}\sum_{i=1}^{r} \left|\varphi_m(x_i) - \varphi_m(x_{i-1})\right| &\leq \sum_{i=1}^{r} \left|\varphi_n(x_i) - \varphi_n(x_{i-1})\right| + \varepsilon \\ &\leq \mathrm{Var}(\varphi_n; I) + \varepsilon.\end{aligned}$$

If we take $n = k_\varepsilon$ and let $m \to \infty$, we conclude that

$$\sum_{i=1}^{r} \left|\varphi(x_i) - \varphi(x_{i-1})\right| \leq \mathrm{Var}(\varphi_{k_\varepsilon}; I) + \varepsilon,$$

whence it follows that $\mathrm{Var}(\varphi; I) \leq \mathrm{Var}(\varphi_{k_\varepsilon}) + \varepsilon$ so that $\varphi \in BV(I)$.

To show that $\|\varphi - \varphi_n\|_{BV} \to 0$, we let $m \to \infty$ in $(I.\pi)$ to get

$$\sum_{i=1}^{r} \left|(\varphi - \varphi_n)(x_i) - (\varphi - \varphi_n)(x_{i-1})\right| \leq \varepsilon \qquad \text{for} \quad n \geq k_\varepsilon.$$

Since this inequality holds for all partitions, then $\mathrm{Var}(\varphi - \varphi_n; I) \leq \varepsilon$ for $n \geq k_\varepsilon$, and since $|\varphi(a) - \varphi_n(a)| \to 0$, we conclude that $\|\varphi - \varphi_n\|_{BV} \to 0$.

(f) The space $AC(I)$ is complete under the norm in $(I.\gamma)$.

If (φ_n) is a sequence in $AC(I)$ that is Cauchy under the BV-norm, then it follows from part (e) that there exists a continuous function $\varphi \in BV(I)$ such that $\|\varphi - \varphi_n\|_{BV} \to 0$. It remains to show that $\varphi \in AC(I)$.

Given $\varepsilon > 0$ let k_ε, m, n and $\mathcal{P}$ be as in $(I.\pi)$. Now let $n = k_\varepsilon$ and since $\varphi_{k_\varepsilon} \in AC(I)$, let $\eta_\varepsilon > 0$ be such that if $\{[u_j, v_j]\}_{j=1}^{s}$ is any subpartition of I with $\sum_{j=1}^{s}(v_j - u_j) < \eta_\varepsilon$, then $\sum_{j=1}^{s} |\varphi_{k_\varepsilon}(v_j) - \varphi_{k_\varepsilon}(u_j)| \leq \varepsilon$. Since the $\{[u_j, v_j]\}$ form a subpartition of I, it follows from $(I.\pi)$ and the Triangle Inequality that if $m \geq k_\varepsilon$, then

$$\sum_{j=1}^{s} \left|\varphi_m(v_j) - \varphi_m(u_j)\right| \leq \sum_{j=1}^{s} \left|\varphi_{k_\varepsilon}(v_j) - \varphi_{k_\varepsilon}(u_j)\right| + \varepsilon \leq 2\varepsilon.$$

Now let $m \to \infty$ to conclude that $\varphi \in AC(I)$.

(g) The space $\mathcal{L}^1(I)$ of absolutely integrable functions is complete under the seminorm $\|f\|_1$.

In fact, this was proved in the Completeness Theorem 9.12.

(h) The space of Riemann integrable functions is complete under the uniform norm, but not complete under the seminorm $\|f\|_1 := \int_I |f|$.

The completeness under $\|f\|_\infty$ is well known (see [B-S; p. 237]); it also follows as in the proof of Theorem 8.3.

To see that this space is not complete under $\|f\|_1$, let $f_n(x) := 1/\sqrt{x}$ for $x \in [1/n, 1]$ and $f_n(x) := 0$ otherwise. Then (f_n) is a Cauchy sequence with respect to this norm, and converges in $\mathcal{L}^1(I)$ to the function $f(x) := 1/\sqrt{x}$ if $x \in (0,1]$ and $f(0) := 0$. However, there is no bounded function (and hence no Riemann integrable function) to which this sequence converges.

(i) The space of generalized Riemann-integrable functions is not complete under the seminorm $\|f\|_*$ that is defined in $(I.\varepsilon)$.

Let $I := [0,1]$ and let (Λ_n) be the sequence of increasing, continuous, piecewise-linear functions on I constructed in defining the Cantor-Lebesgue singular function Λ (see 4.17). Each Λ_n is absolutely continuous on I and hence is the indefinite integral of a function in $\mathcal{L}(I) \subset \mathcal{R}^*(I)$; moreover, the sequence (Λ_n) converges uniformly to Λ. However, the Cantor-Lebesgue function Λ has the property that $\Lambda(\Gamma) = [0,1]$, where Γ is the Cantor set. Hence Λ maps a null set to a set with positive measure, so Theorem 14.20, due to Xu and Lu, implies that Λ is not the indefinite integral of any function in $\mathcal{R}^*(I)$. Therefore, $\mathcal{R}^*(I)$ is not complete under this seminorm.

NEWS FLASH: Just recently, J. Kurzweil [K-5] has published a small book that has bearing on this question. He considers the space P of indefinite integrals of functions in $\mathcal{R}^*(I)$, thought of as functions of intervals in I; thus $F(J) := \int_J f$ for J a subinterval of I and $f \in \mathcal{R}^*(I)$. Kurzweil shows that P can be equipped with a topology under which it becomes a complete topological vector space. (However, it is not locally convex under that topology.)

(j) The space $\mathcal{L}^p(I)$ $(1 \le p < \infty)$ is complete under the seminorm $\|f\|_p$.

As noted before, this was proved in Theorem 9.12 for $p = 1$. We now give an argument that handles all of the cases $1 \le p < \infty$.

Let (f_n) be a Cauchy sequence relative to $\|f\|_p$. We claim that (f_n) is Cauchy in measure in the sense of Definition 11.5. For, if $\alpha > 0$, then

$$\alpha^p \cdot \big|\{|f_m - f_n| \ge \alpha\}\big| \le \int_I |f_m - f_n|^p = \|f_m - f_n\|_p^p,$$

so that $\big|\{|f_m - f_n| \ge \alpha\}\big| \to 0$ as $m, n \to \infty$. The Riesz Subsequence Theorem 11.9 implies that there exist a subsequence (f_{n_k}) and a measurable function f such that f_{n_k} converges a.e. to f, whence it follows that $|f_{n_k} - f_n|^p \to |f - f_n|^p$ a.e. on I. By Fatou's Lemma 8.7 we infer that

$$\int_I |f - f_n|^p \le \liminf_{k\to\infty} \int_I |f_{n_k} - f|^p = \liminf_{k\to\infty} \|f_{n_k} - f_n\|_p^p.$$

But, given $\varepsilon > 0$ there exists k_ε such that if $m > n \geq k_\varepsilon$, then $\|f_m - f_n\|_p^p \leq \varepsilon^p$, whence we conclude from the above inequality that

$$\|f - f_n\|_p^p = \int_I |f - f_n|^p \leq \varepsilon^p.$$

But this implies that $f - f_n \in \mathcal{L}^p(I)$, whence $f \in \mathcal{L}^p(I)$ and $\|f - f_n\|_p \to 0$.

(k) The space $\mathcal{L}^\infty(I)$ is complete under the essential supremum norm.

Let (f_n) be a Cauchy sequence and let Z be a null set in I such that

$$|f_n(x)| \leq \|f_n\|_\infty \quad \text{and} \quad |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty$$

for all $x \notin Z$ and $m, n \in \mathbb{N}$. The second inequality shows that the sequence $(f_n(x))$ converges (uniformly) on $I - Z$ and we define $f(x) := \lim f_n(x)$ when $x \notin Z$ and $f(x) := 0$ for $x \in Z$. Consequently, f is a measurable function.

Given $\varepsilon > 0$, let k_ε be such that if $m > n \geq k_\varepsilon$, $x \notin Z$, then

$$|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty \leq \varepsilon.$$

If we let $m \to \infty$, we conclude that $|f(x) - f_n(x)| \leq \varepsilon$ for all $n \geq k_\varepsilon$, $x \notin Z$. Therefore $\|f - f_n\|_\infty \leq \varepsilon$, so that $f = (f - f_n) + f_n \in \mathcal{L}^\infty$. But since $\varepsilon > 0$ is arbitrary, we also conclude that $\|f - f_n\|_\infty \to 0$.

Some Partial Solutions

We offer outlines of solutions of approximately one-third of the exercises. While these "solutions" are not complete ones, they are more than mere hints. The reader should refrain from looking at them until serious efforts have been made (and failed) to solve the problem.

Section 1

1.A Indeed, $I_i \subseteq [t_i - \delta_1(t_i), t_i + \delta_1(t_i)] \subseteq [t_i - \delta_2(t_i), t_i + \delta_2(t_i)]$.

1.D (a) Yes. Since $\delta(t) \leq \delta_1(t)$, every δ-fine partition is δ_1-fine.

(b) Yes. The third δ_1-interval is $[\frac{3}{20}, \frac{21}{20}]$, which contains $[\frac{1}{2}, 1]$.

(c) No. The third δ_1-interval is $[\frac{1}{8}, \frac{7}{8}]$, which does not contain $[\frac{1}{2}, 1]$.

1.G If $0 < t \leq \frac{1}{2}$, then $\eta(t) = \frac{1}{2}t$, so that $[t - \eta(t), t + \eta(t)] = [\frac{1}{2}t, \frac{3}{2}t]$ does not contain 0 or 1. If $\frac{1}{2} < t < 1$, then $\eta(t) = \frac{1}{2}(1 - t)$, so that $[t - \eta(t), t + \eta(t)] = [\frac{3}{2}t - \frac{1}{2}, \frac{1}{2}t + \frac{1}{2}]$ does not contain 0 or 1.

1.J If $t \in I$, then $t \in J_k = (a_k, b_k)$ for some k and we define $\delta(t) := \frac{1}{2}\min\{t - a_k, b_k - t\}$, so that δ is a gauge on I. Let $\dot{\mathcal{P}} = \{(I_i, t_i)\}$ be a δ-fine partition of I. Then $I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$. If k is the choice corresponding to the point t_i, then $I_i \subseteq J_k$.

1.M If $\dot{\mathcal{P}} \ll \delta_\varepsilon$, it follows as in Example 1.3(c) that $t_1 = 0$ so that $0 \leq S(f; \dot{\mathcal{P}}) = f(0)(x_1 - 0) = x_1 \leq \delta_\varepsilon(0) = \varepsilon$.

1.P Let $\delta_\varepsilon(1/n) := \varepsilon/n2^n$ for $n \in \mathbb{N}$ and $\delta_\varepsilon(t) := 1$ elsewhere on $[0,1]$. If $\dot{\mathcal{P}} \ll \delta_\varepsilon$, then the point $1/n$ can be the tag of at most two subintervals with total length $\leq \varepsilon/n2^n$. Thus $0 \leq S(k;\dot{\mathcal{P}}) \leq \varepsilon/2 + \varepsilon/2^2 + \cdots = \varepsilon$.

1.S Since degenerate intervals have length 0, we may suppose that the I_j are nondegenerate, so that $a_j < b_j$. Relabeling, if necessary, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_m$. Since the intervals are nonoverlapping, we have

$$\alpha < a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m < \beta,$$

where $J = (\alpha, \beta)$. It follows that $b_j - a_j \leq a_{j+1} - a_j$ for $j = 1, \cdots, m-1$ and $b_m - a_m < \beta - a_m$. Adding these inequalities, we get $(b_1 - a_1) + (b_2 - a_2) + \cdots + (b_m - a_m) < \beta - a_1 < \beta - \alpha$.

1.V If $f \in \mathcal{R}^*(I)$, there exists $D \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a gauge δ_ε such that if $\dot{\mathcal{P}} \ll \delta_\varepsilon$ (that is, $t_i \in I_i \subseteq [t_i - \delta_\varepsilon(t_i), t_i + \delta_\varepsilon(t_i)]$) then $|S(f;\dot{\mathcal{P}}) - D| \leq \varepsilon$. It follows from Exercise 1.U(b) that if $\Delta_\varepsilon(t) := [t_i - \delta_\varepsilon(t_i), t_i + \delta_\varepsilon(t_i)]$, then Δ_ε is an interval gauge and $\dot{\mathcal{P}}$ is Δ_ε-fine.

Conversely, suppose that for every $\varepsilon > 0$ there exists an interval-gauge Δ_ε such that if $\dot{\mathcal{P}}$ is Δ_ε-fine, then $|S(f;\dot{\mathcal{P}}) - D| \leq \varepsilon$. Now let δ_ε be defined as in Exercise 1.T(b). If $\dot{\mathcal{P}} \ll \delta_\varepsilon$, then it follows from Exercise 1.U(a) that $\dot{\mathcal{P}}$ is Δ_ε-fine; consequently, we have $|S(f;\dot{\mathcal{P}}) - D| \leq \varepsilon$.

Section 2

2.A Indeed, $0 \leq x_{i-1}^2 \leq x_i x_{i-1} \leq x_i^2$ whence we have $x_{i-1}^2 \leq v_i^2 \leq x_i^2$. If $0 \leq v_i < x_{i-1}$, then $v_i^2 < x_{i-1}^2$, and if $x_i < v_i$, then $x_i^2 < v_i^2$. The final assertion follows by direct multiplication.

2.D By the Mean Value Theorem there exists $w_i \in [x_{i-1}, x_i]$ such that $Q(x_i) - Q(x_{i-1}) = w_i^3(x_i - x_{i-1})$. Adding, we obtain that $Q(b) - Q(a) = \sum_{i=1}^n w_i^3(x_i - x_{i-1})$, from which the displayed equation holds. Since $w_i^3 - t_i^3 = (w_i^2 + w_i t_i + t_i^2)(w_i - t_i)$, it follows that $|w_i^3 - t_i^3| \leq 3c^2 \cdot 2\delta$.

2.G As in Example 2.2(a), we may assume that c tags the abutting subintervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$ where $x_k = c$. Here $S(f_1;\dot{\mathcal{P}}) = \alpha(x_{k+1} - a) + \beta(b - x_{k+1})$, so that $S(f_1;\dot{\mathcal{P}}) - [\alpha(c-a) + \beta(b-c)] = (\alpha - \beta)(x_{k+1} - c)$. Since $\dot{\mathcal{P}} \ll \delta_\varepsilon$, then $0 < x_{k+1} - c \leq \delta_\varepsilon = \varepsilon/|\beta - \alpha|$, and so $|S(f_1;\dot{\mathcal{P}}) - [\alpha(c-a) + \beta(b-c)]| \leq \varepsilon$.

2.J Note that $g_1(x) = g(x)$ for all $x \in I - \{c, d\}$. Now apply Example 2.2(b) and Exercise 1.R.

2.M If the $\{J_k\}$ satisfy Definition 2.4(a), then for each k let K_k be the closed interval obtained by adjoining the endpoints of J_k.

Conversely, if $\{K_k\}$ are closed intervals satisfying the hypothesis, let J_k be obtained by deleting the endpoints of K_k. Take open intervals around the endpoints of K_k with length $\leq \varepsilon/2^{k+1}$. The resulting collection of open intervals has total length $\leq 2\varepsilon$.

2.P Let $\{r_k\}$ be an enumeration of C and $\varepsilon > 0$. Let $\delta_\varepsilon(r_k) := \varepsilon/2^{k+1}M$ and $\delta_\varepsilon(t) := 1$ if $t \in I - C$. If $\dot{\mathcal{P}} \ll \delta_\varepsilon$, note that if $t_i \in I - C$ then $h(t_i) = 0$ and the contribution to $S(h;\dot{\mathcal{P}})$ is 0. Also the total contribution to $S(h;\dot{\mathcal{P}})$ of the (at most two) intervals with tag r_k is $\leq |2h(r_k) \cdot \varepsilon/2^{k+1}M| \leq \varepsilon/2^k$.

2.S Note that the same values of M and $m(\varepsilon)$ can be used for the h_n as for h. Thus the definition of δ_ε makes sense for all functions h_n. In treating h_n, since $a_k = 0$ for $k \geq n$, it follows that $T_k = 0$ for all $k \geq n$, so the estimates given for h remain valid for h_n.

Section 3

3.A Show that $S(cf;\dot{\mathcal{P}}) = cS(f;\dot{\mathcal{P}})$. Hence if $|S(f;\dot{\mathcal{P}}) - A| \leq \varepsilon$, then $|S(cf;\dot{\mathcal{P}}) - cA| \leq \varepsilon|c|$.

3.D Let $Z_1 := \{x \in I : f(x) > g(x)\}$ so that Z_1 is a null set. Now let $f_1(x) := f(x)$ and $g_1(x) := g(x)$ for $x \in I - Z_1$ and let $f_1(x) := 0 =: g_1(x)$ for $x \in Z_1$. Corollary 3.3 and Exercise 3.C imply that $\int_I f = \int_I f_1 \leq \int_I g_1 = \int_I g$.

3.G Let $Z := \{x \in I : |f(x)| > g(x)\}$, so that Z is a null set, and let $f_1(x) := f(x)$ and $g_1(x) := g(x)$ for $x \in I - Z$ and $f_1(x) := 0 =: g_1(x)$ for $x \in Z$. Now apply Exercises 3.C and 3.F.

3.J Theorem 3.7 implies that $g|[c,b] = f|[c,b]$ belongs to $\mathcal{R}^*([c,b])$ and that $\int_c^b g = \int_c^b f$. Also, $g|[a,c] = 0$ a.e. on $[a,c]$ so that $g \in \mathcal{R}^*([a,c])$ and $\int_a^c g = 0$. By Theorem 3.7, we have $g \in \mathcal{R}^*([a,b])$ and $\int_a^b g = \int_a^c g + \int_c^b g = 0 + \int_c^b f$.

3.M. Let κ and λ be as in Example 2.8(a,b), so that $\pm\kappa \in \mathcal{R}^*([0,1])$ and $\lambda \notin \mathcal{R}^*([0,1])$. Note that $\lambda = \max\{\kappa, -\kappa\}$.

3.P Let $I := [a,b]$ with $a < b$. Since $f \in \mathcal{R}^*(I)$ by 3.18, it follows from Corollary 3.4 that $\inf_{x\in I} f(x) \leq (b-a)^{-1}\int_I f \leq \sup_{x\in I} f(x)$. Since f is continuous on I, Bolzano's Intermediate Value Theorem [B-S; p. 133] implies that there exists $c \in I$ with $f(c) = (b-a)^{-1}\int_I f$.

3.S Since g is not bounded on $[0,1]$, the preceding exercise implies that g is not regulated.

3.V (a) If there exists a sequence (s_n) of complex-valued step functions that converge uniformly to f on I, then since $|\text{Re } s_n - \text{Re } f| \le |\text{Re}(s_n - f)| \le |s_n - f|$, it follows that $(\text{Re } s_n)$ converges uniformly to $g := \text{Re } f$, so g is regulated. Similarly, $(\text{Im } s_n)$ converges uniformly to $h := \text{Im } f$, so h is regulated. Finally, $f = g + ih$.

(b) Since $g := \text{Re } f$ and $h := \text{Im } f$ are regulated functions on I to $\mathbb{R}$, they are both in $\mathcal{R}^*(I)$, and Exercise 1.W implies that $f = g + ih \in \mathcal{R}^*(I)$.

Section 4

4.A (a) The signum function never assumes the values $\pm\frac{1}{2}$.

(b) Indeed $H'(x) = 1$ for $x > 0$ and $H'(x) = -1$ for $x < 0$, while the right hand derivative $H'_+(0) = 1$ and the left hand derivative $H'_-(0) = -1$.

(d) $H_{-1}(x) = |x| - 1$, $H_2(x) = |x| - 2$.

4.D If $F'(x) = f(x)$ for $x \notin E_1$ and $G'(x) = g(x)$ for $x \notin E_2$, then $(F+G)'(x) = (f+g)(x)$ for $x \notin E_1 \cup E_2$.

4.G (a) Let b be an irrational number and let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $1/n_0 < \varepsilon$. There are only a finite number of rationals in the interval $(b-1, b+1)$ having denominator less than n_0, so we can choose $\delta > 0$ such that $(b-\delta, b+\delta)$ contains none of these rationals. Since h is a null function, it is integrable on $[0,1]$.

(b) Since h does not have the Intermediate Value Property, it cannot be the derivative of any function on $[0,1]$ by Darboux's Theorem [B-S; p. 174]. However, the zero function is a c-primitive of h on $[0,1]$ and it is the indefinite integral of h with base point 0.

(c) If H is an indefinite integral of h on $[0,1]$, then $H(x) = C$ for some $C \in \mathbb{R}$. Therefore $H'(x) = 0 \neq h(x)$ for $x \in \mathbb{Q} \cap (0,1]$.

4.J By 4.10, continuous functions always have primitives. If the regulated function f is not continuous at $c \in [a,b]$, then $f(c)$ is not equal to at least one of the one-sided limits $f(c-), f(c+)$. Then f does not have the Intermediate Value Property on some interval containing c, so f cannot have a primitive on $[a,b]$.

4.M Since sgn has left hand limit equal to -1 at $x = 0$ and right hand limit equal to $+1$, Theorem 4.8 implies that the indefinite integral F_u has a left hand derivative equal to -1 and a right hand derivative equal to $+1$.

4.P Theorem 4.7 implies that $F(x) - F(u) = \int_u^x f$. Hence $F - F(u)$ is the indefinite integral of f with base point $u \in [a, b]$.

4.S If $n \in \mathbb{Z}$, a calculation shows that the right and left hand limits of F at n both equal $\frac{1}{2}n(n-1) = F(n)$, so F is continuous on $\mathbb{R}$. Also $F'(x) = n = \lfloor x \rfloor$ for $x \in (n, n+1)$, so F is a c-primitive of f on $\mathbb{R}$. The right hand derivative of F at n equals n, while the left hand derivative of F there equals $n - 1$.

4.V (a) The nonhorizontal lines in the graph of Λ_n are obtained by joining two points whose y-coordinates differ by $1/2^n$ and whose x-coordinates differ by $1/3^n$.

(b) If $x \in \Gamma_1$, then either $0 \leq x \leq 1/3$ and $0 \leq \Lambda_1(x) \leq 1/2$, or $2/3 \leq x \leq 1$ and $1/2 \leq \Lambda_1(x) \leq 1$. Suppose that $x \in \Gamma_n$ and there exist $a_1, a_2, \cdots, a_n$ in $\{0, 2\}$ such that x satisfies

$$\gamma_n := \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_n}{3^n} \leq x \leq \gamma_n + \frac{1}{3^n},$$

and such that $\Lambda_n(x)$ satisfies

$$\delta_n := \frac{a_1/2}{2} + \frac{a_2/2}{2^2} + \cdots + \frac{a_n/2}{2^n} \leq \Lambda_n(x) \leq \delta_n + \frac{1}{2^n}.$$

Thus, the first digits in the ternary expansion of x are $(.a_1a_2\cdots a_n)_3$, and the first digits in the binary expansion of $\Lambda_n(x)$ are $(.(a_1/2)(a_2/2)\cdots(a_n/2))_2$.

When Γ_n is trisected, the left third is the interval $[\gamma_n + 0/3^{n+1}, \gamma_n + 1/3^{n+1}]$ and the function Λ_{n+1} satisfies $\delta_n + 0/2^{n+1} \leq \Lambda_{n+1}(x) \leq \delta_n + 1/2^{n+1}$, while the right third is the interval $[\gamma_n + 2/3^{n+1}, \gamma_n + 3/3^{n+1}]$ and the function Λ_{n+1} satisfies $\delta_n + 1/2^{n+1} \leq \Lambda_{n+1}(x) \leq \delta_n + 1/2^n$.

Section 5

5.A By combining abutting intervals in Π_1, suppose that the intervals $J_j := [a_j, b_j]$ $(j = 1, \cdots, s)$ are pairwise disjoint. We order these intervals so that $b_j < a_{j+1}$ for $j = 1, \cdots, s-1$. The subpartition Π_2 consists of the intervals $K_j := [b_j, a_{j+1}]$ for $j = 1, \cdots, s-1$, together with $[a, a_1]$ or $[b_s, b]$ when these intervals are nondegenerate.

5.D Here $f(t_i)l(I_i) - \int_{I_i} f = (-2)(c - x_{k-1})$ if $i = k$ and $= 0$ otherwise.

5.G If F is any indefinite integral of f, there exists $C \in \mathbb{R}$ such that $F(x) = C + \int_a^x f$ for all $x \in [a, b]$.

5.J Let $J := (a, b)$ and $m \in \mathbb{N}$. Exercise 1.S implies that $\sum_{j=1}^{m} l(J_j) \leq l(J) \leq b - a$. Since m is arbitrary, it follows that $\sum_{j=1}^{\infty} l(J_j) \leq b - a$.

5.M Indeed, the left hand derivative $F'_-(x) \neq f(x)$ if and only if there exists $\beta(x) > 0$ such that for any $s > 0$ there exists $u_{x,s} \in I$ with $x - s < u_{x,s} < x$ such that $|[F(u_{x,s}) - F(x)]/[u_{x,s} - x] - f(x)| > \beta(x)$.

5.P Recall that $\Lambda'(x) = 0$ if $x \in [0,1] - \Gamma$. If $\Lambda \in NV_I(\Gamma)$, it follows from 5.12 that $\int_0^x \Lambda' = 0 = \Lambda(x)$ for all $x \in [0,1]$, a contradiction.

5.S Let $f_1 := \text{Re}\, f$ and $f_2 := \text{Im}\, f$. It follows from Exercise 1.W that the real-valued functions f_j are in $\mathcal{R}^*([a,b])$ and since $\dot{\mathcal{P}} \ll \delta_\varepsilon$ that $|S(f_j;\dot{\mathcal{P}}) - \int_I f_j| \leq |S(f;\dot{\mathcal{P}}) - \int_I f| \leq \varepsilon$. We now apply Corollary 5.4 to both f_1 and f_2 and use the fact that $|x + iy| \leq |x| + |y|$ for $x, y \in \mathbb{R}$.

Section 6

6.A In the notation of 3.13, if $s(x) = \alpha_i$ for $x \in (c_{i-1}, c_i)$, then $(\varphi \circ s)(x) = \varphi(\alpha_i)$.

6.D Let $\lambda_n(x) := \lambda(x)$ for $x \in [0, c_n)$ and $\lambda_n(x) := 0$ for $x \in [c_n, 1]$. Then λ_n is a step function on $[0,1]$. Moreover, $\lambda_n(x) \to \lambda(x)$ for all $x \in [0,1]$.

6.G The continuous function $h(x) := -\frac{1}{4}(\alpha - \beta)(x^3 - 3x) + \frac{1}{2}(\alpha + \beta)$ has the property that $h(1) = \alpha$, $h(-1) = \beta$, $h'(1) = 0 = h'(-1)$. It is strictly increasing on $[-1,1]$ when $\alpha > \beta$ and strictly decreasing when $\alpha < \beta$. By translating and dilating this function one can smooth the discontinuities of an approximating step function to obtain a function having a continuous derivative.

6.J Let $m(x) := (-1)^{k+1}$ for $x \in [c_{k-1}, c_k)$, where $c_k := 1 - 1/2^k$, $k = 0, 1, \cdots$, and $m(1) := 0$. Since m is the limit on $[0,1]$ of a sequence of step functions, it belongs to $\mathcal{M}([0,1])$. However, $m \cdot \kappa = |\kappa| = \lambda$, the function in Example 2.8(b).

6.M (a) If $x \in U$, then since x belongs to infinitely many of the sets E_n, given any $n \in \mathbb{N}$ there exists $k(x,n) \geq n$ such that $x \in E_{k(x,n)}$ so that $x \in \bigcup_{k=n}^{\infty} E_k$. Since this is true for all $n \in \mathbb{N}$, we conclude that $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$.

Conversely, if $y \notin U$, then y belongs to at most a finite number of the sets, say $E_{n_1}, \cdots, E_{n_r}$. If $m(y) > \max\{n_1, \cdots, n_r\}$, then $y \notin \bigcup_{k=m(y)}^{\infty} E_k$ so that $y \notin \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k$.

The proof of (b) is similar.

6.P If $I \subseteq J$ are compact intervals containing E, then the characteristic function $1_{E,I}$ of E on I is the restriction to I of the characteristic function

$\mathbf{1}_{E,J}$ of E on J. Now apply the Additivity Theorem 3.7 to appropriate intervals.

If I_1, I_2 are compact intervals containing E, there exists a compact interval J containing $I_1 \cup I_2$.

6.S If E is a null set in I, then its characteristic function is a null function. By Example 2.6(a), $\mathbf{1}_E \in \mathcal{R}^*(I)$ so that $E \in \mathbb{I}(I)$ and $|E| = \int_I \mathbf{1}_E = 0$.

Conversely if E is an integrable set, then $\mathbf{1}_E \in \mathcal{R}^*(I)$. If $0 = |E| = \int_I \mathbf{1}_E$, since $|\mathbf{1}_E| = \mathbf{1}_E$, it follows from Theorem 5.10(e) that $\mathbf{1}_E$ is a null function so that E is a null set.

6.V Since $f \cdot \mathbf{1}_E$ and $f \cdot \mathbf{1}_F$ belong to $\mathcal{R}^*(I)$ and $\mathbf{1}_{E\cup F} = \mathbf{1}_E + \mathbf{1}_F$, it follows from Theorem 3.1(a) that $f \cdot \mathbf{1}_{E\cup F} \in \mathcal{R}^*(I)$ and

$$\int_{E\cup F} f = \int_I f \cdot \mathbf{1}_{E\cup F} = \int_I f \cdot \mathbf{1}_E + \int_I f \cdot \mathbf{1}_F = \int_E f + \int_F f.$$

Section 7

7.A (a) If $x \in (a, b)$, consider the partition $\mathcal{P} := \{a, x, b\}$.

(b) Take the partition $\mathcal{P} := \{a, b\}$.

7.D (b) Let $\varphi(x) := \sqrt{x}$ for $x \in [0, 1]$.

7.G If $\varphi \in BV([a, b])$, it follows from Exercise 7.B(a) that φ belongs to $BV([a, c]) \cap BV([c, d])$. Conversely, let $\mathcal{P}$ be an arbitrary partition of $[a, b]$ and consider $\mathcal{P} \cup \{c\}$.

7.J (a) If $a \le x \le y \le b$, then Exercises 7.B(a) and 7.H imply that $\pm(\varphi(y) - \varphi(x)) \le \mathrm{Var}(\varphi; [x, y]) = V(y) - V(x)$, whence $W(x) \le W(y)$. Evidently, $\varphi = V - W$. Part (b) is similar.

7.M (a) By 7.B(b), we have $|\varphi(x) - \varphi(y)| \le V(y) - V(x)$ when $a \le x \le y \le b$.

(b) Using the hint, we have $\mathrm{Var}(\varphi; [c, b]) \le |\varphi(y) - \varphi(c)| + \mathrm{Var}(\varphi; [y, b]) + \varepsilon$, so that $V(b) - V(c) \le V(b) - V(y) + 2\varepsilon$, whence $V(y) - V(c) \le 2\varepsilon$.

(c) If φ is left continuous at $c \in (a, b]$, given $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in [c - \delta, c]$ then $|\varphi(x) - \varphi(c)| < \varepsilon$. Let $\mathcal{P} = \{x_i\}$ be a partition of $[a, c]$ with $|x_{n-1} - c| \le \delta$ and $\mathrm{Var}(\varphi; [a, c]) \le \sum |\varphi(x_i) - \varphi(x_{i-1})| + \varepsilon$. Therefore $V(c) - V(a) \le \varepsilon + V(x) - V(a) + \varepsilon$ so that $V(c) - V(x) \le 2\varepsilon$.

7.P Let $\varphi(x) := x^2 \sin(\pi/x)$ on $(0,1]$ and $\varphi(0) := 0$. Then $|\varphi'(x)|$ is bounded on $[0,1]$ so $\varphi \in BV([0,1])$. Let $\mathcal{P}$ be any partition of $[0,1]$. If $\varphi(x_1) \geq 0$, let k be an odd natural number such that $z_1 := 2/(2k+1) < x_1$, whence $\varphi(z_1) < 0$. Show that the sum corresponding to $\mathcal{P} \cup \{z_1\}$ strictly exceeds that corresponding to $\mathcal{P}$.

7.S Lemma 6.5(b) gives f^+, f^- in terms of $f, |f|$. Also $|f| = 2f^+ - f$, $f^- = f^+ - f$. The other four possibilities are similar.

7.V The constant functions $g_1 := n$, $g_2 := -n$ are in $\mathcal{L}([a,b])$. By 7.12, $f \wedge g_i$ and $g_1 \wedge g_2$ are in $\mathcal{L}([a,b])$. Now apply 7.12 and 6.5(d, e).

Section 8

8.A (a) Let $f_k(x) := g_k(x)$ and $f(x) := g(x)$ for $x \in I - Z$ and $:= 0$ on Z. By Theorem 8.3 and Exercise 3.C we have $\int_I g_k = \int_I f_k \to \int_I f = \int_I g$.

(b) Let Z be a null set such that $g(x) = \lim g_k(x)$ for all $x \in I - Z$. Define f_k, f as in (a).

(c) Let Z be a null set such that $\alpha(x) \leq g_k(x)$ and that $\liminf g_k(x) \in \mathbb{R}$ for all $x \in I - Z$. Let f_k, f be as in (a) and $\tilde{\alpha}(x) := \alpha(x)$ for all $x \in I - Z$ and $:= 0$ on Z. Then $\tilde{\alpha}(x) \leq f_k(x)$ on all of I, $\liminf f_k = \liminf g_k$ a.e., and $\liminf \int_I f_k = \liminf \int_I g_k$.

(d) Let Z be a null set such that $\alpha(x) \leq g_k(x) \leq \omega(x)$ and $g(x) = \lim g_k(x)$ for all $x \in I - Z$. Now argue as in (c).

8.D If $x \in (0,1]$, then $p_k(x) = 0$ for $k > 2/x$ and $p_k(0) = 0$. Here $\int_I f_k = k$.

8.G (a) If (f_k) is increasing, let $\varphi_k := f_k - f_1 \in \mathcal{R}^*(I)$. Since $\varphi_k \geq 0$ and $\varphi_k \in \mathcal{R}^*(I)$, it belongs to $\mathcal{L}(I)$.

(b) Let $\psi_k := g_k - g_1$ so that $|\psi_k| \leq \omega - \alpha \in \mathcal{L}(I)$.

8.J Here $f_k(x) \to 0$ everywhere and $\int_I \lim f_k = 0$, while $\int_I f_k = -1$ for all $k \in \mathbb{N}$. The sequence (f_k) is not bounded below by a function in $\mathcal{R}^*(I)$.

8.M Since $\alpha_k \leq f_k \leq \omega_k$, then $\int_I \alpha_k \leq \int_I f_k \leq \int_I \omega_k$, so that $\liminf \int_I f_k \leq \lim \int_I \omega_k = \int_I \omega < \infty$ and similarly $-\infty < \limsup \int_I f_k$. Exercise 8.L implies that $f = \lim f_k \in \mathcal{R}^*(I)$ and

$$-\infty < \int_I f \leq \liminf \int_I f_k \leq \limsup \int_I f_k \leq \int_I f < \infty,$$

whence $\int_I f = \lim \int_I f_k$.

8.P If $g_k(x) := \min\{1, k\max\{f(x) - r, 0\}\}$, then Theorem 7.12 implies that $g_k \in \mathcal{L}(I)$. If $0 \le f(x) \le r$, then $f(x) - r \le 0$ so that $g_k(x) = 0$, while if $f(x) > r$ then (by the Archimedean Property of $\mathbb{R}$) we have $k(f(x) - r) > 1$ for sufficiently large $k \in \mathbb{N}$, whence $g_k(x) = 1$. Therefore $e_r = \lim g_k$ and since $0 \le g_k \le 1$, it follows from the Dominated Convergence Theorem 8.8 that $e_r \in \mathcal{L}(I)$ so that $E_r := \{x \in I : e_r(x) = 1\} = \{x \in I : f(x) > r\}$ is a measurable set. The preceding exercise implies that $e_r \cdot f \in \mathcal{L}(I)$. Since $re_r \le e_r \cdot f \le f$, the stated inequality holds.

8.S (a) It follows from the Additivity Theorem 3.7 that $f_c \in \mathcal{R}^*(I)$ and that $\int_I f_c = \int_c^b f$. The remaining assertion follows from the continuity of the indefinite integral $F(x) := \int_a^x f$ (see Theorem 5.6).

(b) Since F is uniformly continuous on I, given $\varepsilon > 0$ there exists $\zeta > 0$ such that if $|x-y| \le \zeta$, $x, y \in I$, then $|F(x) - F(y)| \le \varepsilon$. Let δ_ε be a gauge on I with $\delta_\varepsilon(x) \le \zeta$ for $x \in I$ and such that if $\dot{\mathcal{P}} \ll \delta_\varepsilon$, then $|S(f;\dot{\mathcal{P}}) - \int_a^b f| \le \varepsilon$. Let $\dot{\mathcal{P}}_c$ be the subpartition of $\dot{\mathcal{P}}$ having tags $\le c$, so that the union of the intervals in $\dot{\mathcal{P}}_c$ is an interval $[a, y]$ with $|y - c| \le \zeta$. By the Saks-Henstock Lemma, we have

$$\begin{aligned}\left|S(f_c;\dot{\mathcal{P}}) - \int_a^b f_c\right| &= \left|S(f;\dot{\mathcal{P}}_c) - \int_a^c f\right| \\ &\le \left|S(f;\dot{\mathcal{P}}_c) - \int_a^y f\right| + \left|\int_y^c f\right| \le 2\varepsilon.\end{aligned}$$

8.V (a) Given $\varepsilon > 0$, let δ_ε be as in Definition 8.10. If $c \in [a, b]$ and if $|f(c)| < M$ for $f \in \mathcal{F}$, let $\delta'_\varepsilon(x) := \min\{\delta_\varepsilon(x), \frac{1}{2}|x - c|\}$ if $x \ne c$, and $\delta'_\varepsilon(c) := \min\{\delta_\varepsilon(c), \varepsilon/(M+1)\}$. If $0 < h < \delta'_\varepsilon(c)$ and if $\dot{\mathcal{P}}_0$ is the δ'_ε-fine subpartition consisting of the pair $([c, c+h], c)$, then the Saks-Henstock Lemma implies that $|f(c)h - (F(c+h) - F(c))| \le \varepsilon$, so that $|F(c+h) - F(c)| \le \varepsilon + |f(c)|h \le 2\varepsilon$. Hence $\mathcal{F}^i$ is equicontinuous on the right at c, and a similar argument applies on the left.

(b) Let $\mathcal{F}$ be equicontinuous at every point of $I := [a, b]$. Thus, given $\varepsilon > 0$ and $c \in I$, there exists $\delta(c) > 0$ such that if $|x - c| \le 2\delta(c)$, $x \in I$ and $F \in \mathcal{F}$, then $|F(x) - F(c)| \le \frac{1}{2}\varepsilon$. Thus δ is a gauge on I. If $\dot{\mathcal{P}} = \{(I_i, t_i)\}$ is δ-fine, let $\eta_\varepsilon := \min\{\delta(t_1), \cdots, \delta(t_n)\} > 0$. If $x, y \in I$ and $|x - y| \le \eta_\varepsilon$, there exists i with $|y - t_i| \le \delta(t_i)$. Since $|x - t_i| \le |x - y| + |y - t_i| \le \eta_\varepsilon + \delta(t_i) \le 2\delta(t_i)$, we have $|F(x) - F(y)| \le |F(x) - F(t_i)| + |F(t_i) - F(y)| \le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$ for all $F \in \mathcal{F}$.

8.Y If $x > 0$, then for $k > 3/x$ we have $g_k(x) = 0$, so $g(x) := \lim g_k(x) = 0$. Also $\int_0^3 g_k = 0$. If $\delta_k(x) := \frac{1}{2}\operatorname{dist}(x, \{1/k, 2/k, 3/k\})$ for $x \notin \{1/k, 2/k, 3/k\}$ and $\delta_k(x) := 1/k^2$ otherwise, then if $\dot{\mathcal{P}} \ll \delta_k$, then we

have $|S(g_k;\dot{\mathcal{P}})| \le 2/k$ and so $|S(g_k;\dot{\mathcal{P}})-S(g;\dot{\mathcal{P}})| \le 2/k$. However, the graph of G_k is the piecewise linear function passing through $(0,0), (1/k,0), (2/k,1), (3/k,0), (3,0)$, so $\{G_k\}$ is not uniformly equicontinuous on $[0,3]$.

Section 9

9.A ($\Rightarrow$) Let $f_n := n\mathbf{1}_Z$ so that $f_n \in \mathcal{L}(I)$, (f_n) is increasing, and $f_n(x) \to \infty$ for all $x \in Z$. Moreover, $\int_I f_n = n\int_I \mathbf{1}_E = 0$ for all $n \in \mathbb{N}$, so $(\int_I f_n)$ converges to 0.

($\Leftarrow$) Since $(\int_I f_n)$ converges in $\mathbb{R}$, it is a bounded sequence. The Increasing Sequence Theorem 9.3 implies that $Z := \{x \in I : f_n(x) \to \infty\}$ is a null set.

9.D Let Z_1 be a null set such that $\sum_{n=1}^{\infty} f_n(x) = 0$ for all $x \in I - Z_1$. By Beppo Levi's Theorem there exists a null set Z_2 such that this series is absolutely convergent for all $x \in I - Z_2$ to some function $\tau \in \mathcal{L}(I)$ such that $\int_I \tau = \sum_{n=1}^{\infty}\int_I f_n$. Hence $\tau(x) = 0$ for all $x \in I - (Z_1 \cup Z_2)$ so $\int_I \tau = 0$, whence $\sum_{n=1}^{\infty}\int_I f_n = 0$.

9.G Let $g_n := f_1 \vee \cdots \vee f_n$ so that (g_n) is increasing on I and, by hypothesis, $\int_I g_n \le M$. The Increasing Sequence Theorem 9.3 implies that there exists a null set Z such that if $g(x) := \lim g_n(x)$ for $x \in I - Z$ and $g(x) := 0$ for $x \in Z$, then $g \in \mathcal{L}(I)$. Since $0 \le f_n \le g_n$ a.e., the Monotone Convergence Theorem 8.5 implies that $\lim \int_I f_n = \int_I 0 = 0 = \lim \|f_n\|$.

9.J Indeed, $\sigma(u,v) = N(u-v) \ge 0$ and $\sigma(u,u) = N(0) = 0$. Also $\sigma(u,v) = N(u-v) = N((-1)(v-u)) = N(v-u) = \sigma(v,u)$, and $\sigma(u,v) = N(u-v) = N((u-w)+(w-v)) \le N(u-w)+N(w-v) = \sigma(u,w)+\sigma(w,v)$. Finally, $\sigma(u,v) = 0 \Leftrightarrow N(u-v) = 0$.

9.M Indeed, $\big||s-t|-|s|-|t|\big| \le \big||s-t|-|s|\big|+|t| \le |(s-t)-s|+|t| = 2|t|$ for $s,t \in \mathbb{R}$. Thus $\big||f_n-f|-|f_n|-|f|\big|$ is dominated by $2|f| \in \mathcal{L}(I)$. Since it converges a.e. to 0, then $\big|\|f_n-f\|-\|f_n\|-\|f\|\big| = \big|\int_I(|f_n-f|-|f_n|-|f|)\big| \to 0$.

9.P Let $f_n := \varphi_n - \varphi_1$, so that $f_n \in \mathcal{L}(I)$ and, since $\|f_m - f_n\| = \|\varphi_m - \varphi_n\| \to 0$, the proof of the Completeness Theorem 9.12 implies that there exists a subsequence (f_{n_k}) of (f_n) that converges a.e. and in mean to some $f \in \mathcal{L}(I)$. Now let $\varphi := f + \varphi_1 \in \mathcal{R}^*(I)$ so that $\varphi_{n_k} = f_{n_k} + \varphi_1 \to \varphi$ a.e. and $\|\varphi_{n_k} - \varphi\| = \|f_{n_k} - f\| \to 0$. Part (b) follows as in 9.12 and 9.8.

9.S (a) Indeed, $\|f\|_1 = 4$ and $\|f\|_2 = 2$, while $\|g\|_1 = \frac{1}{4}$ and $\|g\|_2 = \frac{1}{2}$.

(b) Since $0 \le |f| \le 1 + |f|^2$, the integrability of $|f|^2$ implies that of $|f|$ and that $\int_I |f| \le (b-a) + \int_I |f|^2$.

(c) Since $|st| \le \frac{1}{2}(s^2+t^2)$ for $s, t \in \mathbb{R}$, we have $|fg| \le \frac{1}{2}(|f|^2 + |g|^2)$. Thus, $f, g \in \mathcal{L}^2(I)$ imply that $fg \in \mathcal{L}^1(I)$ and $\int_I |fg| \le \frac{1}{2}(\int_I |f|^2 + \int_I |g|^2)$.

(d) Since $(s+t)^2 \le 2(s^2+t^2)$ for $s, t \in \mathbb{R}$, we have $|f+g|^2 \le 2(|f|^2+|g|^2)$. Thus, if $f, g \in \mathcal{L}^2(I)$, then $f+g \in \mathcal{L}^2(I)$ and $\int_I |f+g|^2 \le 2(\int_I |f|^2 + \int_I |g|^2)$.

9.V ($\Rightarrow$) By the Density Theorem 9.15, for every $n \in \mathbb{N}$ there exists a step [resp., continuous] function s_n such that $\|s_n - f\| \le 1/n$. Therefore (s_n) converges in mean to f and (s_n) is a mean Cauchy sequence. By Corollary 9.13, by dropping to a subsequence, we may assume that $s_n \to f$ a.e. Since $|\int_I s_n - \int_I f| \le \int_I |s_n - f| = \|s_n - f\| \to 0$, it follows that $\|s_m - s_n\| \to 0$.

($\Leftarrow$) Since (s_n) is a mean Cauchy sequence, the Completeness Theorem 9.12 implies that it converges in mean to a function $h \in \mathcal{L}(I)$. It follows from Corollary 9.13 that some subsequence of (s_n) converges a.e. to h, whence $f = h$ a.e. Therefore $f \in \mathcal{L}(I)$.

Section 10

10.A It follows from (10.β) that $|E_1 \cup E_2| \le |E_1| + |E_2|$. Since $E_1 \cup \cdots \cup E_n \cup E_{n+1} = (E_1 \cup \cdots \cup E_n) \cup E_{n+1}$, we have $|E_1 \cup \cdots \cup E_n \cup E_{n+1}| \le |E_1 \cup \cdots \cup E_n| + |E_{n+1}| \le |E_1| + \cdots + |E_n| + |E_{n+1}|$. For the second part, use 10.1(b).

10.D (a) Since $E_* = \bigcup_{n=1}^{\infty} F_n$, where $F_n := \bigcap_{k=n}^{\infty} E_k$, it follows from Exercise 10.B that $\mathbf{1}_{E_*} = \sup_{n\ge1}\{\mathbf{1}_{F_n}\}$ and $\mathbf{1}_{F_n} = \inf_{k\ge n}\{\mathbf{1}_{E_k}\}$. Thus $\mathbf{1}_{E_*} = \sup_{n\ge1}\{\inf_{k\ge n} \mathbf{1}_{E_k}\} = \liminf_k \mathbf{1}_{E_k}$. The second part is similar.

(b) If $E_k \in \mathbb{M}(I)$ for all $k \in \mathbb{N}$, then $0 \le \mathbf{1}_{E_k} \in \mathcal{L}(I)$ for all $k \in \mathbb{N}$. By Exercise 8.K with $f_k = \mathbf{1}_{E_k}$, $\omega = 1$, since $\int_I f_k \ge 0$, and from part (a), we have $\limsup_k |E_k| = \limsup_k \int_I f_k \le \int_I \limsup_k f_k = \int_I \mathbf{1}_{E^*} = |E^*|$.

10.G Let $E_n := \{|f| > n\} \in \mathbb{M}(I)$ so that $E_{n+1} \subseteq E_n$ for $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} E_n = \emptyset$. Theorem 10.2(c) implies that $\lim |E_n| = 0$. Hence, given $\gamma > 0$ there exists $n(\gamma)$ such that if $n \ge n(\gamma)$, then $|E_n| < \gamma$. Now take $E_\gamma := E_{n(\gamma)}$.

10.J (a) These are routine verifications. We note that a point belongs to $E \triangle F$ if and only if it belongs to exactly one of the sets E, F. Also, a point belongs to $E \triangle (F \triangle G)$ if and only if it belongs to (i) exactly one, or (ii) all three of the sets E, F, G.

(b) Indeed, $\mathbf{1}_{E\triangle F} = |\mathbf{1}_E - \mathbf{1}_F|$ so that $\rho(E,F) = |E\triangle F| = \int_I \mathbf{1}_{E\triangle F} = \int_I |\mathbf{1}_E - \mathbf{1}_F| = \|\mathbf{1}_E - \mathbf{1}_F\|$.

(c) Since ρ is the restriction to $\mathbb{M}(I)$ of the semimetric corresponding to the seminorm $\|\cdot\|$ on $\mathcal{L}(I)$, it gives a semimetric on $\mathbb{M}(I)$. The completeness of $\mathbb{M}(I)$ under ρ follows from Exercise 10.I.

10.M Since $r\mathbf{1}_{E_r} \le f\cdot\mathbf{1}_{E_r}$, Corollary 3.3 implies that $r|E_r| = \int_I r\mathbf{1}_{E_r} \le \int_{E_r} f \le \|f\|$. More generally, $r\mathbf{1}_{F_{rs}} \le f\cdot\mathbf{1}_{F_{rs}} \le s\mathbf{1}_{F_{rs}}$ implies that $r|F_{rs}| \le \int_{F_{rs}} f \le s|F_{rs}|$.

10.P (a) The function $\operatorname{sgn}\circ f$ equals 1 on the set $\{f>0\}$, equals 0 on the set $\{f=0\}$, and equals -1 on the set $\{f<0\}$. All three of these sets are measurable by Theorem 10.4.

(b) If $\varphi_n(x) := (2/\pi)\operatorname{Arctan}(nf(x))$ for $x\in[a,b]$, then $\varphi_n \in \mathcal{M}([a,b])$ by Theorem 6.3(d). Note that $\varphi_n(x) \to \operatorname{sgn}(f(x))$ for $x\in[a,b]$ and apply Theorem 9.2.

10.S (a) If the set A is a null set, then $|f| \le a$ a.e., which implies that $\|f\|_\infty \le a$, a contradiction. By Exercise 10.P, $\operatorname{sgn} f$ is measurable, so the function g_a is measurable, bounded by $1/|A|$, and belongs to $\mathcal{L}^1(I)$. Moreover, $\|g_a\|_1 = \int_I |g_a| = \int_I (1/|A|)\mathbf{1}_A = 1$. Since $a/|A| \le fg_a$ on A, then $a \le \int_A fg_a = \int_I fg_a$.

(b) Corollary 3.5 and Exercise 10.R(d) imply that if $g\in\mathcal{L}^1(I)$ and $\|g\|_1 \le 1$, then $|\int_I fg| \le \int_I |fg| \le \|f\|_\infty$. On the other hand, if $a < \|f\|_\infty$, then part (a) shows the existence of $g_a \in \mathcal{L}^1(I)$ with $\|g_a\|_1 = 1$ such that $a \le |\int_I fg_a|$. Hence the supremum is $\ge \|f\|_\infty$.

(c) If $f\in\mathcal{L}^\infty(I)$ satisfies $\|f\|_\infty \le 1$, then $|f|\le 1$ a.e., so that $|\int_I fg| \le \int_I |fg| \le \int_I |g| \le \|g\|_1$. Conversely, if $f_1 := \operatorname{sgn} g$, then $f_1 \in \mathcal{L}^\infty(I)$, $\|f_1\|_\infty \le 1$ and $|\int_I f_1 g| = \int_I |g| = \|g\|_1$.

(d) Let $b_1 := 1$; since $f\notin\mathcal{L}^\infty(I)$, there exists a natural number $b_2 \ge 2^2$ such that the set B_1 is not a null set. Given $b_n\in\mathbb{N}$, choose a natural number $b_{n+1} > \max\{b_n, n^2\}$ such that B_n is not a null set. Clearly $\sum 1/b_n$ converges. Since the sets B_n are pairwise disjoint, the function $\tilde{g}$ is well-defined and $\int_I |\tilde{g}| = \sum 1/b_n$. Since $b_n < |f|$ on B_n, then $1/|B_n| \le |\tilde{g}f|$ on the set B_n, whence $\tilde{g}f \notin \mathcal{L}^1(I)$.

(e) Apply part (d). (f) Take $f := 1$.

Section 11

11.A Since the limit functions are not continuous on the entire interval, the convergence is not uniform.

(a) Let $f(0) := 0$ and $f(x) := 1$ for $x \in (0,1]$. Then $|f_n(x) - f(x)| \leq 1/(1+nx) \leq 1/(1+n\gamma)$ for $x \notin E_\gamma := [0,\gamma]$.

(b) Let $g(x) := 0$ for $0 \leq x < 1$, $g(1) := 1/2$, and $g(x) := 1$ for $1 < x \leq 2$. If $x \in [0,a]$ for $0 < a < 1$, then $|g_n(x) - g(x)| \leq a^n$, while if $x \in [b,2]$ for $b > 1$, then $|g_n(x) - g(x)| \leq 1/b^n$. Thus we take $E_\gamma := [1-\gamma/2, 1+\gamma/2]$.

(c) Let $h(x) := 1$ for $x \in [0,1)$ and $h(1) := 1/2$. Then $|h_n(x) - h(x)| \leq \gamma^n$ for $x \in [0,\gamma]$, $0 \leq \gamma < 1$. Thus we take $E_\gamma := [1-\gamma, 1]$.

(d) Let $\varphi(x) := 1$ for $x \in [0,1)$, $\varphi(1) := 1/2$, and $\varphi(x) := 0$ for $x \in (1,2]$. If $x \in [a,2]$, $1 < a < 2$, then $|\varphi_n(x) - \varphi(x)| \leq 1/a^n$. Thus we take $E_\gamma := [1-\gamma/2, 1+\gamma/2]$.

11.D (a, b, c) As in Exercise 11.C.

(d) Here $\|g_n - g\|_2 = (\int_I |g_n - g|^2)^{1/2} = 1 \not\to 0$.

11.G (a) $\Leftrightarrow$ (b) Indeed, part (b) is merely the ε-formulation of the assertion: for every $\alpha > 0$, we have $\lim_{n\to\infty} |\{|f_n - f| \geq \alpha\}| = 0$.

(b) $\Rightarrow$ (c) This is trivial since $\{|f_n - f| > \alpha\} \subseteq \{|f_n - f| \geq \alpha\}$.

(c) $\Rightarrow$ (b) Given $\alpha > 0$, $\varepsilon > 0$, if $n \geq M_{\alpha/2,\varepsilon}$ then $|\{|f_n - f| \geq \alpha\}| \leq |\{|f_n - f| > \alpha/2\}| \leq \varepsilon$.

(b) $\Rightarrow$ (d) Take $\varepsilon = \alpha$.

(d) $\Rightarrow$ (e) As before $\{|f_n - f| > \alpha\} \subseteq \{|f_n - f| \geq \alpha\}$.

(e) $\Rightarrow$ (b) Given $\alpha > 0$, $\varepsilon > 0$, let $\gamma := \min\{\alpha/2, \varepsilon\}$ and let $n \geq Q_\gamma$. Then $|\{|f_n - f| \geq \alpha\}| \leq |\{|f_n - f| > \alpha/2\}| \leq |\{|f_n - f| > \gamma\}| \leq \gamma \leq \varepsilon$.

11.J ($\Rightarrow$) If $f_n \to f$ [meas], then any subsequence $(f_{n'})$ also converges in measure to f. By the Riesz Subsequence Theorem there is a further subsequence that converges a.e. to f.

($\Leftarrow$) If (f_n) does not converge in measure to f, by Exercise 11.H there exists $\alpha > 0$ and a subsequence $(f_{n'})$ of (f_n) such that $|\{|f_{n'} - f| \geq \alpha\}| > \alpha$. Now, if $(f_{n'})$ has a further subsequence that converges a.e. to f, then Egorov's Theorem implies that this subsequence converges a.u. to f, contradicting the above inequality.

11.M Given $\gamma > 0$, Exercise 10.G implies that there exist a set $A_\gamma \in \mathbf{M}$ with $|A_\gamma| \leq \gamma/2$ and an $M_0 \in \mathbb{N}$ such that $|f(x)| \leq M_0$ for $x \in [a,b] - A_\gamma$. By Egorov's Theorem there exist a set $B_\gamma \in \mathbf{M}$ with $|B_\gamma| \leq \gamma/2$ and an N such that if $x \in [a,b] - B_\gamma$ and $n \geq N$, then $|s_n(x) - f(x)| \leq 1$, so that $|s_n(x)| \leq M_0 + 1$. Since each s_n is bounded, there exists $M_\gamma \geq M_0 + 1$ such that $|s_k(x)| \leq M_\gamma$ for $x \in [a,b]$, $k = 1, \cdots, N-1$. Now let $E_\gamma := A_\gamma \cup B_\gamma$ so that $|E_\gamma| \leq \gamma$ and $|s_n(x)| \leq M_\gamma$ for $x \in [a,b] - E_\gamma$.

11.P (a) Let $g_n := \min\{f, f_n\}$ so that $0 \le g_n \le f$, $0 \le g_n \le f_n$ and $|f_n - f| = (f - g_n) + (f_n - g_n)$. Since $f_n \to f$ a.e., then $g_n \to f$ a.e. The Mean Convergence Theorem 8.9 implies that $\int_I |g_n - f| \to 0$, whence $\int_I |f_n - f| \to 0$.

(b) If not, there exist $\varepsilon_0 > 0$ and a subsequence $(f_{n'})$ with $\|f_{n'} - f\| \ge \varepsilon_0$. Now let $(f_{n''})$ be a further subsequence that converges a.e. to f. If we apply (a), we conclude that $\|f_{n''} - f\| \to 0$, contradicting the above inequality.

(c) If $n > 3$, let $f_n(x) := n$ on $[1/n, 2/n)$, $:= -n$ on $[2/n, 3/n)$ and $:= 0$ elsewhere on $[0,1]$. Then $f_n \to 0$ a.e. and in measure and $\int_0^1 f_n = 0$, but $\int_0^1 |f_n - 0| = 2$.

11.S ($\Rightarrow$) Let δ_ε be as in Definition 11.12(a) so that if $E \in \mathbf{M}([a,b])$, $|E| \le \delta_\varepsilon, f \in \mathcal{F}$, then $\int_E |f| \le \varepsilon$. Let $\{I_1, \cdots, I_M\}$ be a partition of I into nonoverlapping intervals with length $\le \delta_1$. Then $\|f\| = \int_I |f| = \sum_{j=1}^M \int_{I_j} |f| \le \sum_{j=1}^M 1 = M$ for $f \in \mathcal{F}$. Let $K \ge M/\delta_\varepsilon$ and $H_{f,K} := \{|f| \ge K\}$. Since $K \cdot \mathbf{1}_{H_{f,K}} \le |f|$, it follows from the fact that $K|H_{f,K}| \le \|f\| \le M$ that $|H_{f,K}| \le M/K \le \delta_\varepsilon$ so that $\|f\|_{H_{f,K}} \le \varepsilon$.

($\Leftarrow$) Given $\varepsilon > 0$, let K be such that if $f \in \mathcal{F}$, then $\|f\|_{H_{f,K}} \le \frac{1}{2}\varepsilon$. Now let $\delta_\varepsilon := \varepsilon/2K$, so that if $|E| \le \delta_\varepsilon$ and $f \in \mathcal{F}$, then

$$\|f\|_E = \|f\|_{E \cap H_{f,K}} + \|f\|_{E - H_{f,K}} \le \|f\|_{H_{f,K}} + K|E| \le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

11.V Theorem 10.4 implies that $\Phi(|f|)$ is measurable for all $f \in \mathcal{M}([a,b])$.

(a) Given $\varepsilon > 0$, let K be such that if $t \ge K$, then $\Phi(t)/t \ge B/\varepsilon$; thus $t \le (\varepsilon/B)\Phi(t)$. Therefore, if $f \in \mathcal{F}$, then

$$\int_{\{|f| \ge K\}} |f| \le \frac{\varepsilon}{B} \int_{\{|f| \ge K\}} \Phi(|f|) \le \frac{\varepsilon}{B} \int_a^b \Phi(|f|) \le \varepsilon,$$

whence $\mathcal{F}$ is uniformly integrable.

(b) As in (a), if $f \in \mathcal{F}$ and $E \in \mathbf{M}([a,b])$, then

$$\int_E |f| \le \int_{E \cap \{|f| \ge K\}} |f| + \int_{E - \{|f| \ge K\}} |f| \le \frac{\varepsilon}{B} \int_a^b \Phi(|f|) + K|E|.$$

Consequently, if $|E| \le \varepsilon/K$, then $\int_E |f| \le 2\varepsilon$ for all $f \in \mathcal{F}$.

(c) Let $\Phi(t) := t^2$ so that $\int_a^b \Phi(|f|) \le A^2$ for all $f \in \mathcal{F}_A$. Now apply (a).

11.Y (a) Indeed, if $f_n(x) \to f(x)$, then $\varphi \circ f_n(x) \to \varphi \circ f(x)$. Thus everywhere and almost everywhere convergence is preserved by composition with a continuous function φ. If $f_n \to f$ [a.u.], then $f_n \to f$ [a.e.], so that $\varphi \circ f_n \to \varphi \circ f$ [a.e.] and Egorov's Theorem applies. If $f_n \to f$ [meas], and if $\varphi \circ f_{n'}$ is any subsequence of $(\varphi \circ f_n)$, Exercise 11.J implies that $(f_{n'})$ has a

subsequence with $f_{n''} \to f$ [a.e.], so that $\varphi \circ f_{n''} \to \varphi \circ f$ [a.e.]. Now apply 11.J again.

(b) If φ is not continuous at $c \in \mathbb{R}$, there exist a sequence (c_n) and $\varepsilon_0 > 0$ such that $c_n \to c$ but $|\varphi(c_n) - \varphi(c)| \geq \varepsilon_0$. Now let $f_n(x) := c_n$ and $f(x) := c$ for $x \in [a, b]$, so that (f_n) converges uniformly to f, but $(\varphi \circ f_n)$ does not converge anywhere or in measure to $\varphi \circ f$.

(c) Given $\varepsilon > 0$ there exists $\gamma_\varepsilon > 0$ such that if $|s - t| \leq \gamma_\varepsilon$, then $|\varphi(s) - \varphi(t)| \leq \varepsilon$. Since $(f_n) \to f$ uniformly on $[a, b]$, there exists N_ε such that if $n \geq N_\varepsilon$, then $|f_n(x) - f(x)| \leq \gamma_\varepsilon$, whence $|\varphi \circ f_n(x) - \varphi \circ f(x)| \leq \varepsilon$.

(d) If φ is not uniformly continuous, there exists $\varepsilon_0 > 0$ and two sequences (s_n) and (t_n) such that $|s_n - t_n| \leq 1/n$, but $|\varphi(s_n) - \varphi(t_n)| \geq \varepsilon_0$. For convenience, let $[a, b] = [0, 1]$. If $n \geq 1$, let $f(x) := t_n$ on $(1/2^n, 1/2^{n-1}]$ and $f(0) := 0$, and let $f_n(x) := s_n$ on $(1/2^n, 1/2^{n-1}]$ and $f_n(x) := f(x)$ elsewhere on $[0, 1]$. Then $|f_n(x) - f(x)| \leq 1/n$ for all $x \in [0, 1]$, but $|\varphi \circ f_n(x) - \varphi \circ f(x)| \geq \varepsilon_0$ for $x \in (1/2^n, 1/2^{n-1}]$.

Section 12

12.A Since $G(t_i) = G(x_{i-1}) + \big[G(t_i) - G(x_{i-1})\big]$, the term in (12.$\zeta_2$) equals

$$\begin{aligned}
&\sum_{i=1}^{n} \Big| f(t_i)G(x_{i-1})(x_i - x_{i-1}) + f(t_i)\big[G(t_i) - G(x_{i-1})\big](x_i - x_{i-1}) \\
&\qquad\qquad\qquad - G(x_{i-1})\big[F(x_i) - F(x_{i-1})\big]\Big| \\
&\qquad \leq \sum_{i=1}^{n} |G(x_{i-1})| \cdot \Big| f(t_i)(x_i - x_{i-1}) - \big[F(x_i) - F(x_{i-1})\big]\Big| \\
&\qquad\qquad\qquad + \sum_{i=1}^{n} |f(t_i)| \cdot \big|G(t_i) - G(x_{i-1})\big|(x_i - x_{i-1}).
\end{aligned}$$

We now use the fact that $|G(x_{i-1})| \leq M$ and (12.ε_1) to bound the first term on the right, and the first term in (12.δ) to bound the second term. We conclude that the term is (12.ζ_2) is dominated by $M(\varepsilon/4M) + (\varepsilon/4M)(b - a) \leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2$.

12.D Direct calculation shows that

$$\begin{aligned}
(Fg)(x) &= x^{1/2} \sin(\pi/x)\big[\tfrac{1}{2}x^{-1/2}\cos(\pi/x) + x^{1/2}(\sin(\pi/x))(\pi/x^2)\big] \\
&= \tfrac{1}{2}\sin(\pi/x)\cos(\pi/x) + (\pi/x)\sin^2(\pi/x) \\
&= \tfrac{1}{4}\sin(2\pi/x) - \tfrac{1}{2}(\pi/x)\cos(2\pi/x) + \tfrac{1}{2}(\pi/x),
\end{aligned}$$

where we have used the identities $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$. Now the first two terms in the last right hand side belong to $\mathcal{R}^*([0,1])$, by Exercise 6.T. However, as seen in Exercise 3.L, the function $x \mapsto 1/x$ does not belong to $\mathcal{R}^*([0,1])$. Hence $Fg \notin \mathcal{R}^*([0,1])$.

12.G (a) Since $\sin(\alpha x + \pi/2) = \cos\alpha x$, the Riemann-Lebesgue Lemma 9.17 implies that $\int_a^b f(x)\cos\alpha x\,dx \to 0$ as $\alpha \to \infty$.

(b) Let $G_1(x) := \int_a^x \cos nx\,dx = (1/n)(\sin nx - \sin na)$, so that $g_1(x) := G_1'(x) = \cos nx$ and Fg_1 is continuous and in $\mathcal{R}^*([a,b])$ and

$$\int_a^b f(x)G_1(x)\,dx = F(b)G_1(b) - \int_a^b F(x)\cos nx\,dx.$$

It follows that

$$\frac{1}{n}\int_a^b f(x)\sin nx\,dx - \frac{1}{n}\sin na\int_a^b f(x)\,dx$$
$$= \frac{1}{n}F(b)(\sin nb - \sin na) - \int_a^b F(x)\cos nx\,dx,$$

so that $(1/n)\int_a^b f(x)\sin nx\,dx = (1/n)F(b)\sin nb - \int_a^b F(x)\cos nx\,dx$. Consequently $(1/n)\int_a^b f(x)\sin nx\,dx \to 0$.

12.J (a) Let $g_1(a) := A$, $g_1(x) := g(x)$ for $x \in (a,b)$, and $g_1(b) := B$. Then g_1 is increasing on $[a,b]$ and the Second Mean Value Theorem applies.

(b) If g is decreasing, let $g_2(a) := A \geq g(a)$, let $g_2(x) := g(x)$ for $x \in (a,b)$, and $g_2(b) := B \leq g(b)$. We obtain $\int_a^b fg = \int_a^b fg_2 = A\int_a^\xi g + B\int_\xi^b g$.

12.M (a) Since g is increasing, then $g_n(x_{n,k-1}) = g(x_{n,k-1}) \leq g(x_{n,k}) = g_n(x_{n,k})$, so that g_n is also increasing on each interval $[x_{n,k-1}, x_{n,k}]$.

(b) Since g and g_n are increasing on the interval $[x_{n,k-1}, x_{n,k}]$ and agree at its endpoints, we have $|g(x) - g_n(x)| \leq |g(x_{n,k-1}) - g(x_{n,k})|$. Thus, if g is continuous at x, then $g_n(x) \to g(x)$. Since g is continuous c.e., we have $g_n(x) \to g(x)$ c.e., and hence a.e.

(c) By Exercise 12.L there exist $\xi_n \in [a,b]$ such that $\int_a^b fg = g(a)\int_a^{\xi_n} f + g(b)\int_{\xi_n}^b f$. Now drop to a convergent subsequence of (ξ_n) and use the continuity of the maps $x \mapsto \int_a^x f$ and $x \mapsto \int_x^b f$, and the fact that $|fg_n| \leq M|f|$, where $M := \max\{|g(a)|, |g(b)|\}$, so that the Dominated Convergence Theorem applies.

12.P If $\gamma \in (0,1)$, then the restriction of h to $[0,\gamma]$ is a step function and so belongs to $\mathcal{R}^*([0,\gamma])$. If $\gamma \in [c_n, c_{n+1})$, then $\int_0^\gamma h = a_1 + \cdots + a_n + r_\gamma$,

where $|r_\gamma| \le |a_{n+1}|$. If the series converges to A, then $|a_{n+1}| \to 0$, so that

$$\lim_{\gamma \to 1-} \int_0^\gamma h = \lim_{n \to \infty} \sum_{k=1}^n a_k = \sum_{k=1}^\infty a_k = A.$$

If $h \in \mathcal{R}^*([0,1])$, then $\lim_{n\to\infty} \int_0^{c_n} h = \lim_{n\to\infty} \sum_{k=1}^n a_k = A$.

12.S (a) The function f is continuous on $(0, \pi/2]$ and so is measurable on $[0, \pi/2]$. Also $|f(x)| \le 1$, so the Integrability Theorem 9.1 implies that $f \in \mathcal{L}([0, \pi/2])$. Note the oscillation of f as $x \to 0+$.

(b) The function g is measurable but not bounded on $[0, \pi/2]$. If we let $u = \csc x$, it is seen that

$$\int_{c_1}^{c_2} g(x)\, dx = \int_{\csc c_2}^{\csc c_1} \frac{\sin u\, du}{\sqrt{u^2 - 1}}.$$

If we apply Exercise 12.K(b), the absolute value of the right side is dominated by $2/\sqrt{\csc^2 c_2 - 1} = 2 \tan c_2$, which approaches 0. Thus the Cauchy Criterion implies that the limit of $\int_c^{\pi/2} g(x)\, dx$ as $c \to 0+$ exists, so Hake's Theorem applies.

12.V (a) Since both f and f_t are continuous on $[-1, 1] \times [0, 1]$, we can apply Exercise 12.U to obtain $F'(t) = (1/t) \int_{-1}^1 [te^x/(1 + te^x)]\, dx = (1/t) \ln(1 + te^x)\big|_{x=-1}^{x=1} = (1/t) \ln[(1 + te)/(1 + t/e)]$ for $t > 0$ and $F'(0) = e - 1/e$.

(b) Again, both f and f_t are continuous on $[0, \pi] \times [0, 1]$, so that

$$\begin{aligned} F'(t) &= \int_0^\pi e^{tx} \sin x\, dx = \frac{e^{tx}}{t^2 + 1} \left[t \sin x - 1 \cos x \right] \Big|_{x=0}^{x=\pi} \\ &= \frac{e^{\pi t}}{t^2 + 1}(0 + 1) - \frac{e^0}{t^2 + 1}(0 - 1) = \frac{e^{\pi t} + 1}{t^2 + 1}. \end{aligned}$$

(c) Define the integrand to be 0 for $x = 0$. The map $x \mapsto x^{-1/2} \cos(tx)$ is clearly measurable; also $|f(x,t)| \le 1/\sqrt{x}$ and $|f_t(x,t)| = |-\sqrt{x} \sin(tx)| \le \sqrt{x}$. By the Differentiation Theorem 12.13, $F'(t) = -\int_0^1 \sqrt{x} \sin(tx)\, dx$.

(d) Define the integrand to be 0 for $x = 0$. The map $x \mapsto x^{1/2} \cos(t/x)$ is measurable and $|f(x,t)| \le \sqrt{x}$ and $|f_t(x,t)| = |-x^{-1/2} \sin(t/x)| \le 1/\sqrt{x}$. By 12.13, we have $F'(t) = -\int_0^1 x^{-1/2} \sin(t/x)\, dx$.

Section 13

13.A (a) Let $\Phi(x) := 4 + x^2$ on $[0, 3]$, so Φ is 1-1 and strictly increasing and $\varphi(x) = 2x$. Let $f(u) := \frac{1}{2}\sqrt{u}$, so $F(u) = \frac{1}{3} u^{3/2}$ for $u \in [2, 13]$. Use

Theorem 13.1 (or 13.5) to infer that the integral equals $\int_0^3 (f \circ \Phi) \cdot \varphi = \int_4^{13} f = \frac{1}{3} u^{3/2} \big|_2^{13} = \frac{1}{3}[13^{3/2} - 4^{3/2}] \approx 12.957$.

(b) The integral equals 0, since the integrand is odd. The same argument as in (a) can be used, although here Φ is 2-1 and 13.1 is used.

13.D (a) Let $\Phi(x) := \sqrt{x+1}$ for $x \in [1,3]$ so Φ is continuous and strictly increasing and $\Phi'(x) > 0$. Here $\Psi(u) = u^2 - 1$ so $\psi(u) = 2u$. Let $f(u) := 1/[(u^2-1)u]$ for $u \in [\sqrt{2}, 2]$, so f is continuous and decreasing. Use Theorem 13.3 (or 13.7) and the fact that $2/(u^2-1) = 1/(u-1) - 1/(u+1)$ to conclude that the integral equals

$$\int_{\sqrt{2}}^{2} \frac{2u\, du}{(u^2-1)u} = \big[\ln(u-1) - \ln(u+1)\big]\Big|_{\sqrt{2}}^{2} = \ln(1 + 2\sqrt{2}/3) \approx 0.664.$$

(b) If $0 \le x \le 3$, then $x\sqrt{x+1} \le 2x$ and so $1/(2x) \le 1/(x\sqrt{x+1})$. Since $\int_0^3 (1/x)\, dx$ is divergent, the integral is divergent.

Alternatively, the argument in (a) gives $\int_a^3 (1/x\sqrt{x+1})\, dx = \ln(a + 2 + 2\sqrt{a+1}) - \ln 3a$, which converges to ∞ as $a \to 0+$. By Hake's Theorem 12.8, the integral diverges.

13.G (a) Let $\Phi(x) := \sqrt{x-1}$ for $x \in [1,2]$, so Φ is continuous and strictly increasing and $\Phi'(x) > 0$ for $x \in (1,2]$. Also $\Psi(u) = u^2 + 1$, so $\psi(u) = 2u$ for $u \in [0,1]$. Let $f(u) := u/(u^2+1)$, so f is continuous and strictly decreasing on $[0,1]$. Use Theorem 13.3 (or 13.7) to conclude that the integral equals

$$\int_0^1 \frac{u}{u^2+1} \cdot 2u\, du = 2\big[u - \operatorname{Arctan} u\big]\Big|_0^1 = 2(1 - \pi/4) \approx 0.429.$$

(b) The integrand is not real-valued for $x \in (0,1)$, but is equal to $(i/x)\sqrt{1-x}$, which might lead to a complex-valued integral. However, if $0 < x < 3/4$, then $1/2 < \sqrt{1-x}$ and so $1/2x < (1/x)\sqrt{1-x}$. Therefore the integral over $[0,1]$ is divergent.

13.J The result follows immediately from the fact that $D_x[\ln(2+\sin x)] = \cos x/(2+\sin x)$, or from Theorem 13.1. It is natural to substitute $u = \Phi(x) := 2 + \sin x$ and $f(u) := 1/u$, but Theorem 13.5 is never applicable when $b - a > \pi$. (Why not?) However Φ is continuous and φ has a finite number of roots on $[a,b]$, so that Theorem 13.8 can be applied to $f(u) := 1/u$ to give the value of the integral to be $F(u)\big|_{2+\sin a}^{2+\sin b}$, where $F(u) := \ln u$.

13.M (a) Let $u = \Phi(x) := x^{3/2}$, so Φ is continuous and increasing on $[0,1]$ and $\varphi(x) = (3/2)x^{1/2}$. Let $f(u) := 1/\sqrt{1-u^2}$ and use Theorem 13.1

(or 13.5) to obtain

$$\int_{\Phi(0)}^{\Phi(1)} \frac{(2/3)\,du}{\sqrt{1-u^2}} = \tfrac{2}{3}\operatorname{Arcsin} u\Big|_0^1 = \tfrac{2}{3}\cdot\tfrac{\pi}{2} = \tfrac{\pi}{3} \approx 1.047.$$

(b) Let Φ be as in (a), $f(u) := 1/(1+u^2)$ and use Theorem 13.1 (or 13.5) to obtain

$$\int_{\Phi(0)}^{\Phi(1)} \frac{(2/3)\,du}{1+u^2} = \tfrac{2}{3}\operatorname{Arctan} u\Big|_0^1 = \tfrac{2}{3}\cdot\tfrac{\pi}{4} = \tfrac{\pi}{6} \approx 0.524.$$

13.P (a) The integrand has a continuous extension to $[0,1]$. Further

$$\int_0^1 \sqrt{x}\ln x\,dx = \tfrac{2}{3}x^{3/2}\ln x\Big|_0^1 - \tfrac{2}{3}\int_0^1 x^{3/2}(1/x)\,dx = -\tfrac{2}{3}\int_0^1 x^{1/2}\,dx = -\tfrac{4}{9}.$$

(b) Since $\sqrt{x}\ln x/(x-1) \to 1$ as $x \to 1-$, then $1/2(1-x) \le |1/\sqrt{x}\ln x|$ for x near 1. Since the integrand is always negative, the integral over $[0,1]$ does not exist.

13.S Let $t > 0$ and let $y = \Phi(x) := x^t$, so that Φ is continuous and strictly increasing on $[0,1]$ and $\varphi(x) = tx^{t-1}$ for $x \in (0,1)$. Let $f(y) := (1/t)y^{(s-t+1)/t}\sin(\pi/y)$ so that

$$f(\Phi(x))\varphi(x) = (1/t)(x^t)^{(s-t+1)/t}\sin(\pi/x^t)\cdot tx^{t-1} = x^s\sin(\pi/x^t).$$

Theorem 13.5 and Example 6.13(b)–(d) show that $x \mapsto x^s\sin(\pi/x^t)$ belongs to $\mathcal{R}^*([0,1])$ if and only if $(s-t+1)/t > -2$, which is the case if and only if $s+t > -1$.

13.V (a) The substitution given in Exercise 13.T gives

$$\int_0^{\pi/2} \frac{dx}{2+\cos x} = \int_0^1 \frac{1}{2+(1-y^2)/(1+y^2)}\cdot\frac{2}{1+y^2}\,dy = \int_0^1 \frac{2\,dy}{3+y^2}$$
$$= \frac{2}{\sqrt{3}}\tan\Big(\frac{y}{\sqrt{3}}\Big)\Big|_0^1 = \frac{2}{\sqrt{3}}\operatorname{Arctan}\Big(\frac{1}{\sqrt{3}}\Big) \approx 0.605.$$

(b) We have

$$\int_0^{\pi/2} \frac{dx}{1+\sin x+\cos x} = \int_0^1 \frac{1}{1+\frac{2y}{1+y^2}+\frac{1-y^2}{1+y^2}}\cdot\frac{2}{1+y^2}\,dy$$
$$= \int_0^1 \frac{2\,dy}{2+2y} = \ln(1+y)\Big|_0^1 = \ln 2 \approx 0.693.$$

13.Y If $x = \Phi(u) := \tan u$, then $u = \Psi(x) = \operatorname{Arctan} x$ so $\psi(x) = 1/(1+x^2)$. Thus the integral equals

$$\int_0^{\pi/4} \ln(1+\tan u)\,du = \int_0^{\pi/4} \ln(\cos x + \sin u)\,du - \int_0^{\pi/4} \ln(\cos u)\,du.$$

If we let $u = \pi/4 - v$, we get

$$\begin{aligned}\int_0^{\pi/4} \ln(\cos u)\,du &= -\int_{\pi/4}^{0} \ln\big(\cos(\pi/4)\cos v + \sin(\pi/4)\sin v\big)\,dv \\ &= \int_0^{\pi/4} \ln\big((\cos v + \sin v)/\sqrt{2}\big)\,dv \\ &= \int_0^{\pi/4} \ln(\cos v + \sin v)\,dv - \tfrac{\pi}{4}\cdot \ln(\sqrt{2}).\end{aligned}$$

If we combine these expressions, we conclude that

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \int_0^{\pi/4} \ln(1+\tan u)\,du = \tfrac{\pi}{8}\cdot \ln 2 \approx 0.272.$$

Section 14

14.A Since $\sum_j |F(v_j) - F(u_j)| \le \sum_j M|v_j - u_j|$, one can take $\eta_\varepsilon := \varepsilon/M$.

14.D If $F \in AC(I)$ and $\sum_j^\infty |v_j - u_j| \le \eta_\varepsilon$, then we have $\sum_j^n |v_j - u_j| \le \eta_\varepsilon$ for all $n \in \mathbb{N}$, so that $\sum_j^n |F(v_j) - F(u_j)| \le \varepsilon$ for all $n \in \mathbb{N}$, and therefore $\sum_j^\infty |F(v_j) - F(u_j)| \le \varepsilon$.

Conversely, if the condition is satisfied and if $\sum_j^s |v_j - u_j| \le \theta_\varepsilon$, then take points $u_k = v_k$ for $k > s$, whence $\sum_j^s |F(v_j) - F(u_j)| \le \varepsilon$.

14.G If F is increasing, adapt the argument in Exercise 14.E(c).

14.J (a) Let $F(x) := (-1)^{n-1}$ if $x \in (1/2^n, 1/2^{n-1}]$ for $n \in \mathbb{N}$ and let $F(0) := 1$. If n is even and $x_0 := 0$, $x_1 := 1/2^{n-1}, \cdots, x_{n-1} := 1/2$, $x_n := 1$, then $|F(x_1) - F(x_0)| + |F(x_2) - F(x_1)| + \cdots + |F(x_n) - F(x_{n-1})| = 2n$, so $F \notin BV([0,1])$, but $|F|(x) = 1$ for all $x \in [0,1]$ so $|F| \in AC([0,1])$.

(b) Let $G(x) := (-1)^{n-1}x$ if $x \in (1/2^n, 1/2^{n-1}]$ for $n \in \mathbb{N}$ and $G(0) := 0$, so that $|G|(x) = x$ and $|G| \in AC([0,1])$. Also $\operatorname{Var}(G;[0,1]) \le 3/2$, but G is not continuous so it does not belong to $AC([0,1])$.

14.M If $x > 0$, then $H'(x) = rx^{r-1}\sin(1/x^s) - sx^{r-s-1}\cos(1/x^s)$. If $r > s > 0$, then $r - 1 > -1$ and $r - s - 1 > -1$, so that both terms in H' belong to $\mathcal{L}(I)$ and so $H \in AC(I)$.

Conversely, if $r \geq s$, then it is seen as in Examples 7.6(b, c) that the function H is not in $BV([0,1])$.

14.P Use Exercise 14.N. Here $F(x) := \sqrt{x}$ does not satisfy a Lipschitz condition and K is not monotone.

14.S ($\Leftarrow$) Since $|F(v_j)-F(u_j)| \leq \omega_F([u_j,v_j])$, the given condition implies that $F \in AC(I)$.

($\Rightarrow$) Suppose that $F \in AC(I)$ and that η_ε and $\{[u_j,v_j]\}_{j=1}^s$ are as in Definition 14.4. For each j there exists a subinterval $[\tilde{u}_j, \tilde{v}_j]$ of $[u_j, v_j]$ such that

$$\omega_F([u_j, v_j]) - \varepsilon/2^j \leq |F(\tilde{v}_j) - F(\tilde{u}_j)|.$$

But since $F \in AC(I)$, it follows that $\sum_{j=1}^s |F(\tilde{v}_j) - F(\tilde{u}_j)| \leq \varepsilon$, so that

$$\sum_{j=1}^{s} \omega_F([u_j,v_j]) \leq \sum_{j=1}^{s} |F(\tilde{v}_j) - F(\tilde{u}_j)| + \varepsilon \leq 2\varepsilon.$$

14.V (a) If η_ε^i "works" for F_i $(i = 1, \cdots, n)$ in Definition 14.4, we take $\eta_\varepsilon := \min\{\eta_\varepsilon^1, \cdots, \eta_\varepsilon^n\}$.

(b) Given $\varepsilon > 0$, suppose that $\mathrm{Var}(F_m - F_n; I) \leq \varepsilon$ for $m > n \geq K$. If $\{[u_j,v_j]\}_{j=1}^s$ is any subpartition of I and $m \geq K$, then

$$\sum_{j=1}^{s} |F_m(v_j) - F_m(u_j)| \leq \sum_{j=1}^{s} |F_K(v_j) - F_K(u_j)| + \varepsilon.$$

Now take $\eta_\varepsilon := \min\{\eta_\varepsilon^1, \cdots, \eta_\varepsilon^{K-1}, \eta_{\varepsilon/2}^K\}$.

(c) Since $f_m - f_n \in \mathcal{L}(I)$, it follows from Exercise 14.L that we have $\mathrm{Var}(F_m - F_n; I) = \int_I |f_m - f_n| \to 0$ as $m, n \to \infty$. Now apply part (b).

Section 16

16.A Suppose that there exist $C' \neq C''$ satisfying Definition 16.2 and let $\varepsilon := |C'' - C'|/3 > 0$. Let δ'_ε and δ''_ε be corresponding gauges and let $\delta_\varepsilon := \min\{\delta'_\varepsilon, \delta''_\varepsilon\}$. If $\dot{\mathcal{P}}$ is a δ_ε-fine partition of $[a,\infty]$, then $\dot{\mathcal{P}}$ is both δ'_ε-fine and δ''_ε-fine, so that $3\varepsilon = |C' - C''| \leq |C' - S(f;\dot{\mathcal{P}})| + |S(f;\dot{\mathcal{P}}) - C''| \leq 2\varepsilon$, which is a contradiction.

16.D By the extension of the Additivity Theorem 3.7, the integral over $[0,\infty]$ exists if and only if the integrals over $[0,1]$ and $[1,\infty]$ exist. The first integral does not exist for $p \geq 1$ and the second integral does not exist for $p \leq 1$. Thus the integral over $[0,\infty]$ does not exist for any value of $p > 0$.

16.G If $c \in (a, \infty)$, then F is a c-primitive of f on the interval $[a, c]$. By the Fundamental Theorem 4.7, we conclude that $f \in \mathcal{R}^*([a, c])$ and that $\int_a^c f = F(c) - F(a)$. Now let $c \to \infty$ and apply Hake's Theorem to conclude that $f \in \mathcal{R}^*([a, \infty])$ and that $\int_a^\infty f = \lim_{c\to\infty} F(x) - F(a) = F(\infty) - F(a)$.

16.J If $f \in \mathcal{R}^*([a, \infty])$ then, by Hake's Theorem, $I := \lim_{c\to\infty} \int_a^c f$ exists. Therefore, given $\varepsilon > 0$ there exists $K(\varepsilon) > a$ such that if $p \geq K(\varepsilon)$, then $|I - \int_a^p f| \leq \frac{1}{2}\varepsilon$. Thus, if $q > p$, we have $|\int_p^q f| = |\int_a^q f - I| + |I - \int_a^p f| \leq \varepsilon$.

Conversely, if the condition holds, then $\lim_{c\to\infty} \int_a^c f$ exists and Hake's Theorem implies that $f \in \mathcal{R}^*([a, \infty])$.

16.M (a) Note that the integrand approaches 0 as $x \to 0$. Integrating by parts, we have

$$\Big|\int_p^q \frac{\sin x}{\sqrt{x}}\,dx\Big| = \Big| -x^{-1/2}\cos x\Big|_p^q - \tfrac{1}{2}\int_p^q x^{-3/2}\cos x\,dx\Big|$$

$$\leq q^{-1/2} + p^{-1/2} + \tfrac{1}{2}\int_p^q x^{-3/2}\,dx \leq 2(q^{-1/2} + p^{-1/2}) \leq 4p^{-1/2}.$$

If p is sufficiently large, then the final term is $\leq \varepsilon$.

(b) If $u = \Phi(x) := x^2$, then $\big|\int_p^q \sin(x^2)\,dx\big| = \big|\frac{1}{2}\int_{p^2}^{q^2} u^{-1/2}\sin u\,du\big|$. Now use part (a).

(c) Note that the integrand approaches 0 as $x \to 0$. Also

$$\Big|\int_p^q x^{-1/4}\sin x\,dx\Big| = \Big| -x^{-1/4}\cos x\Big|_p^q - \tfrac{1}{4}\int_p^q x^{-5/4}\cos x\,dx\Big|$$

$$\leq q^{-1/4} + p^{-1/4} + x^{-1/4}\Big|_p^q \leq 4p^{-1/4}.$$

If p is sufficiently large, the final term is $\leq \varepsilon$.

(d) If $u = \Phi(x) = x^2$, then $\big|\int_p^q x^{1/2}\sin(x^2)\,dx\big| = \big|\frac{1}{2}\int_{p^2}^{q^2} u^{-1/4}\sin u\,du\big|$. Now use part (c).

16.P (a) The integrand is even, but it is not clear that the integral exists. If $a < 0 < b$, then $\int_a^b e^{-|x|}\,dx = \int_a^0 e^x\,dx + \int_0^b e^{-x}\,dx = (1 - e^a) - (e^{-b} - 1)$. Since $e^a \to 0$ as $a \to -\infty$ and $e^{-b} \to 0$ as $b \to \infty$, the integral equals 2.

(b) The integrand is odd, but it is not clear that the integral exists. If $a < 0 < b$, then $\int_0^b xe^{-x^2}\,dx = \frac{1}{2}(1 - e^{-b^2}) \to \frac{1}{2}$. Similarly, $\int_a^0 xe^{-x^2}\,dx \to -\frac{1}{2}$, so that the integral equals 0.

(c) The integrand approaches 1 as $x \to 0$. Since $e^x/(e^x - e^{-x}) \to 1$ as $x \to \infty$, we have $0 < 2x/(e^x - e^{-x}) \leq 4xe^{-x}$ for x sufficiently large. Thus the integrand is in $\mathcal{R}^*([0, \infty])$ and similarly on $[-\infty, 0]$, since it is even.

(d) The integrand is even. As seen in Exercise 16.H(b), $\int_0^\infty x^2e^{-x}\,dx = 2$. Also $\int_{-\infty}^0 x^2e^x\,dx = 2$, so the integral equals 4.

16.S Since $\varphi(x) \geq 0$, then $0 \leq |\varphi(x)\sin x| \leq \varphi(x)$, so that $x \mapsto \varphi(x)\sin x$ is absolutely integrable on $[a,\infty)$. Similarly if sin is replaced by cos.

(b) Let $c_n := n\pi$ for $n \in \mathbb{N}$. Since $\varphi \geq 0$ is decreasing, if $m > n$ then

$$\begin{aligned}\int_{c_n}^{c_m} |\varphi(x)\sin x|\,dx &= \int_{c_n}^{c_{n+1}} |\varphi(x)\sin x|\,dx + \cdots + \int_{c_{m-1}}^{c_m} |\varphi(x)\sin x|\,dx\\ &\geq [\varphi(c_{n+1}) + \cdots + \varphi(c_m)]\cdot\int_0^\pi |\sin x|\,dx \geq (2/\pi)\int_{c_{n+1}}^{c_{m+1}} \varphi(x)\,dx.\end{aligned}$$

Since $\int_a^\infty \varphi$ is divergent, it follows that $\int_c^\infty |\varphi(x)\sin x|\,dx$ is also divergent.

16.V (a) Note that $(\operatorname{Arctan} x)/\sqrt{x}$ is ultimately decreasing to 0.

(b) Take $f(x) := (\cos x)/x$ and use the fact that $\tanh x$ and $\coth x$ are monotone and converge to 1 as $x \to \infty$.

(c) Let $f(x) := \sin(x^2)$ and $\varphi(x) := x/(x+1)$, so that $f \in \mathcal{R}^*([0,\infty])$ by Exercise 16.M(b) and φ increases to 1.

(d) Let $f(x) := \sqrt{x}\sin(x^2)$ and $\varphi(x) := \sqrt{x/(x+1)}$, so that $f \in \mathcal{R}^*([0,\infty])$ by Exercise 16.M(d) and φ increases to 1.

Section 18

18.A (a) If $A \in \mathcal{D}$, then $\emptyset = A - A \in \mathcal{D}$. Since $A \cap B = A - (A - B)$, then $A \cap B \in \mathcal{D}$. Since $A - B$ and $B - A$ belong to $\mathcal{D}$, so does $A \Delta B$.

(b) Use induction.

18.D (a) If $E, F \in \mathcal{D}_1 \cap \mathcal{D}_2$, then $E, F \in \mathcal{D}_i$, so that $E \cup F$ and $E - F$ are in $\mathcal{D}_i$. Hence $E \cup F$ and $E - F$ are in $\mathcal{D}_1 \cap \mathcal{D}_2$.

(b) Let Ω be an infinite set and let $\mathcal{D}_1 := \{\emptyset, \Omega\}$ and $\mathcal{D}_2$ be the ring of all finite subsets of Ω. The union $\mathcal{D}_1 \cup \mathcal{D}_2$ is not a ring, since if $\emptyset \neq E \in \mathcal{D}_2$, then $E^c = \Omega - E$ does not belong to $\mathcal{D}_1 \cup \mathcal{D}_2$.

(c) If $\mathcal{A}_i$ are σ-algebras of sets in Ω, then they are both rings; therefore, $\mathcal{A}_1 \cap \mathcal{A}_2$ is a ring of sets by (a). Since $\Omega \in \mathcal{A}_1 \cap \mathcal{A}_2$, then $\mathcal{A}_1 \cap \mathcal{A}_2$ is an algebra of sets. If the sequence $(E_n)_{n=1}^\infty$ belongs to $\mathcal{A}_i$, then $\bigcup_{n=1}^\infty E_n$ belongs to $\mathcal{A}_i$, so it belongs to $\mathcal{A}_1 \cap \mathcal{A}_2$.

18.G If $C := \{x_i\}_{i=1}^\infty$ and $\varepsilon > 0$, take $J_i := (x_i - \varepsilon/2^{i+1}, x_i + \varepsilon/2^{i+1})$ so that $|J_i| = \varepsilon/2^i$ and hence $\sum_i |J_i| = \varepsilon$.

If $\mathbb{R}$ is countable, then $\mathbb{R}$ is a null set. Therefore $I := [0,1] \subset \mathbb{R}$ is a null set and $|I| = 0$ (by 18.10), contrary to the fact that $|I| = 1$.

18.J (b) Let $U := \bigcup_{n=1}^{\infty}(n, n+1/2^n)$.

18.M (a) If A is open and $x \in A$, then there exists $r_x > 0$ such that $(x - r_x, x + r_x) \subseteq A$. Therefore, x is an interior point of A.

Conversely, if A is a set such that for every point $z \in A$ there exists $r_z > 0$ such that $(z - r_z, z + r_z) \subseteq A$, then the set A is open.

(b) If $G \subseteq \mathbb{R}$ is a nonempty open set, let $x \in G$ and $r > 0$ be such that $J := (x - r, x + r) \subseteq G$. If $G \in \mathbb{I}(\mathbb{R})$, then since $J \subseteq G$ it follows from Theorem 18.3(e) that $0 < 2r = |J| \leq |G| = \lambda(G)$. If $G \notin \mathbb{I}(\mathbb{R})$, then since $G \in \mathbb{M}(\mathbb{R})$, we have $\lambda(G) = \infty > 0$.

18.P (a) Since every point of $\mathbb{R}$ is an interior point of $\mathbb{R}$, then $F_1 \neq \mathbb{R}$, so there exists $y_1 \in F_1^c$. By Exercise 18.O(b) there exists a nondegenerate closed interval J_1 with center y_1 such that $F_1 \cap J_1 = \emptyset$. Evidently we may also assume that the length of J_1 is ≤ 1. Now F_2 does not contain J_1, and we let $y_2 \in F_2^c \cap J_1$; by the argument in 18.O(b) there exists a nondegenerate closed interval J_2 with center y_2 such that $F_2 \cap J_2 = \emptyset$. We may also assume that $J_2 \subseteq J_1$ and that the length of J_2 is $\leq 1/2$.

(b) Suppose that closed intervals $J_1, \cdots, J_n$ have been chosen as indicated. Now F_{n+1} does not contain J_n, and we let $y_{n+1} \in F_{n+1}^c \cap J_n$. As before there exists a nondegenerate closed interval J_{n+1} with center y_{n+1} such that $F_{n+1} \cap J_{n+1} = \emptyset$ and we may assume that $J_{n+1} \subseteq J_n$ and that the length of J_{n+1} is $\leq 1/(n+1)$.

(c) Since the nondegenerate closed intervals (J_n) are nested and their length approaches 0, the Nested Intervals Theorem implies that there exists $\xi \in \bigcap_{n=1}^{\infty} J_n$. Since $F_n \cap J_n = \emptyset$, it follows that $\xi \notin F_n$ for any $n \in \mathbb{N}$, so that $\bigcup_{n=1}^{\infty} F_n \neq \mathbb{R}$, a contradiction.

18.S If $t \in G$, let $\delta(t) > 0$ be such that $(t - 2\delta(t), t + 2\delta(t)) \subseteq G$ and let $\delta(t) := 1$ for $t \notin G$, so that δ is a gauge on $\mathbb{R}$. Let $\{(J_i, t_i)\}$ be as in 18.15 so that the J_i are nonoverlapping compact intervals and $G \subseteq \bigcup_i J_i$.

From the construction in the proof, for each i there exists t_i such that $J_i = I_{n(t_i)} \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \subseteq (t_i - 2\delta(t_i), t_i + 2\delta(t_i)) \subseteq G$, whence it follows that $\bigcup_i J_i \subseteq G$. Therefore $G = \bigcup_i J_i$.

18.V (a) Since K is compact, it is bounded, say by B. Theorem 18.18 implies that there exists an open set G with $K \subseteq G$ and $|G - K| < |K|$, so that $|G| = |(G - K) \cup K| \leq |G - K| + |K| < 2|K|$. If G is not bounded, replace it by $G \cap (-2B, 2B)$. Since K is closed and G is open, then $K \subset G$.

(b) If $\delta = 0$, then there exist sequences $(k_n) \subset K$ and $(x_n) \subset G^c$ with $|k_n - x_n| \to 0$. By the Bolzano-Weierstrass Theorem [B-S; p. 78] we may assume that $k_n \to k_0$ and $x_n \to x_0$. Since K and G^c are closed, then $k_0 \in K$

and $x_0 \in G^c$. But $|k_0 - x_0| = 0$ implies that $k_0 = x_0$, contrary to the fact that $K \subset G$. Therefore $\delta > 0$.

Let $|y| < \delta$. If K_y intersects G^c, there exist $k_1 \in K$, $z \in G^c$ such that $k_1 + y = z$, so that $|y| = |z - k_1| \geq \delta$, contrary to the hypothesis that $|y| < \delta$.

(c) For, if $K_y \cap K = \emptyset$, then since both K_y and K are compact, they are integrable and 18.3(c) and 18.21 imply that $|K_y \cup K| = |K_y| + |K| = 2|K|$. On the other hand, part (b) implies that $K_y \cup K \subset G$, so that $|K_y \cup K| \leq |G| < 2|K|$, which is a contradiction.

(d) If $y \in (-\delta, \delta)$ is arbitrary, then part (c) implies that $K_y \cap K \neq \emptyset$. Therefore, there exist $k_1, k_2 \in K$ such that $y + k_1 = k_2$, so that $y = k_2 - k_1$ is in $\Delta(K)$.

(e) Since $\lambda(E) = \lim_n |E_n|$, where $E_n := E \cap [-n, n]$ and $\Delta(E_n) \subseteq \Delta(E)$, it suffices to suppose that $E \in \mathbb{I}(\mathbb{R})$. By Exercise 18.T(b), there exists a compact set $K \subseteq E$ such that $|E| - |K| = |E - K| \leq \frac{1}{2}|E|$ so that $|K| > 0$. By part (d) and Exercise 18.U(a) it follows that $(-\delta, \delta) \subseteq \Delta(K) \subseteq \Delta(E)$.

Section 19

19.A (a) (i) $y \in f(C \cap E) \Rightarrow$ there exists $x \in C \cap E$ with $y = f(x) \Rightarrow$ there exists $x \in C$ and $x \in E$ such that $y = f(x) \Rightarrow y \in f(C)$ and $y \in f(E) \Rightarrow y \in f(C) \cap f(E)$. Since y is arbitrary, $f(C \cap E) \subseteq f(C) \cap f(E)$.

(ii) We have: $y \in f(C \cup E) \iff$ there exists $x \in C \cup E$ with $y = f(x) \iff$ either $x \in C$ with $y = f(x) \in f(C)$ or $x \in E$ with $y = f(x) \in f(E) \iff$ either $y \in f(C)$ or $y \in f(E) \iff y \in f(C) \cup f(E)$. Since y is arbitrary, $f(C \cup E) = f(C) \cup f(E)$.

(iii) $y \in f(C) - f(E) \Rightarrow y \in f(C)$ but $y \notin f(E) \Rightarrow$ there exists $x \in C$ with $y = f(x)$ but $x \notin E$ (else $y = f(x) \in f(E)$) $\Rightarrow x \in C - E$ with $y \in f(C - E)$. Since y is arbitrary, $f(C) - f(E) \subseteq f(C - E)$.

Also: $y \in f(C - E) \Rightarrow$ there exists $x \in C - E$ with $y = f(x) \Rightarrow$ there exists $x \in C$ with $y = f(x) \Rightarrow y \in f(C)$. Since y is arbitrary, $f(C - E) \subseteq f(C)$.

(b) Let $f(x) := x^2$ for $x \in \mathbb{R}$ and let $C := [-1, 0]$ and $E := [0, 1]$. Then $f(C \cap E) = \{0\}$, while $f(C) \cap f(E) = [0, 1]$. Also, $f(C) - f(E) = \emptyset$, $f(C - E) = (0, 1]$, and $f(C) = [0, 1]$.

19.D (a) Suppose that $r > 0$ and that $\omega_f(c) < r$. Then there exists s with $0 < s < r$ such that $0 \leq \omega_f(c) < s$. Thus there exists $\delta_s > 0$ such that $\sup\{|f(x) - f(y)| : x, y \in (c - \delta_s, c + \delta_s)\} < s$. Let $|\tilde{c} - c| < \frac{1}{2}\delta_s$ so that if $x, y \in (\tilde{c} - \frac{1}{2}\delta_s, \tilde{c} + \frac{1}{2}\delta_s) \subseteq (c - \delta_s, c + \delta_s)$ then $|f(x) - f(y)| < s$,

whence $\sup\{|f(x) - f(y)| : x, y \in (\tilde{c} - \delta_s, \tilde{c} + \delta_s)\} \le s < r$. Consequently, $\omega_f(\tilde{c}) < r$ for $|\tilde{c} - c| < \frac{1}{2}\delta_s$, showing that the set $\{x : \omega_f(x) < r\}$ is an open set. Therefore $\{x : \omega_f(x) \ge r\}$ is closed.

(b) From Exercise 19.C(a) the set D where f is discontinuous equals $D = \{x : \omega_f(x) > 0\} = \bigcup_{n=1}^{\infty}\{x : \omega_f(x) \ge 1/n\}$. By part (a), D is an F_σ-set.

(c) The set where f is continuous is the complement of D; hence it is a G_δ-set.

19.G We use Exercise 19.F. If the functions f_n are continuous, then the functions $|f_n - f_m|$ are continuous so that the sets $\{|f_n - f_m| \le 1/k\}$ are closed in $\mathbb{R}$. Therefore the intersections of these sets for $n, m \ge N$ are also closed. Consequently, the union for $N \in \mathbb{N}$ yields an F_σ-set, and the intersection for $k \in \mathbb{N}$ is an $F_{\sigma\delta}$-set.

19.J (a) If V is a nonmeasurable subset of $\mathbb{R}$ (see Theorem 18.22), let $f(x) := 1$ for $x \in V$ and $f(x) := -1$ for $x \in \mathbb{R} - V$.

(b) The function $\varphi(x) := \operatorname{sgn} x$ is Borel-measurable, since the sets $\{\varphi > c\}$ all have the form $\mathbb{R}, [0, \infty)$ or $(0, \infty)$. By Theorem 19.12(b), the function $\operatorname{sgn} \circ f$ is Lebesgue measurable when f is.

19.M Since $\mathcal{A}$ is a σ-algebra and $f^{-1}(\emptyset) = \emptyset \in \mathcal{A}$ then $\emptyset \in \mathcal{C}$. Similarly, $f^{-1}(Y) = X \in \mathcal{A}$ implies that $Y \in \mathcal{C}$. If $H \in \mathcal{C}$, then $f^{-1}(H) \in \mathcal{A}$, so that $f^{-1}(Y - H) = X - f^{-1}(H) \in \mathcal{A}$, which implies that $Y - H \in \mathcal{C}$. If $H_n \in \mathcal{C}$ for $n \in \mathbb{N}$, then $f^{-1}(\bigcup_{n=1}^{\infty} H_n) = \bigcup_{n=1}^{\infty} f^{-1}(H_n) \in \mathcal{A}$, which implies that $\bigcup_{n=1}^{\infty} H_n \in \mathcal{C}$. Thus $\mathcal{C}$ is a σ-algebra.

19.P (a) In the preceding exercise it was shown that Ψ is continuous and V is Lebesgue measurable, but $W = \Psi(V)$ is not measurable.

(b) Since $\Phi^{-1} = \Psi$, this follows from (a).

(c) In the notation of the preceding exercise, if $f := \mathbf{1}_V$ and $g := \Phi$, then f is Lebesgue measurable and g is continuous, but $(f \circ g)^{-1}(\{1\}) = \Phi^{-1}(\mathbf{1}_V^{-1}(\{1\})) = \Psi(V) = W$. Hence $f \circ g$ is not Lebesgue measurable.

19.S (a) Let $f_n(x) := x^n$ for $x \in (-1, 1]$ and have period 2 (i.e., be such that $f_n(x+2) = f_n(x)$ for all $x \in \mathbb{R}$). It is easy to see that $f(x) := \inf\{f_n(x)\}$ equals 1 when $x = 2n + 1,\ n \in \mathbb{Z}$, and that $f(x) = 0$ elsewhere.

(b) Let $\varepsilon > 0$. Since $f(c) = \inf\{f_n(c) : n \in \mathbb{N}\}$, there exists N such that $f_N(c) < f(c) + \varepsilon/2$. Since f_N is continuous, there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f_N(x) - f_N(c)| < \varepsilon/2$. But this implies that if $|x - c| < \delta$, then $f(x) \le f_N(x) \le f_N(c) + \varepsilon/2 < f(c) + \varepsilon$. Thus f is upper semicontinuous at c.

19.V (a) If $(B_i)_{i=1}^n \in \mathcal{A}$ is a decomposition of $E \cup F$, then $(B_i \cap E)_{i=1}^n$ is a decomposition of E and $(B_i \cap F)_{i=1}^n$ is a decomposition of F. Since $(B_i \cap E) \cap (B_i \cap F) \subseteq E \cap F = \emptyset$, then $\gamma(B_i \cap (E \cup F)) = \gamma(B_i \cap E) + \gamma(B_i \cap F)$. Using the triangle inequality and summing, we obtain

$$\sum_{i=1}^n |\gamma(B_i \cap (E \cup F))| \le \sum_{i=1}^n |\gamma(B_i \cap E)| + \sum_{i=1}^n |\gamma(B_i \cap F)| \le |\gamma|(E) + |\gamma|(F),$$

from which it follows that $|\gamma|(E \cup F) \le |\gamma|(E) + |\gamma|(F)$.

On the other hand, given $\varepsilon > 0$ there is a decomposition $(C_i)_{i=1}^n$ of E and a decomposition $(D_i)_{i=1}^m$ of F such that

$$\sum_{i=1}^n |\gamma(C_i)| \ge |\gamma|(E) - \varepsilon/2 \qquad \text{and} \qquad \sum_{i=1}^m |\gamma(D_i)| \ge |\gamma|(F) - \varepsilon/2.$$

If $(B_i)_{i=1}^{n+m}$ is the union of these two decompositions, then it is a decomposition of $E \cup F$, and we have

$$\sum_{i=1}^n |\gamma(C_i)| + \sum_{i=1}^m |\gamma(D_j)| = \sum_{i=1}^{n+m} |\gamma(B_i)|.$$

Now the left side is $\ge |\gamma|(E) + |\gamma|(F) - \varepsilon$, and the right side is $\le |\gamma|(E \cup F)$. Since $\varepsilon > 0$ is arbitrary, we conclude that $|\gamma|(E) + |\gamma|(F) \le |\gamma|(E \cup F)$.

We therefore conclude that $|\gamma|$ is additive on $\mathcal{A}$ and an induction argument shows that $|\gamma|$ is finitely additive in the sense of 18.3(g).

(b) If $E \subseteq F$, then $F = E \cup (F - E)$ and $E \cap (F - E) = \emptyset$, so that $|\gamma|(F) = |\gamma|(E) + |\gamma|(F - E) \ge |\gamma|(E)$, showing that $|\gamma|$ is monotone.

If $(B_i)_{i=1}^n$ is a decomposition of E, we may relabel to obtain $\gamma(B_i) \ge 0$ for $i = 1, \cdots, r$ and $\gamma(B_i) < 0$ for $i = r+1, \cdots, n$. We note that

$$\sum_{i=1}^n |\gamma(B_i)| = \sum_{i=1}^r \gamma(B_i) - \sum_{i=r+1}^n \gamma(B_i) = \gamma\Big(\bigcup_{i=1}^r B_i\Big) - \gamma\Big(\bigcup_{i=r+1}^n B_i\Big).$$

Since both terms on the right have absolute values $\le M$, we have $|\gamma|(E) \le 2M$.

(c) If $(E_k)_{k=1}^\infty$ is a pairwise disjoint sequence in $\mathcal{A}$ with union E, then it follows from the finite additivity of $|\gamma|$ and the fact that $|\gamma|$ is monotone that for any $n \in \mathbb{N}$, we have

$$\sum_{k=1}^n |\gamma|(E_k) = |\gamma|\Big(\bigcup_{k=1}^n E_k\Big) \le |\gamma|(E),$$

so that $\sum_{k=1}^\infty |\gamma|(E_k) \le |\gamma|(E)$.

On the other hand, if $(C_i)_{i=1}^r$ is a decomposition of E, then it follows from the countable additivity of γ that

$$\begin{aligned}\sum_{i=1}^r |\gamma(C_i)| &= \sum_{i=1}^r \Big|\gamma\Big(\bigcup_{k=1}^\infty C_i \cap E_k\Big)\Big| = \sum_{i=1}^r \Big|\sum_{k=1}^\infty \gamma(C_i \cap E_k)\Big| \\ &\le \sum_{i=1}^r \sum_{k=1}^\infty \big|\gamma(C_i \cap E_k)\big| = \sum_{k=1}^\infty \sum_{i=1}^r |\gamma(C_i \cap E_k)| \le \sum_{k=1}^\infty |\gamma|(E_k),\end{aligned}$$

where we have reversed the order of summation in the iterated sum with nonnegative terms. Taking the supremum over all decompositions $(C_i)_{i=1}^r$, we have $|\gamma|(E) \le \sum_{k=1}^\infty |\gamma|(E_k)$.

Therefore the countable additivity of $|\gamma|$ is proved.

Section 20

20.A (a) Since $F = E \cup (F - E)$ and $E \cap (F - E) = \emptyset$, it follows that $m(F) = m(E) + m(F - E) \ge m(E)$. If $m(E)$ is finite, we can subtract it.

(b) Let $B_1 := A_1$ and $B_{n+1} := A_{n+1} - \bigcup_{k=1}^n A_k$ for $n \in \mathbb{N}$, so that $\bigcup_{k=1}^\infty B_n = \bigcup_{k=1}^\infty A_k$. Since the (B_n) are pairwise disjoint and since $m(B_k) \le m(A_k)$, then $m\big(\bigcup_{k=1}^\infty A_k\big) = m\big(\bigcup_{k=1}^\infty B_k\big) = \sum_{k=1}^\infty m(B_k) \le \sum_{k=1}^\infty m(A_k) \le \infty$.

(c) If $m(E_k) = \infty$ for some k, the result is trivial; therefore we suppose otherwise. Let $A_1 := E_1$ and $A_k := E_k - E_{k-1}$ for $n > 1$. Thus (A_k) is a pairwise disjoint sequence in $\mathcal{A}$ and $E_k = \bigcup_{j=1}^k A_j$ and $\bigcup_{k=1}^\infty E_k = \bigcup_{k=1}^\infty A_k$. Now $m(A_k) = m(E_k) - m(E_{k-1})$ for $k > 1$ so that $\sum_{k=1}^N m(A_k) = m(E_N)$. Since m is countably additive, then we have $m\big(\bigcup_{k=1}^\infty E_k\big) = m\big(\bigcup_{k=1}^\infty A_k\big) = \sum_{k=1}^\infty m(A_k) = \lim_N \sum_{k=1}^N m(A_k) = \lim_N m(E_N)$.

(d) Let $E_k := F_1 - F_k$ so that (E_k) is an increasing sequence; since $m(F_k) \le m(F_1) < \infty$, we have $m(E_k) = m(F_1) - m(F_k)$. By part (c), we have

$$m\Big(\bigcup_{k=1}^\infty E_k\Big) = \lim_{k\to\infty} m(E_k) = \lim_{k\to\infty} \big[m(F_1) - m(F_k)\big] = m(F_1) - \lim_{k\to\infty} m(F_k).$$

Since $\bigcup_{k=1}^\infty E_k = F_1 - \bigcap_{k=1}^\infty F_k$, then $m\big(\bigcup_{k=1}^\infty E_k\big) = m(F_1) - m\big(\bigcap_{k=1}^\infty F_k\big)$. Combine this with the displayed equation to obtain the desired conclusion.

(e) Let $F_k := [k, \infty)$ in $\mathbb{M}(\mathbb{R})$, so that $m\big(\bigcap_{k=1}^\infty F_k\big) = m(\emptyset) = 0$, while $m(F_k) = \infty$.

20.D (a) Not [a.u.], not [meas], not [mean]. $T_n(r) = (nr, \infty)$. Not VT or FT.

(b) [a.u.], [meas], not [mean]. $T_n(r) = [0, n]$ for $n < 1/r$ and $T_n(r) = \emptyset$ if $n \geq 1/r$. Has VT and ET.

(c) [a.u.], [meas], [mean]. $T_n(r) = [0, n]$ if $n \leq 1/\sqrt{r}$ and $T_n(r) = \emptyset$ if $n \geq 1/\sqrt{r}$. Has VT and ET.

(d) Not [a.u.], not [meas], not [mean]. $T_n(r) = [n, \infty)$. Not VT or FT.

(e) Not [a.u.], but [meas], [mean]. $T_n(r) = \bigcup_{k=n}^{\infty}[k, k + 1/k]$. Not VT or FT.

(f) Not [a.u.], but [meas], not [mean]. $T_n(r)$ as in (e). Not VT or FT.

(g) [a.u.], [meas], [mean]. $T_n(r) = \bigcup_{k=n}^{\infty}[k, k + 1/k^2]$. Has VT and FT.

(h) [a.u.], [meas], [mean]. $T_n(r)$ as in (g). Has VT and FT.

20.G ($\Rightarrow$) If (f_k) is a uniform Cauchy sequence, then given $r > 0$ there exists $n(r)$ such that if $k, j \geq n(r)$, $x \in X$, then $|f_k(x) - f_j(x)| \leq r$. Thus, if $k, j \geq n(r)$, then we have $\tilde{T}_{n(r)}(r) = \emptyset$, so that (f_k) has the ECT property.

($\Leftarrow$) If (f_k) has the ECT property, then given $r > 0$ there exists $n(r)$ such that $\tilde{T}_{n(r)}(r) = \emptyset$. Therefore, if $k, j \geq n(r), x \in X$, we have $|f_k(x) - f_j(x)| \leq r$. Since this holds for all $r > 0$, the sequence (f_k) is a uniform Cauchy sequence.

20.J (a) Indeed, $f_k(x) \to f(x)$ if and only if $\sup_{k \geq n} |f_k(x) - f(x)| \to 0$ as $n \to \infty$.

(b) If $x \in T_n(r)$, then $|f_k(x) - f(x)| > r$ for some $k \geq n$ so that $\psi_n(x) = \sup_{k \geq n} |f_k(x) - f(x)| > r$.

(c) $T_{n(r)}(r) = \emptyset \iff \{\psi_{n(r)} > r\} = \emptyset \iff \{\psi_{n(r)}(r) \leq r\} = X \iff 0 \leq \psi_{n(r)}(x) \leq r$ for all $x \in X$.

(d) In view of (b), we have $m(T_n(r)) = m(\{\psi_n > r\})$.

(e) In view of (b), we have $m(T_{n(r)}(r)) = m(\{\psi_{n(r)} > r\})$.

20.M (a) $(f_k(x))$ is Cauchy $\iff \sup_{k,j \geq n} |f_k(x) - f_j(x)| \to 0$ as $n \to \infty$.

(b) $x \in \tilde{T}_n(r) \iff |f_k(x) - f_j(x)| > r$ for some $k, j \geq n \iff \tilde{\psi}_n(x) = \sup_{k,j \geq n} |f_k(x) - f_j(x)| > r$.

(c) $\tilde{T}_{n(r)}(r) = \emptyset \iff \{\psi_{n(r)} > r\} = \emptyset$.

(d, e) Use part (b).

20.P (a) If (f_k) has the VCT property, then (by Exercise 20.H(b)) there exists an $\mathcal{A}$-measurable function g such that $f_k \to g$ a.u. and therefore in measure to g. If the sequence (f_k) also converges in measure to some function f, we want to show that $f = g$ a.e. Indeed, if $r > 0$, then since $\{|f - g| > r\} \subseteq \{|f - f_k| > r/2\} \cup \{|g - f_k| > r/2\}$, we have $m(\{|f - g| > r\}) = 0$. Since $\{|f - g| > 0\} = \bigcup_{n=1}^{\infty}\{|f - g| > 1/n\}$, we conclude that $f = g$ a.e.

(b) If (f_k) converges in mean to f, then it also converges in measure, so we can use part (a).

20.S Let $B_k(r) := \{|f_k - f| > r\}$ so $B_k(r) \in \mathbb{M}(\mathbb{R})$ and $|B_k(r)| \leq (1/r)\|f_k - f\|$. But since $T_n(r) = \bigcup_{k=n}^{\infty} B_k(r)$, the hypothesis implies that $|T_n(r)| \leq (1/r)\sum_{k=n}^{\infty} \|f_k - f\| \to 0$ as $n \to \infty$. Therefore f_k, f have the VT property and so (f_k) converges a.u. to f.

20.V (a) Indeed, if $f_n(x) \to f(x)$, then $\varphi \circ f_n(x) \to \varphi \circ f(x)$.

(b) If φ is not continuous at $c \in \mathbb{R}$, there exist $\varepsilon_0 > 0$ and a sequence (c_n) such that $c_n \to c$, but $|\varphi(c_n) - \varphi(c)| \geq \varepsilon_0$. Let $f_n(x) := c_n$ and $f(x) := c$ for $x \in \mathbb{R}$. Then (f_n) converges uniformly to f, but $(\varphi \circ f_n)$ does not converge at any point or in measure to $\varphi \circ f$.

(c) If φ is uniformly continuous and if $|x-y| \leq \delta_\varepsilon$, then $|\varphi(x)-\varphi(y)| \leq \varepsilon$. Thus, if $|f_n(x) - f(x)| \leq \delta_\varepsilon$, then we have $|\varphi \circ f_n(x) - \varphi \circ f(x)| \leq \varepsilon$. Consequently, if $f_n \to f$ uniformly (or a.u.), then $\varphi \circ f_n \to \varphi \circ f$ uniformly (or a.u.). Similarly, $\{|\varphi \circ f_n - \varphi \circ f| > \varepsilon\} \subseteq \{|f_n - f| > \delta_\varepsilon\}$.

(d) If φ is not uniformly continuous, there exist $\varepsilon > 0$ and sequences $(x_n), (y_n)$ in $\mathbb{R}$ such that $|x_n - y_n| \leq 1/n$ but $|\varphi(x_n) - \varphi(y_n)| > \varepsilon$. Let f_n, f be defined on $\mathbb{R}$ by $f_n(x) := 0 =: f(x)$ for $x < 0$ and $f(x) := x_k$ for $x \in [k-1, k)$, $k \in \mathbb{N}$, and $f_n(x) := y_k$ for $x \in [k-1, k)$ and $f_n(x) := f(x)$ elsewhere on $[0, \infty)$. Then (f_n) converges uniformly, and hence almost uniformly and in measure to f, but $(\varphi \circ f_n)$ does not converge uniformly, almost uniformly, or in measure to $\varphi \circ f$.

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